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LIMITS OF BIFRACTIONAL BROWNIAN NOISES

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ABSTRACT. Let $B^{H,K} = \left(B_t^{H,K}, t \geq 0\right)$ be a bifractional Brownian motion with two parameters $H \in (0,1)$ and $K \in (0,1]$. The main result of this paper is that the increment process generated by the bifractional Brownian motion $\left(B_{h+t}^{H,K} - B_h^{H,K}, t \geq 0\right)$ converges when $h \to \infty$ to $\left(2^{(1-K)/2} B_t^{HK}, t \geq 0\right)$, where $\left(B_t^{HK}, t \geq 0\right)$ is the fractional Brownian motion with Hurst index HK. We also study the behavior of the noise associated to the bifractional Brownian motion and limit theorems to $B^{H,K}$.

1. Introduction

Introduced in [4], the bifractional Brownian motion, a generalization of the fractional Brownian motion, has been studied in many aspects (see [1], [3], [6], [7], [8], [9] and [10]). This stochastic process is defined as follows. Let $H \in (0,1)$ and $K \in (0,1]$. Then $B^{H,K} = \left(B_t^{H,K}, t \geq 0\right)$ is a centered Gaussian process with covariance

$$E\left[B_t^{H,K} B_s^{H,K}\right] = 2^{-K}\left((t^{2H} + s^{2H})^K - |t-s|^{2HK}\right).$$

When $K = 1$, it is the fractional Brownian motion $B^H = \left(B_t^H, t \geq 0\right)$ with the Hurst index $H \in (0,1)$. In general, the process $B^{H,K}$ has the following basic properties: It is a selfsimilar stochastic process of order $HK \in (0,1)$, the increments are not stationary and it is a quasi-helix in the sense of [5] since for every $s, t \geq 0$, we have

$$2^{-K}|t-s|^{2HK} \leq E\left[\left(B_t^{H,K} - B_s^{H,K}\right)^2\right] \leq 2^{1-K}|t-s|^{2HK}.$$

The trajectories of the process $B^{H,K}$ are δ-Hölder continuous for any $\delta < HK$ and they are nowhere differentiable.

A better understanding of this process has been presented in the paper [7], where the authors showed a decomposition of $B^{H,K}$ with $H, K \in (0,1)$ as follows. Let $(W_\theta, \theta \geq 0)$ be a standard Brownian motion independent of $B^{H,K}$. For any $K \in (0,1)$, they defined a centered Gaussian process $X^K = \left(X_t^K, t \geq 0\right)$ by

$$X_t^K = \int_0^\infty (1 - e^{-\theta t})\theta^{-(1+K)/2} dW_\theta. \tag{1.1}$$

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Its covariance is

$$E\left[X_t^K X_s^K\right] = \Gamma(1-K)K^{-1}\left(t^K + s^K - (t+s)^K\right). \tag{1.2}$$

Then they showed, by setting

$$X_t^{H,K} := X_{t^{2H}}^K, \tag{1.3}$$

that

$$\left(C_1 X_t^{H,K} + B_t^{H,K}, t \geq 0\right) \overset{\mathrm{d}}{=} \left(C_2 B_t^{HK}, t \geq 0\right), \tag{1.4}$$

where $C_1 = (2^{-K}K(\Gamma(1-K))^{-1})^{1/2}, C_2 = 2^{(1-K)/2}$ and $\overset{\mathrm{d}}{=}$ means equality of all finite dimensional distributions.

The main purpose of this paper is to study the increment process

$$\left(B_{h+t}^{H,K} - B_h^{H,K}, t \geq 0\right)$$

(where $h \geq 0$) of $B^{H,K}$ and the noise generated by $B^{H,K}$ and to see how close this process is to a process with stationary increments. In principle, since the bifractional Brownian motion is not a process with stationary increments, its increment process depends on h. But in this paper we show, by using the decomposition (1.4), that for $h \to \infty$ the increment process $\left(B_{t+h}^{H,K} - B_h^{H,K}, t \geq 0\right)$ converges, modulo a constant, to the fractional Brownian motion with Hurst index HK in the sense of finite dimensional distributions, so the dependence of the increment process depending on h decreases for very large h. Somehow, one can interpret that, for very big starting point, the bifractional Brownian motion has stationary increments. Then we will try to understand this property from the perspective of the "noise" generated by $B^{H,K}$ i.e. the Gaussian sequence $B_{n+1}^{H,K} - B_n^{H,K}$, where $n \geq 0$ is an integer. The behavior of the sequence

$$Y_a(n) = E\left[\left(B_{a+1}^{H,K} - B_a^{H,K}\right)\left(B_{a+n+1}^{H,K} - B_{a+n}^{H,K}\right)\right], \quad a \in \mathbb{N},$$

(which, if $K = 1$, is constant with respect to a and of order n^{2H-2}) is studied with respect to a and with respect to n in order to understand the contributions of B^{HK} and $X^{H,K}$.

We organize our paper as follows. In Section 2 we prove our principal result which says that the increment process of $B^{H,K}$ converges to the fractional Brownian motion B^{HK}. Sections 3-5 contain some consequences and different views of this main result. We analyze the noise generated by the bifractional Brownian motion and we study its asymptotic behavior and we interpret the process $X^{H,K}$ as the difference between "the even part" and "the odd part" of the fractional Brownian motion. Finally, in Section 6 we prove limit theorems to the bifractional Brownian from a correlated non-stationary Gaussian sequence.

2. The Limiting Process of the Bifractional Brownian Motion

In this section, we prove the following main result; it says that the increment process of the bifractional Brownian motion converges to the fractional Brownian motion with Hurst index HK.

