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NUMERIC AND DYNAMIC B-STABILITY, EXACT-MONOTONE AND ASYMPTOTIC TWO-POINT BEHAVIOR OF THETA METHODS FOR STOCHASTIC DIFFERENTIAL EQUATIONS

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Abstract. This paper is devoted to moment stability analysis of the two-point motion of drift-implicit θ-methods for (nonlinear) stochastic differential equations (SDEs) in Itô sense. Two concepts of numeric and dynamic B-stability are presented. Under appropriate conditions, it is shown that the drift-implicit θ-methods with $\theta \geq 0.5$ are mean-square B-stable for all SDEs with mean square dissipative perturbations. Moreover, exponential mean square B-stability for those schemes can be verified for sufficiently small step sizes. A general mean square identity for the two-point motion of numerical methods may explain why symmetric methods such as midpoint-type methods are preferable ones in both deterministic and stochastic settings in order to replicate the qualitative behavior of underlying continuous time SDEs in an adequate manner. Indeed, midpoint-type methods (i.e. $\theta = 0.5$) are contraction-exact methods (i.e. exact-monotone). Asymptotic almost sure and asymptotic mean square B-stability are verified by a discrete invariance principle.

1. Introduction to Numeric Moment B-stability

Stability investigations are as important as convergence ones. This fact is known for a long time. It is also true for stochastic numerical analysis, i.e. numerical analysis for stochastic differential equations (SDEs)

$$dX(t) \quad = \quad a(t, X(t))dt + \sum_{j=1}^{m} b^j(t, X(t))dW_j(t) \tag{1.1}$$

driven by standard independent Wiener processes W_j, where a and b^j are Borel-measurable real-valued vector functions in $\mathbb{R}^d$. Very seldom such equations are explicitly solvable, and most often one has to resort to numerical methods for approximating SDEs (1.1). A comprehensive introduction to numerical methods for ordinary stochastic differential equations is given in the books and review papers by Allen [1], Artemiev and Averina [3], Bouleau and Lepingle [5], Kanagawa

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and Ogawa [11], Milstein [14], Pardoux and Talay [17], Schurz [21], [22], [23], and Talay [26]. Theory of stochastic ODEs is explained in Arnold [2], Dynkin [7], Friedman [8], Gard [9], Gikhman and Skorochod [10], Khasminskij [13], Karatzas and Shreve [12], Oksendal [16], Protter [18], just to name a few traditional textbooks.

After comprehensive studies of convergence (more precisely, local consistency), we focus our main emphasis on the replication of qualitative features of SDEs under discretization. At first, we studied the feature of linear and nonlinear A-stability. As one of the first, mean square stability of implicit Euler-Maruyama method

$$Y_{n+1} \quad = \quad Y_n + a(t_{n+1}, Y_{n+1})h_n + \sum_{j=1}^{m} b^j(t_n, Y_n)\Delta W_n^j \tag{1.2}$$

driven by independent random variables ΔW_n^j with

$$\mathbb{E}\ \Delta W_n^j = 0, \qquad \mathbb{E}\ [\Delta W_n^j]^2 = h_n$$

is discussed in Mitsui and Saito [15], Artemiev and Averina [3] and Schurz [19]. In 1996-1997 [19], [20] establish mean square A-stability of those methods when applied to mean square stable bilinear or nonlinear systems of SDEs in any dimension. Recall that SDE (1.1) with $a(t,0) = b^j(t,0) = 0$ has a mean square stable null solution $X \equiv 0$ if $\exists K_{OB} \leq 0$ (a real constant) such that

$$\forall (t,x) \in \mathbb{R}^{d+1}: \qquad 2\langle x, a(t,x)\rangle_d + \sum_{j=1}^{m} ||b^j(t,x)||_d^2 \leq K_{OB}||x||_d^2. \tag{1.3}$$