Theorem 2.1. *Let $K \in (0,1)$. Then*

$$\left(B^{H,K}_{h+t} - B^{H,K}_{h}, t \geq 0\right) \overset{\mathrm{d}}{\Rightarrow} \left(2^{(1-K)/2} B^{HK}_t, t \geq 0\right) \quad \text{as } h \to \infty,$$

where $\overset{\mathrm{d}}{\Rightarrow}$ *means convergence of all finite dimensional distributions.*

To prove Theorem 2.1, we use the decomposition (1.4). It is enough to show that the increment process associated to $X^{H,K}$ converges to zero; we prove it in the next result, and actually we measure how fast it tends to zero with respect to L^2 norm. It will be useful to compare this rate of convergence with results in the following sections.

Proposition 2.2. *Let $X^{H,K}$ be the process given by* (1.3). *Then as $h \to \infty$*

$$E\left[\left(X^{H,K}_{h+t} - X^{H,K}_{h}\right)^2\right] = \Gamma(1-K)K^{-1}2^K H^2 K(1-K)t^2 h^{2(HK-1)}(1+o(1)).$$

As a consequence,

$$\left(X^{H,K}_{h+t} - X^{H,K}_{h}, t \geq 0\right) \overset{\mathrm{d}}{\Rightarrow} (X(t) \equiv 0, t \geq 0) \quad \text{as } h \to \infty.$$

Proof. Note from (1.2) and (1.3) that

$$E\left[X^{H,K}_t X^{H,K}_s\right] = \Gamma(1-K)K^{-1}\left(t^{2HK} + s^{2HK} - \left(t^{2H} + s^{2H}\right)^K\right)$$

and in particular, for every $t \geq 0$

$$E\left[\left(X^{H,K}_t\right)^2\right] = \Gamma(1-K)K^{-1}(2-2^K)t^{2HK}.$$

We have

$$E\left[\left(X^{H,K}_{h+t} - X^{H,K}_{h}\right)^2\right] = E\left[\left(X^{H,K}_{h+t}\right)^2\right] - 2E\left[X^{H,K}_{h+t}X^{H,K}_{h}\right] + E\left[\left(X^{H,K}_{h}\right)^2\right].$$

Then

$$\begin{aligned}
I &:= K(\Gamma(1-K))^{-1}E\left[\left(X^{H,K}_{h+t} - X^{H,K}_{h}\right)^2\right] \\
&= \Big((2-2^K)(h+t)^{2HK} \\
&\qquad - 2\left((h+t)^{2HK} + h^{2HK} - \left((h+t)^{2H} + h^{2H}\right)^K\right) + (2-2^K)h^{2HK}\Big) \\
&= -2^K\left((h+t)^{2HK} + h^{2HK}\right) + 2\left((h+t)^{2H} + h^{2H}\right)^K \\
&= -2^K h^{2HK}\left(1 + (th^{-1})^{2HK} + 1\right) + 2h^{2HK}\left((1+th^{-1})^{2H} + 1\right)^K.
\end{aligned}$$

Therefore for very large $h > 0$ we obtain by using Taylor's expansion

$$\begin{aligned}
I = &-2^K h^{2HK}\left(2 + 2HKth^{-1} + H(2H-1)t^2h^{-2}(1+o(1))\right) \\
&+ 2h^{2HK}\left(2 + 2Hth^{-1} + H(2H-1)t^2h^{-2}(1+o(1))\right)^K.
\end{aligned}$$

Now we use Taylor expansion for the function $(2+Z)^K$ for Z close to zero. In our case $Z = 2Hth^{-1} + H(2H-1)t^2h^{-2} + r(h)$ with $r(h)h^2 \to 0$ as $h \to \infty$. We obtain

$$\begin{aligned} I &= -2^K h^{2HK}\left(2 + 2HKth^{-1} + H(2H-1)t^2h^{-2}(1+o(1))\right) \\ &\quad + 2h^{2HK}\Big(2^K + K2^{K-1}(2Hth^{-1} + H(2H-1)t^2h^{-2} + r(h)) \\ &\qquad + 2^{-1}K(K-1)2^{K-2}(2Hth^{-1} + H(2H-1)t^2h^{-2} + r(h))^2 + o(h^{-2})\Big) \\ &= h^{2HK}2^K HK(-2HK + 1 + 2H - 1 + HK - H)t^2h^{-2}(1+o(1)) \\ &= h^{2HK}2^K H^2K(1-K)t^2h^{-2}(1+o(1)). \end{aligned}$$

Consequently, we have

$$E\left[\left(X_{h+t}^{H,K} - X_h^{H,K}\right)^2\right] = \Gamma(1-K)K^{-1}2^K H^2K(1-K)t^2h^{2(HK-1)}(1+o(1)),$$

which tends to 0 as $h \to \infty$, since $HK - 1 < 0$. □

3. Bifractional Brownian Noise

By considering the bifractional Brownian noise, which are increments of bifractional Brownian motion, we can understand Theorem 2.1 in a different way. Define for every integer $n \geq 0$, the bifractional Brownian noise

$$Y_n = B_{n+1}^{H,K} - B_n^{H,K}.$$

Remark 3.1. Recall that in the fractional Brownian motion case ($K = 1$) we have for every $a \in \mathbb{N}$ and for every $n \geq 0$, $E\left[Y_aY_{a+n}\right] = E\left[Y_0Y_n\right]$.

Let us denote

$$R(0,n) = E[Y_0Y_n] = E\left[B_1^{H,K}\left(B_{n+1}^{H,K} - B_n^{H,K}\right)\right],$$

$$R(a,a+n) = E\left[Y_aY_{a+n}\right] = E\left[\left(B_{a+1}^{H,K} - B_a^{H,K}\right)\left(B_{a+n+1}^{H,K} - B_{a+n}^{H,K}\right)\right]. \tag{3.1}$$

Let us compute the term $R(a,a+n)$ and understand how different it is from $R(0,n)$. We have for every $n \geq 1$,

$$\begin{aligned} R(a,a+n) &= 2^{-K}\Big(\left((a+1)^{2H} + (a+n+1)^{2H}\right)^K - n^{2HK} \\ &\qquad - \left((a+1)^{2H} + (a+n)^{2H}\right)^K - (n-1)^{2HK} \\ &\qquad - \left(a^{2H} + (a+n+1)^{2H}\right)^K - (n+1)^{2HK} \\ &\qquad + \left(a^{2H} + (a+n)^{2H}\right)^K - n^{2HK}\Big) \\ &:= 2^{-K}(f_a(n) + g(n)), \end{aligned} \tag{3.2}$$

where $f_a(n)$ and $g(n)$ are given by

$$\begin{aligned} f_a(n) &= \left((a+1)^{2H} + (a+n+1)^{2H}\right)^K - \left((a+1)^{2H} + (a+n)^{2H}\right)^K \\ &\quad - \left(a^{2H} + (a+n+1)^{2H}\right)^K + \left(a^{2H} + (a+n)^{2H}\right)^K, \end{aligned}$$

$$g(n) = (n+1)^{2HK} + (n-1)^{2HK} - 2n^{2HK}.$$

Remark 3.2. (i) The function g is, modulo a constant, the covariance function of the fractional Brownian noise with Hurst index HK. Indeed, for $n \geq 1$,

$$g(n) = 2E\left[B_1^{HK}(B_{n+1}^{HK} - B_n^{HK})\right]. \tag{3.3}$$

(ii) g vanishes if $2HK = 1$.

(iii) The function f_a is a "new function" specific to the bifractional Brownian case. (Note that f_a vanishes in the case $K = 1$.) It corresponds to the noise generated by $X^{H,K}$. Indeed, it follows easily from (1.4) that

$$\begin{aligned} f_a(n) &= -2^K C_1^2 E\left[\left(X_{a+1}^{H,K} - X_a^{H,K}\right)\left(X_{a+n+1}^{H,K} - X_{a+n}^{H,K}\right)\right] \\ &= -2^K C_1^2 R^{X^{H,K}}(a, a+n) \end{aligned} \tag{3.4}$$

for every a and $n \in \mathbb{N}$.

We need to analyze the function f_a to understand "how far" the bifractional Brownian noise is from the fractional Brownian noise. In other words, how far is the bifractional Brownian motion from a process with stationary increments?

Theorem 3.3. *For each n it holds that as $a \to \infty$*

$$f_a(n) = 2H^2K(K-1)a^{2(HK-1)}(1+o(1)).$$

Therefore $\lim_{a\to\infty} f_a(n) = 0$ *for each n.*

The bifractional Brownian noise is not stationary. However, the meaning of the theorem above is that it converges to a stationary sequence.

Proof. We have, for $a \to \infty$,

$$\begin{aligned} f_a(n) &= a^{2HK}\Big[\big\{(1+a^{-1})^{2H} + (1+(n+1)a^{-1})^{2H}\big\}^K \\ &\quad - \big\{(1+a^{-1})^{2H} + (1+na^{-1})^{2H}\big\}^K - \big\{1 + (1+(n+1)a^{-1})^{2H}\big\}^K \\ &\quad + \big\{1 + (1+na^{-1})^{2H}\big\}^K\Big] \\ &= a^{2HK}\Big[\big\{1 + 2Ha^{-1} + H(2H-1)a^{-2}(1+o(1)) \\ &\qquad + 1 + 2H(n+1)a^{-1} + H(2H-1)(n+1)^2a^{-2}(1+o(1))\big\}^K \\ &\quad - \big\{1 + 2Ha^{-1} + H(2H-1)a^{-2}(1+o(1)) \\ &\qquad + 1 + 2Hna^{-1} + H(2H-1)n^2a^{-2}(1+o(1))\big\}^K \\ &\quad - \big\{1 + 1 + 2H(n+1)a^{-1} + H(2H-1)(n+1)^2a^{-2}(1+o(1))\big\}^K \\ &\quad - \big\{1 + 1 + 2H(n+1)a^{-1} + H(2H-1)(n+1)^2a^{-2}(1+o(1))\big\}^K\Big] \\ &= 2a^{2HK}\Big[\big\{1 + H(n+2)a^{-1} \\ &\qquad + 2^{-1}H(2H-1)(1+(n+1)^2)a^{-2}(1+o(1))\big\}^K \\ &\quad - \big\{1 + H(n+1)a^{-1} + 2^{-1}H(2H-1)(1+n^2)a^{-2}(1+o(1))\big\}^K \end{aligned}$$