Moreover, the trivial solution $X \equiv 0$ is exponentially mean square stable if $K_{OB} < 0$. Several stability and implementation issues of one-point motion process $Y = (Y_n)_{n\in\mathbb{N}}$ are also discussed in Burrage and Mitsui [6]. At the first glance, classic stability investigations of numerical methods usually center around the behavior of the one-point motion process related to them. Digging into that topic more deeply, investigators study the two-point motion process $X_n - Y_n$ of two identical copies of one and the same numerical method, but started at different initial values X_0 and Y_0. That process is important for the propagation and control of initial value perturbations during the course of numerical integration. The standard concept of B-stability is known from deterministic numerical analysis and aiming at gaining control on the numerical two-point motion process.

The second most interesting question on numerical stability (after A-stability) is whether there are mean square B-stable numerical methods for SDEs. For the first time, this has been positively answered in [20] by the verification of that property for fully drift-implicit Euler methods (1.2). In contrast to A-stability which describes the behavior of the one-point motion, B-stability is a requirement on the two-point motion process. The two-point motion process determines the perturbative behavior of numerical methods. It analyzes the temporal evolution of growth or decay of initial perturbations during the course of numerical integration. It is very important to know whether initial perturbations accumulate to essential errors at later times or whether a perturbation in initial errors is under control during numerical integration (cf. axiomatic approach to numerical analysis in [22] and its requirement of control on nonexpansive perturbations on Hilbert spaces).

So, for the same numerical method with distinct initial data X_0 and Y_0, we rather look here at the evolution of

$$||X_n - Y_n||_d$$

as time $n \to +\infty$ (i.e. $t_n \to +\infty$). For this purpose, consider the following naturally induced stochastic counterpart of moment B-stability: Let $p \neq 0$ be a real constant.

Definition 1.1 (Numerical B-Stability)**.** A numerical method with the scheme $Y_{n+1} = Y_n + \Phi_n(Y)$ is called (globally) *numerically p-th moment B-stable* iff

$$\forall n \in \mathbb{N} \qquad \mathbb{E}\, ||X_{n+1} - Y_{n+1}||_d^p \leq \mathbb{E}\, ||X_n - Y_n||_d^p \tag{1.4}$$

for $\mathcal{F}_0$-adapted initial data with $\mathbb{E}\, ||X_0 - Y_0||_d^p < +\infty$. If $p = 2$, we additionally speak of *(numerical) mean square B-stability.* A numerical method with scheme $Y_{n+1} = Y_n + \Phi_n(Y)$ is called (numerically) *exponentially p-th moment B-stable* iff

$$\forall n \in \mathbb{N}\ \exists \alpha_n > 0 \qquad \mathbb{E}\, ||X_{n+1} - Y_{n+1}||_d^p \leq \exp(-\alpha_n h_n)\mathbb{E}\, ||X_n - Y_n||_d^p \tag{1.5}$$

for $\mathcal{F}_0$-adapted initial data with $\mathbb{E}\, ||X_0 - Y_0||_d^p < +\infty$ and

$$\sum_{n=0}^{\infty} \alpha_n h_n = +\infty$$

with any nonrandom step sizes h_n with $\sup_{n\in\mathbb{N}} h_n < +\infty$.

The aim of this paper is to verify mean square B-stability and asymptotic behavior of two-point motion process of drift-implicit Theta methods with $\theta_n \geq 0.5$, i.e. for the scheme

$$Y_{n+1} \quad = \quad Y_n + \Big[\theta_n a(t_{n+1}, Y_{n+1}) + (1-\theta_n) a(t_n, Y_n)\Big] h_n + \sum_{j=1}^{m} b^j(t_n, Y_n)\Delta W_n^j \tag{1.6}$$

whenever the underlying SDE has mean square contractive perturbations. Moreover, we shall establish exponential mean square and almost sure asymptotic B-stability of those methods. Some key general identities for the mean square perturbative behavior of numerical methods and a more general invariance principle leading to the concept of V-stability along Lyapunov-type functionals for the two-point motion will serve us to reach that goal. Furthermore, a special role of midpoint-type and monotone methods is to be revealed (i.e. their superiority with respect to adequate contraction-monotonicity). In contrast to many other treatments, we shall not confine ourselves to the case of equidistant partitions with uniform mesh size h. All our results will be still valid for this case as we may obviously treat it as a very special case of possibly variable and adapted step sizes h_n.

The paper is organized as follows. In Section 2 we establish a mean square identity for numerical perturbations. This answers the question on the existence of contraction-exact numerical methods constructively (by midpoint-type methods). Section 3 shows exponential mean square B-stability of both backward Euler methods with all $\theta_n = 1$ and drift-implicit Theta methods with $0 \leq \theta_n \leq 1$ while using sufficiently small step sizes h_n. Section 4 proves V-stability along Lyapunov-type functionals V of the two-point motion process under appropriate conditions. We establish asymptotic mean square and a.s. B-stability of Theta methods with all $\theta_n \geq 0.5$ as a by-product of a more general invariance principle (cf. appendix). Section 5 discusses the applicability of some findings with a series of linear and nonlinear test examples. Finally, Section 6 briefly summarizes our major results. In the appendix, a discrete invariance principle (DIP) for the verification of B-stability is formulated and proved.

Some standard notations: Assume that $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ is a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (that is, it is increasing and right-continuous, and $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets). The process $W = (W_1(t), \dots, W_m(t))^T_{t\geq 0}$ is supposed to be a standard m-dimensional Wiener process defined on the probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ with mutually independent coordinates W_i throughout the paper. Furthermore, let $0 \leq t_0 < T < \infty$, $\mathcal{B}(S)$ be the Borel σ-algebra of subsets of set S. $||.||_d$ and $\langle .,. \rangle_d$ denote the Euclidean vector norm and scalar product in $\mathbb{R}^d$. $||.||_{L^2}$ represents the naturally induced norm of the Hilbert space $L^2(\Omega, \mathcal{F}, \mathbb{P})$. $[a]_+$ is the nonnegative part of a.

2. Contraction Identity, Contractivity and Exact Monotone of NMs

2.1. Two-point motion process and general contraction-identity. One is able to establish a general contraction identity for numerical methods. This will show that midpoint-type and more general exact monotone methods are designed to adequately replicate the increasing or decreasing evolution of perturbations for random initial data. Recall that any one-step difference method for the approximation of any (explicit) differential equation is constructed from general scheme-structure

$$X_{n+1} \quad = \quad X_n + \Phi_n(X) \tag{2.1}$$

with $\Phi_n(X)$ representing the **increment functional** of related numerical method. Recall $\bar{X}_n = (X_{n+1} + X_n)/2$. Let X and Y denote the stochastic processes belonging to the same numerical scheme (2.1) and started at $X_0 \in \mathbb{R}^d$ and $Y_0 \in \mathbb{R}^d$, respectively. We shall study the dynamic behavior of the **two-point motion process** (X, Y) along the same numerical method governed by the schemes

$$\begin{aligned} X_{n+1} &= X_n + \Phi_n(X) \\ Y_{n+1} &= Y_n + \Phi_n(Y) \end{aligned}$$

with one and the same increment functional Φ_n along one and the same partition $(t_n)_{n\in\mathbb{N}}$. Recall that $||.||$ denotes the Euclidean vector norm in $\mathbb{R}^d$ and $\langle .,. \rangle_d$ the Euclidean scalar product in $\mathbb{R}^d$.