$$
\begin{aligned}
&\quad - \left\{1 + H(n+1)a^{-1} + 2^{-1}H(2H-1)(n+1)^2a^{-2}(1+o(1))\right\}^K \\
&\quad + \left\{1 + Hna^{-1} + 2^{-1}H(2H-1)n^2a^{-2}(1+o(1))\right\}^K\Big] \\
&= 2a^{2HK}\Big[\big\{1 + K(H(n+2)a^{-1} \\
&\qquad + 2^{-1}H(2H-1)(1+(n+1)^2)a^{-2}(1+o(1))) \\
&\qquad + 2^{-1}K(K-1)(H(n+2)a^{-1}(1+o(1)))^2(1+o(1))\big\} \\
&\quad - \big\{1 + K(H(n+1)a^{-1} + 2^{-1}H(2H-1)(1+n^2)a^{-2}(1+o(1))) \\
&\qquad + 2^{-1}K(K-1)(H(n+1)a^{-1}(1+o(1)))^2(1+o(1))\big\} \\
&\quad - \big\{1 + K(H(n+1)a^{-1} \\
&\qquad + 2^{-1}H(2H-1)(1+(n+1)^2)a^{-2}(1+o(1))) \\
&\qquad + 2^{-1}K(K-1)(H(n+1)a^{-1}(1+o(1)))^2(1+o(1))\big\} \\
&\quad + \big\{1 + K(Hna^{-1} + 2^{-1}H(2H-1)n^2a^{-2}(1+o(1))) \\
&\qquad + 2^{-1}K(K-1)(Hna^{-1}(1+o(1)))^2(1+o(1))\big\}\Big] \\
&= 2a^{2HK}\Big[(KH(n+2) - KH(n+1) - KH(n+1) + KHn)a^{-1} \\
&\quad + 2^{-1}KH(2H-1)(1+(n+1)^2) + 2^{-1}K(K-1)H^2(1+(n+1)^2) \\
&\quad - 2^{-1}KH(2H-1)(n^2+1) - 2^{-1}K(K-1)H^2(n+1)^2 \\
&\quad - 2^{-1}KH(2H-1)(n+1)^2 - 2^{-1}K(K-1)H^2(n+1)^2 \\
&\quad + 2^{-1}KH(2H-1)n^2 + 2^{-1}K(K-1)H^2n^2)\}a^{-2}(1+o(1))\Big] \\
&= 2H^2K(K-1)a^{2(HK-1)}(1+o(1)).
\end{aligned}
$$

Since $HK - 1 < 0$, the last term tends to 0 when a goes to the infinity. $\square$

Remark 3.4. The fact that the term $f_a(n)$ converges to zero as $a \to \infty$ could be seen by Proposition 2.2 since, using Hölder inequalities,

$$
R^{X^{H,K}}(a, a+n) \le \left(E\left[\left(X^{H,K}_{a+1} - X^{H,K}_{a}\right)^2\right]\right)^{1/2} \left(E\left[\left(X^{H,K}_{a+n+1} - X^{H,K}_{a+n}\right)^2\right]\right)^{1/2}
$$

and both factors on the right hand side above are of order a^{HK-1}. The result actually confirms that for large a, $X^{H,K}_{a+n+1} - X^{H,K}_{a+n}$ is very close to $X^{H,K}_{a+1} - X^{H,K}_{a}$.

4. The Behavior of Increments of the Bifractional Brownian Motion

In this section we continue the study of the bifractional Brownian noise (3.1). We are now interested in the behavior with respect to n (as $n \to \infty$). We know that as $n \to \infty$ the fractional Brownian noise with Hurst index HK behaves as $HK(2HK - 1)n^{2(HK-1)}$. Given the decomposition (1.4) it is natural to ask what the contribution of the bifractional Brownian noise to this is and what the contribution of the process $X^{H,K}$ is. We have the following.

Theorem 4.1. *For integers $a, n \geq 0$, let $R(a, a+n)$ be given by* (3.1). *Then for large n,*

$$R(a,a+n) = 2^{-K}\Big(2HK(2HK-1)n^{2(HK-1)} \\ +HK(K-1)\left((a+1)^{2H}-a^{2H}\right)n^{2(HK-1)+(1-2H)}+\cdots\Big).$$

Proof. Recall first that by (3.2) and (3.3),

$$R(a,a+n) = 2^{-K}(f_a(n)+g(n))$$

and the term $g(n)$ behaves as $2HK(2HK-1)n^{2(HK-1)}$ for large n. Let us study the behavior of the term $f_a(n)$ for large n. We have

$$\begin{aligned}
f_a(n) &= \left((a+1)^{2H}+(a+n+1)^{2H}\right)^K - \left((a+1)^{2H}+(a+n)^{2H}\right)^K \\
&\quad - \left(a^{2H}+(a+n+1)^{2H}\right)^K + \left(a^{2H}+(a+n)^{2H}\right)^K \\
&= n^{2HK}\Big[\Big(\left((a+1)n^{-1}\right)^{2H}+\left((a+1)n^{-1}+1\right)^{2H}\Big)^K \\
&\qquad - \Big(\left((a+1)n^{-1}\right)^{2H}+\left(an^{-1}+1\right)^{2H}\Big)^K \\
&\qquad - \Big(\left(an^{-1}\right)^{2H}+\left((a+1)n^{-1}+1\right)^{2H}\Big)^K \\
&\qquad + \Big(\left(an^{-1}\right)^{2H}+\left(an^{-1}+1\right)^{2H}\Big)^K\Big] \\
&= n^{2HK}\Bigg[\Bigg((a+1)^{2H}n^{-2H}+1 \\
&\qquad + \sum_{j=0}^{\infty}((j+1)!)^{-1}2H(2H-1)\cdots(2H-j)(a+1)^{j+1}n^{-j-1}\Bigg)^K \\
&\qquad - \Bigg((a+1)^{2H}n^{-2H}+1 \\
&\qquad + \sum_{j=0}^{\infty}((j+1)!)^{-1}2H(ZH-1)\cdots(2H-j)a^{j+1}n^{-j-1}\Bigg)^K \\
&\qquad - \Bigg(a^{2H}n^{-2H}+1 \\
&\qquad + \sum_{j=0}^{\infty}((j+1)!)^{-1}2H(2H-1)\cdots(2H-j)(a+1)^{j+1}n^{-j-1}\Bigg)^K \\
&\qquad + \Bigg(a^{2H}n^{-2H}+1 \\
&\qquad + \sum_{j=0}^{\infty}((j+1)!)^{-1}2H(2H-1)\cdots(2H-j)^{j+1}a^{j+1}n^{-j-1}\Bigg)^K\Bigg].
\end{aligned}$$