Theorem 2.1 (General Contraction Identity)**.** *For all numerical methods in $\mathbb{R}^d$ satisfying (2.1) with increment functional Φ_n, we have*

$$||X_{n+1} - Y_{n+1}||^2 \quad = \quad ||X_n - Y_n||^2 + 2\langle \bar{X}_n - \bar{Y}_n, \Phi_n(X) - \Phi_n(Y) \rangle_d \tag{2.2}$$

where $||.||$ is the Euclidean norm in $\mathbb{R}^d$ and $\langle .,. \rangle_d$ the Euclidean scalar product.

Proof. First, for the Euclidean norm $||.||$ and scalar product $\langle .,. \rangle_d$, recall the identity $||u + v||^2 = ||u||^2 + 2\langle u, v \rangle_d + ||v||^2$ for all vectors $u, v \in \mathbb{R}^d$. Note that, for numerical methods (2.1) with increment functional Φ_n, we have

$$\Phi_n(X) \quad = \quad X_{n+1} - X_n.$$

Second, analyzing the Euclidean norm of the two-point motion process gives the chain of identities

$$\begin{aligned} ||X_{n+1} - Y_{n+1}||^2 \; &= \; ||X_n - Y_n + \Phi_n(X) - \Phi_n(Y)||^2 \\ = \quad & ||X_n - Y_n||^2 + 2\langle X_n - Y_n, \Phi_n(X) - \Phi_n(Y)\rangle_d + ||\Phi_n(X) - \Phi_n(Y)||^2 \\ = \quad & ||X_n - Y_n||^2 + \langle X_n - Y_n, \Phi_n(X) - \Phi_n(Y)\rangle_d \\ & + \langle X_{n+1} - \Phi_n(X) - Y_{n+1} + \Phi_n(Y), \Phi_n(X) - \Phi_n(Y)\rangle_d + ||\Phi_n(X) - \Phi_n(Y)||^2 \\ = \quad & ||X_n - Y_n||^2 + \langle X_{n+1} + X_n - Y_{n+1} - Y_n, \Phi_n(X) - \Phi_n(Y)\rangle_d \\ & - \langle \Phi_n(X) - \Phi_n(Y), \Phi_n(X) - \Phi_n(Y)\rangle_d + ||\Phi_n(X) - \Phi_n(Y)||^2 \\ = \quad & ||X_n - Y_n||^2 + 2\langle \bar{X}_n - \bar{Y}_n, \Phi_n(X) - \Phi_n(Y)\rangle_d \end{aligned}$$

which confirm the validity of contraction identity (2.2). □

Remark 2.2. For stochastic numerical methods, the contraction identity (2.2) holds almost surely too (with their increment functional Φ_n which is random). This identity (2.2) also explains why midpoint-type numerical integrators with $\Phi_n = \Phi_n(\bar{X}_n)$ form a preferable base for adequate construction of numerical methods from a dynamical point of view. They may preserve the monotone character of contractions (perturbations) of two-point motion process along scalar products (as long as all increment functionals Φ_n are monotone).

Definition 2.3 (Exact-Monotone NMs)**.** A numerical method Z with scheme $Z_{n+1} = Z_n + \Phi_n$ is called *exact contraction-monotone* iff the following implications while discretizing ODEs $dx/dt = f(t, x)$ with Caratheodory functions f can be established

$$\forall x, y \in \mathbb{R}^d, t \in \mathbb{R}^1 : \; \langle f(t, x) - f(t, y), x - y\rangle_d \leq 0$$
$$\Longrightarrow \forall n \in \mathbb{N}\colon ||X_0 - Y_0|| \geq ||X_1 - Y_1|| \geq ... \geq ||X_n - Y_n|| \geq ||X_{n+1} - Y_{n+1}|| \geq ...$$

and

$$\forall x, y \in \mathbb{R}^d, t \in \mathbb{R}^1 : \; \langle f(t, x) - f(t, y), x - y\rangle_d \geq 0$$
$$\Longrightarrow \forall n \in \mathbb{N}\colon ||X_0 - Y_0|| \leq ||X_1 - Y_1|| \leq ... \leq ||X_n - Y_n|| \leq ||X_{n+1} - Y_{n+1}|| \leq ...$$

for all adapted random initial values $X_0, Y_0 \in \mathbb{R}^d$ of method Z (where X denotes the realization of Z-values started at X_0 and Y of Z-values started at Y_0, respectively, for the same method Z).

Theorem 2.4 (Exact Contraction-Property of Midpoint Methods)**.** *All midpoint-type methods X with increments $\Phi(\bar{X}_n) = f(t_n^*, \bar{X}_n)h_n$, $\bar{X}_n = (X_n + X_{n+1})/2$, and any sample time-points $t_n^* \in \mathbb{R}^1$ are exact contraction-monotone for all ODEs with adapted random initial values X_0 and any choice of step sizes h_n.*

Proof. Apply the contraction identity (2.2) to midpoint methods with increment functional $\Phi(\bar{X}_n) = f(t_n^*, \bar{X}_n)h_n$ and any sample time-points $t_n^* \in \mathbb{R}^1$. For them, this identity reads as

$$||X_{n+1} - Y_{n+1}||^2 \quad = \quad ||X_n - Y_n||^2 + 2\langle \bar{X}_n - \bar{Y}_n, f(t_n^*, \bar{X}_n) - f(t_n^*, \bar{Y}_n\rangle_d \, h_n.$$

Obviously, by taking the square root, this relation is equivalent to

$$||X_{n+1} - Y_n|| \quad = \quad \sqrt{||X_n - Y_n||^2 + 2\langle \bar{X}_n - \bar{Y}_n, f(t_n^*, \bar{X}_n) - f(t_n^*, \bar{Y}_n)\rangle_d \, h_n}.$$

Now, the uniform monotonicity of Euclidean scalar product

$$\langle x - y, f(t,x) - f(t,y)\rangle_d$$

with respect to $x, y \in \mathbb{R}^d$ and positivity of h_n imply the exact contraction-monotonicity of related midpoint methods X. For example, if $\langle x - y, f(t,x) - f(t,y)\rangle_d \leq 0$ for all $x, y \in \mathbb{R}^d$ then we have

$$||X_{n+1} - Y_{n+1}|| \quad \leq \quad ||X_n - Y_n||$$

for all $n \in \mathbb{N}$. Complete induction on $n \in \mathbb{N}$ yields the non-increasing evolution of Euclidean norms $||X_n - Y_n||$ in n. Similarly, we can verify the monotonicity (i.e. the nondecreasing property) for $\langle x - y, f(t,x) - f(t,y)\rangle_d \geq 0$ for all $x, y \in \mathbb{R}^d$. Thus, the proof of Theorem 2.4 is completed. □

Remark 2.5. The situation with fully random increment functionals Φ_n (e.g. with $\Phi_n(X) = a(t_n, X_n)h_n + b(t_n, X_n)\Delta W_n$ for Euler methods) is somewhat more complicated (due to the non-monotone character of Wiener processes W) and requires further research. However, an extension to p-th mean contractions (appropriate for the concept of B-stability) gives some more insight for stochastic Theta methods (see next subsections).

2.2. P-th mean contractivity and non-expansivity of backward Euler methods. Let $X_{s,x}(t)$ denote the value of the stochastic process X at time $t \geq s$, provided that it has started at the value $X_{s,x}(s) = x$ at prior time s. x and y are supposed to be adapted initial values. Let Π denote an ordered time-scale (discrete ($\Pi = \mathbb{N}$) or continuous ($\Pi = [0, +\infty)$)) and $p \neq 0$ be a nonrandom constant.