By the asymptotic behavior of the function $(1+y)^K$ when $y \to 0$ we obtain

$$
\begin{aligned}
f_a(n) = n^{2HK} \Bigg[& 1 + \sum_{\ell=0}^{\infty} ((l+1)!)^{-1} K(K-1)\cdots(K-\ell) \\
& \times \Bigg((a+1)^{2H} n^{-2H} + 1 \\
& \quad + \sum_{j=0}^{\infty} ((j+1)!)^{-1} 2H(2H-1)\cdots(2H-j)(a+1)^{j+1} n^{-j-1} \Bigg)^{\ell+1} \\
& - 1 - \sum_{\ell=0}^{\infty} ((l+1)!)^{-1} K(K-1)\cdots(K-\ell) \\
& \times \Bigg((a+1)^{2H} n^{-2H} + 1 \\
& \quad + \sum_{j=0}^{\infty} ((j+1)!)^{-1} 2H(2H-1)\cdots(2H-j) a^{j+1} n^{-j-1} \Bigg)^{\ell+1} \\
& - 1 - \sum_{\ell=0}^{\infty} ((l+1)!)^{-1} K(K-1)\cdots(K-\ell) \\
& \times \Bigg(a^{2H} n^{-2H} + 1 \\
& \quad + \sum_{j=0}^{\infty} ((j+1)!)^{-1} 2H(2H-1)\cdots(2H-j) a^{j+1} n^{-j-1} \Bigg)^{\ell+1} \\
& + 1 + \sum_{\ell=0}^{\infty} ((\ell+1)!)^{-1} K(K-1)\cdots(K-\ell) \\
& \times \Bigg(a^{2H} n^{-2H} + 1 \\
& \quad + \sum_{j=0}^{\infty} ((j+1)!)^{-1} 2H(2H-1)\cdots(2H-j) a^{j+1} n^{-j-1} \Bigg)^{\ell+1} \Bigg] \\
= n^{2HK} \Bigg[& 1 + \sum_{\ell=1}^{\infty} ((l+1)!)^{-1} K(K-1)\cdots(K-\ell) \\
& \times \Bigg((a+1)^{2H} n^{-2H} + 1 \\
& \quad + \sum_{j=0}^{\infty} ((j+1)!)^{-1} 2H(2H-1)\cdots(2H-j)(a+1)^{j+1} n^{-j-1} \Bigg)^{\ell+1}
\end{aligned}
$$

$$
\begin{aligned}
&- 1 - \sum_{\ell=1}^{\infty}((l+1)!)^{-1}K(K-1)\cdots(K-\ell) \\
&\times \Bigg((a+1)^{2H}n^{-2H} + 1 \\
&\quad + \sum_{j=0}^{\infty}((j+1)!)^{-1}2H(2H-1)\cdots(2H-j)a^{j+1}n^{-j-1}\Bigg)^{\ell+1} \\
&- 1 - \sum_{\ell=1}^{\infty}((l+1)!)^{-1}K(K-1)\cdots(K-\ell) \\
&\times \Bigg(a^{2H}n^{-2H} + 1 \\
&\quad + \sum_{j=0}^{\infty}((j+1)!)^{-1}2H(2H-1)\cdots(2H-j)a^{j+1}n^{-j-1}\Bigg)^{\ell+1} \\
&+ 1 + \sum_{\ell=1}^{\infty}((l+1)!)^{-1}K(K-1)\cdots(K-\ell) \\
&\times \Bigg(a^{2H}n^{-2H} + 1 \\
&\quad + \sum_{j=0}^{\infty}((j+1)!)^{-1}2H(2H-1)\cdots(2H-j)a^{j+1}n^{-j-1}\Bigg)^{\ell+1}\Bigg] \\
= {}& 2^{-1}K(K-1)n^{2HK}\Big[\big((a+1)^{2H}n^{-2H} + 2H(a+1)n^{-1} \\
&\qquad\qquad + H(2H-1)(a+1)^2n^{-2}\big)^2 \\
&- \big((a+1)^{2H}n^{-2H} + 2Han^{-1} + H(2H-1)a^2n^{-2}\big)^2 \\
&- \big(a^{2H}n^{-2H} + H(a+1)n^{-1} + 2H(2H-1)(a+1)^2n^{-2}\big)^2 \\
&+ \big(a^{2H}n^{-2H} + Han^{-1} + 2H(2H-1)a^2n^{-2}\big)^2\Big] \\
&+ \cdots \\
= {}& HK(K-1)\left((a+1)^{2H} - a^{2H}\right)n^{2(HK-1)+(1-2H)} + \cdots .
\end{aligned}
$$

This completes the proof. □

Let us discuss some consequences of the theorem above.

Remark 4.2. What is the main term in $R(a, a+n)$? Note that $H > \frac{1}{2}$ if and only if $2(HK-1) > 2(HK-1) + (1-2H)$. Consequently the dominant term for $R(a, a+n)$ is of order n^{2HK-2} if $H > \frac{1}{2}$ and of order $n^{2HK-1-2H}$ if $H < \frac{1}{2}$.

Another interesting observation is that, although the main term of the covarinace function $R(a, a+n)$ changes depending on whether H is bigger or less than one half, the long-range dependence of the process $B^{H,K}$ depends on the value of the product HK.

Corollary 4.3. *For integers $a \geq 1$ and $n \geq 0$, let $R(a, a+n)$ be given by* (3.1). *Then for every $a \in \mathbb{N}$, we have*

$$\sum_{n\geq 0} R(a, a+n) = \infty \qquad \text{if } 2HK > 1$$

and

$$\sum_{n\geq 0} R(a, a+n) < \infty \qquad \text{if } 2HK \leq 1.$$

Proof. Suppose first that $2HK > 1$. Then it forces H to be bigger than $\frac{1}{2}$ and the dominant term of $R(a, a+n)$ is n^{2HK-2} when n is large. So the series diverges.

Suppose that $2HK < 1$. If $H > \frac{1}{2}$, the main term of $R(a, a+n)$ is n^{2HK-2} and the series converges. If $H < \frac{1}{2}$, then the main term is $n^{2HK-2H-1}$ and the series converges again.

If $2HK = 1$ (and thus $H > \frac{1}{2}$) then $R(a, a+n)$ behaves as $n \to \infty$ as n^{-1-2H} and the series is convergent. $\square$

Corollary 4.4. *Let $R^{X^{H,K}}(a, a+n)$ be the noise defuned in* (3.4). *Then*

$$\begin{aligned} &R^{X^{H,K}}(a, a+n) \\ &\quad = \Gamma(1-K)K^{-1}\left(-4HK(K-1)\left((a+1)^{2H} - a^{2H}\right)n^{2(HK-1)+(1-2H)} + \cdots\right). \end{aligned}$$

Proof. It follows from Theorem 4.1 and the fact that the covariance function of the fractional Brownian motion with Hurst parameter HK behaves as $HK(2HK - 1)n^{2(HK-1)}$ when $n \to \infty$. $\square$

5. More on the Process $X^{H,K}$

We will give few additional properties of the process X^K defined in (1.1). Recall (1.2) that for every $s, t \geq 0$

$$R^{X^K}(s,t) := E[X^K_s X^K_t] = \Gamma(1-K)K^{-1}(t^K + s^K - (t+s)^K).$$

Denote by $B^{K/2} = (B^{K/2}_t, t \in \mathbb{R})$ a fractional Brownian motion with Hurst index $K/2$ defined for all $t \in \mathbb{R}$ and let

$$B^{o,K/2}_t = 2^{-1}\left(B^{K/2}_t - B^{K/2}_{-t}\right), \qquad B^{e,K/2}_t = 2^{-1}\left(B^{K/2}_t + B^{K/2}_{-t}\right).$$

The processes $B^{o,K/2}$ and $B^{e,K/2}$ are respectively the odd part and the even part of the fractional Brownian motion $B^{K/2}$. Denote by $R^{o,K/2}$ the covariance of the process $B^{o,K/2}_t$, by $R^{e,K/2}$ the covariance of the $B^{e,K/2}$ and by $R^{B^{K/2}}$ the covariance of the fractional Brownian motion $B^{K/2}$. We have the following facts:

$$R^{X^K}(t,s) = C_3 R^{B^{K/2}}(t,-s) = C_3 R^{B^{K/2}}(s,-t)$$

where $C_3 = 2\Gamma(1-K)K^{-1}$, and

$$R^{o,K/2}(t,s) = \frac{1}{2}\left(R^{B^{K/2}}(t,s) - R^{B^{K/2}}(t,-s)\right)$$

and

$$R^{e,K/2}(t,s) = \frac{1}{2}\left(R^{B^{K/2}}(t,s) + R^{B^{K/2}}(t,-s)\right).$$

As a consequence

$$R^{e,K/2}(t,s) - R^{o,K/2}(t,s) = R^{B^{K/2}}(t,-s) = C_3^{-1}R^{X^K}(t,s).$$

From the above computations, we obtain

Proposition 5.1. *We have the following equality*

$$C_3^{-1/2}X^K + B^{e,K/2} \stackrel{\mathrm{d}}{=} B^{o,K/2}$$

if X^K and $B^{e,K/2}$ are independent.

Let us go back to the bifractional Brownian noise $R(a,a+n)$ given in (3.1). By the decomposition (1.4), we have

$$C_1X^{H,K} + B^{H,K} \stackrel{\mathrm{d}}{=} C_2B^{HK},$$

where C_1 and C_2 are as before, and thus we get

$$\begin{aligned} R(a,a+n) &= C_2^2R^{B^{HK}}(a,a+n) - C_1^2R^{X^{H,K}}(a,a+n) \\ &= C_2^2R^{B^{HK}}(0,n) - C_3\left(R^{e,K/2,H}(a,a+n) - R^{o,K/2,H}(a,a+n)\right) \end{aligned}$$

where $R^{e,K/2,H}(a,a+n)$ is the noise of the process $B^{e,K/2}_{t^{2H}}, t \geq 0$.

Remark 5.2. The fact that the covariance function $R^{X^K}(a,a+n)$ of the process $X^{K/2}$ converges to zero as $a \to \infty$ can be interpreted as "the covariance of the odd part" $C_3R^{B^{o,K/2}}(a,a+n)$ and "the covariance of the even part" $C_3R^{B^{e,K/2}}(a,a+n)$ have the same limit $2^{-1}C_1^2R^{B^{K/2}}(0,n)$ when a tends to infinity.

6. Limit Theorems to the Bifractional Brownian Motion

In this section, we prove two limit theorems to the bifractional Brownian motion. Throughout this section, we use the following notation. Let $0 < H < 1, 0 < K < 1$ such that $2HK > 1$ and let $(\xi_j, j = 1, 2, ...)$ be a sequence of standard normal random variables. Define a function $g(x,y), x \geq 0, y \geq 0$ by

$$\begin{aligned} g(x,y) &= 2^{2-K}H^2K(K-1)(x^{2H}+y^{2H})^{K-2}(xy)^{2H-1} \\ &\qquad + 2^{1-K}HK(2HK-1)|x-y|^{2HK-2} \\ &:= g_1(x,y) + g_2(x,y), \end{aligned} \tag{6.1}$$

for (x,y) with $x \neq y$ and $x \neq 0$ and $y \neq 0$.

Proposition 6.1. *Under the notation above, assume that $E[\xi_i\xi_j] = g(i,j)$. Then*

$$\left(n^{-HK}\sum_{j=1}^{[nt]}\xi_j, t \geq 0\right) \stackrel{\mathrm{d}}{\Rightarrow} \left(B_t^{H,K}, t \geq 0\right).$$

To prove this, we need a lemma.

Lemma 6.2.

$$\int_0^t \int_0^s g(u,v)dudv = 2^{-K}\left[(t^{2H}+s^{2H})^K - |t-s|^{2HK}\right].$$

Proof. It follows easily from the fact that $\frac{\partial^2 R}{\partial x \partial y}(x,y) = g(x,y)$ for every $x, y \geq 0$ and by using that $2HK > 1$.

□

Proof. (*Proof of Proposition* 6.1.) It is enough to show that as $n \to \infty$,

$$\begin{aligned} I_n &:= E\left[\left(n^{-HK}\sum_{i=1}^{[nt]}\xi_i\right)\left(n^{-HK}\sum_{j=1}^{[ns]}\xi_j\right)\right] \\ &\to E[B_t^{H,K}B_s^{H,K}] = 2^{-K}\left((t^{2H}+s^{2H})^K - |t-s|^{2HK}\right). \end{aligned}$$

We have

$$I_n = n^{-2HK}\sum_{i=1}^{[nt]}\sum_{j=1}^{[ns]} E[\xi_i\xi_j] = n^{-2HK}\sum_{i=1}^{[nt]}\sum_{j=1}^{[ns]} g(i,j).$$

Observe that

$$\begin{aligned} g\left(\frac{i}{n},\frac{j}{n}\right) &= 2^{2-K}H^2K(K-1)\left(\left(\frac{i}{n}\right)^{2H}+\left(\frac{j}{n}\right)^{2H}\right)^{K-2}\left(\frac{ij}{n^2}\right)^{2H-1} \\ &\qquad + 2^{1-K}HK(2HK-1)\left|\frac{i}{n}-\frac{j}{n}\right|^{2HK-2} \\ &= 2^{2-K}H^2K(K-1)n^{-2H(K-2)-2(2H-1)}(i^{2H}+j^{2H})^{K-2}(ij)^{2H-1} \\ &\qquad + 2^{1-K}HK(2HK-1)n^{-2HK+2}|i-j|^{2HK-2} \\ &= n^{2(1-HK)}\Big(2^{2-K}H^2K(K-1)(i^{2H}+j^{2H})^{K-2}(ij)^{2H-1} \\ &\qquad\qquad + 2^{1-K}HK(2HK-1)|i-j|^{2HK-2}\Big) \\ &= n^{2(1-HK)}g(i,j). \end{aligned} \tag{6.2}$$

Thus, as $n \to \infty$,

$$\begin{aligned} I_n &= n^{-2HK}\sum_{i=1}^{[nt]}\sum_{j=1}^{[ns]} n^{2HK-2} g\left(\frac{i}{n},\frac{j}{n}\right) = n^{-2}\sum_{i=1}^{[nt]}\sum_{j=1}^{[ns]} g\left(\frac{i}{n},\frac{j}{n}\right) \\ &\to \int_0^t\int_0^s g(u,v)dudv = 2^{-K}\left((t^{2H}+s^{2H})^K - |t-s|^{2HK}\right) \\ &= E[B_t^{H,K}B_s^{H,K}]. \end{aligned}$$

□

Remark 6.3. This result seems easy to be generalized to more general Gaussian selfsimilar processes such that their covariance R satisfies $\frac{\partial R}{\partial x \partial y} \in L^2\left((0,\infty)^2\right)$.

We next consider more general sequence of nonlinear functional of standard normal random variables. Let f be a real valued function such that $f(x)$ does not vanish on a set of positive measure, $E[f(\xi_1)] = 0$ and $E[(f(\xi_1))^2] < \infty$. Let $H_k(x)$ denotes the k-th Hermite polynomial with highest coefficient 1. We expand f as follows (see e.g. [2]):

$$f(x) = \sum_{k=1}^{\infty} c_k H_k(x),$$

where $\sum_{k=1}^{\infty} c_k^2 k! < \infty, c_k = E[f(\xi_i)H_k(\xi_j)]$. This expansion is possible under the assumption $Ef(\xi_1) = 0$ and $E[(f(\xi_1))^2] < \infty$. Assume that $c_1 \neq 0$. Now consider a new sequence

$$\eta_j = f(\xi_j), j = 1, 2, ...,$$

where $(\xi_j, j = 1, 2, ...)$ is the same sequence of standard normal random variables as before.

Proposition 6.4. *Under the same assumptions of Proposition 6.1, we have*

$$\left(n^{-HK} \sum_{j=1}^{[nt]} \eta_j, t \geq 0 \right) \overset{\mathrm{d}}{\Rightarrow} \left(c_1 B_t^{H,K}, t \geq 0 \right).$$

Proof. Note that $\eta_j = f(\xi_j) = c_1 \xi_j + \sum_{k=2}^{\infty} c_k H_k(\xi_j)$. We have

$$n^{-HK} \sum_{j=1}^{[nt]} \eta_j = c_1 n^{-HK} \sum_{j=1}^{[nt]} \xi_j + n^{-HK} \sum_{j=1}^{[nt]} \sum_{k=2}^{\infty} c_k H_k(\xi_j).$$

By Proposition 6.1, it is enough to show that

$$E\left[\left(n^{-HK} \sum_{j=1}^{[nt]} \sum_{k=2}^{\infty} c_k H_k(\xi_j) \right)^2 \right] \to 0 \quad \text{as } n \to \infty.$$

We have

$$\begin{aligned} J_n &:= E\left[\left(n^{-HK} \sum_{j=1}^{[nt]} \sum_{k=2}^{\infty} c_k H_k(\xi_j) \right)^2 \right] \\ &= n^{-2HK} \sum_{i=1}^{[nt]} \sum_{j=1}^{[nt]} \sum_{k=2}^{\infty} \sum_{\ell=2}^{\infty} c_k c_\ell E[H_k(\xi_j) H_\ell(\xi_j)]. \end{aligned}$$

In general, if ξ and η are jointly Gaussian random variables with $E[\xi] = E[\eta] = 0$, $E[\xi^2] = E[\eta^2] = 1$ and $E[\xi\eta] = r$, then

$$E[H_k(\xi) H_\ell(\eta)] = \delta_{k,\ell} r^k k!,$$

where

$$\delta_{k,\ell} = \begin{cases} 1, & k = \ell, \\ 0, & k \neq \ell. \end{cases}$$

Thus

$$
\begin{aligned}
J_n &= n^{-2HK}[nt]\sum_{\ell=2}^{\infty} c_\ell^2 \ell! + n^{-2HK}\sum_{i,j=1;i\neq j}^{[nt]}\sum_{\ell=2}^{\infty} c_\ell^2 (E[\xi_i\xi_j])^\ell \ell! \\
&= n^{-2HK}[nt]\sum_{\ell=2}^{\infty} c_\ell^2 \ell! + n^{-2HK}\sum_{i,j=1;i\neq j}^{[nt]}\sum_{\ell=2}^{\infty} c_\ell^2 g(i,j)^\ell \ell!.
\end{aligned}
$$

Since for every $i,j\geq 1$ $(i\neq j)$ one has $|g(i,j)|\leq \left(E[\xi_i^2]\right)^{1/2}\left(E[\xi_j^2]\right)^{1/2}=1$, we get

$$
\begin{aligned}
J_n &\leq n^{-2HK}[nt]\sum_{\ell=2}^{\infty} c_\ell^2 \ell! + n^{-2HK}\sum_{\ell=2}^{\infty} c_\ell^2 \ell! \sum_{i,j=1;i\neq j}^{[nt]} g(i,j)^2 \\
&\leq tn^{1-2HK}\sum_{\ell=2}^{\infty} c_\ell^2 \ell! + n^{2(HK-1)}\left(\sum_{\ell=2}^{\infty} c_\ell^2 \ell!\right)\left(n^{-2}\sum_{i,j=1;i\neq j}^{[nt]} g\left(\frac{i}{n},\frac{j}{n}\right)^2\right),
\end{aligned}
$$

where we have used (6.2)

Here as $n\to\infty$, since $\sum_{\ell=2}^{\infty} c_\ell^2\ell! < \infty$ and $2HK>1$ we obtain that $tn^{1-2HK}\sum_{\ell=2}^{\infty} c_\ell^2\ell!$ converges to zero as $n\to\infty$. On the other hand, with C an absolute positive constant and g_1 and g_2 given by (6.1),

$$
n^{-2}\sum_{i,j=1;i\neq j}^{[nt]} g\left(\frac{i}{n},\frac{j}{n}\right)^2 \leq Cn^{-2}\left(\sum_{i,j=1;i\neq j}^{[nt]} g_1\left(\frac{i}{n},\frac{j}{n}\right)^2 + \sum_{i,j=1;i\neq j}^{[nt]} g_2\left(\frac{i}{n},\frac{j}{n}\right)^2\right).
$$

The first sum $n^{-2}\sum_{i,j=1;i\neq j}^{[nt]} g_1\left(\frac{i}{n},\frac{j}{n}\right)^2$ is a Riemann sum converging to the integral $\int_0^t\int_0^t g_1^2(x,y)dxdy$. Note that this integral is finite because $|g_1(x,y)|\leq C(xy)^{HK-1}$ and the integral $\int_0^t\int_0^t |x-y|^{2HK-2}dxdy$ is finite when $2HK>1$. Since $n^{2(HK-1)}\to 0$ we easily get

$$
n^{2(HK-1)}n^{-2}\sum_{i,j=1;i\neq j}^{[nt]} g_1\left(\frac{i}{n},\frac{j}{n}\right)^2 \to 0
$$

as $n\to\infty$.

The second sum involving g_2 appears in the classical fractional Brownian case because it is, modulo a constant, the second derivative of the covariance of the fractional Brownian motion with Hurst parameter HK. The convergence of

$$
n^{2(HK-1)}n^{-2}\sum_{i,j=1;i\neq j}^{[nt]} g_2\left(\frac{i}{n},\frac{j}{n}\right)^2
$$

has been already proved in e.g. [2]. The proof is completed. □

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