Definition 2.6 (P-th Mean Contractive SPs)**.** A stochastic process $X = (X(t))_{t\in\Pi}$ with stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\Pi}, \mathbb{P})$ is said to be uniformly *p-th mean (forward) contractive* on $\mathbb{R}^d$ iff $\exists K_C^X \in \mathbb{R}\ \forall t \geq s \in \Pi\ \forall x, y \in \mathbb{R}^d$

$$\mathbb{E}\left[||X_{s,x}(t) - X_{s,y}(t)||^p \Big| \mathcal{F}_s \right] \quad \leq \quad \exp\Big(pK_C^X(t-s)\Big)||x-y||^p \tag{2.3}$$

with *p-th mean contractivity constant* K_C^X. In the case $K_C^X < 0$, we speak of *strict p-th mean contractivity.* Moreover, X is said to be a process with *p-th mean non-expansive perturbations* iff $\forall t \geq s \in \Pi\ \forall x, y \in \mathbb{R}^d$

$$\mathbb{E}\left[||X_{s,x}(t) - X_{s,y}(t)||^p \Big| \mathcal{F}_s \right] \quad \leq \quad ||x-y||^p. \tag{2.4}$$

If $p = 2$ then we speak of *mean square contractivity* and *mean square non-expansivity.*

For strictly contractive processes, adapted perturbations in the initial data have no significant impact on its long-term dynamic behavior. Adapted perturbations of non-expansive processes are under control along the entire time-scale Π. These concepts are important for the long-term control of error propagation through numerical methods. It turns out that they are meaningful to test numerical methods while applying to SDEs with monotone coefficient systems.

Let $p > 0$ be a nonrandom constant.

Definition 2.7 (P-th Mean Monotone Coefficient Systems)**.** A coefficient system (a, b^j) of SDEs (1.1) and its SDE are said to be *(strictly uniformly) p-th mean monotone* on $\mathbb{R}^d$ iff $\exists K_{UC} \in \mathbb{R}\ \forall t \in \mathbb{R}\ \forall x, y \in \mathbb{R}^d$

$$\langle a(t,x) - a(t,y), x - y\rangle_d + \frac{1}{2}\sum_{j=1}^{m} ||b^j(t,x) - b^j(t,y)||^2 + \frac{p-2}{2}\sum_{j=1}^{m}\frac{\langle b^j(t,x) - b^j(t,y), x - y\rangle_d^2}{||x-y||^2} \le K_{UC}||x-y||^2. \quad (2.5)$$

If $p = 2$ then we speak of *mean square monotonicity.*

In passing, of course, this definition and related B-stability analysis makes only sense for SDEs with p-th mean monotone coefficient systems with $K_{UC} \le 0$ (cf. requirement (2.5)). This definition is consistent with definition 1.1.

Lemma 2.8. *Assume that X satisfies SDE (1.1) with p-th mean monotone coefficient system (a, b^j).*
Then X is p-th mean contractive for all $p \ge 2$ and its p-th mean contractivity constant K_C^X can be estimated by

$$K_C^X \le K_{UC}.$$

As an exercise, this lemma can be proved by Dynkin's formula (see Dynkin [7], i.e. averaged Itô formula). Let us discuss the possible "worst case effects" on perturbations of numerical methods under condition (2.5) with $p = 2$. It turns out that the (drift-implicit) backward Euler methods are mean square contractive for SDEs with monotone drift and Lipschitz continuous diffusion terms.

Theorem 2.9. *Assume that*

(i) $\forall n \in I\!N : \theta_n = 1$,
(ii) $0 < \inf_{n\in I\!N} h_n \le \sup_{n\in I\!N} h_n < +\infty$, *all h_n nonrandom (i.e. only admissible step sizes),*
(iii) $\exists K_a \le 0 \forall x, y \in R^d\ \forall t \ge 0\ :\ \langle a(t,x) - a(t,y), x - y\rangle_d \le K_a||x-y||^2$,
(iv) $\exists K_b \forall x, y \in R^d\ \forall t \ge 0\ :\ \sum_{j=1}^{m} ||b^j(t,x) - b^j(t,y)||^2 \le K_b||x-y||^2$.

Then, the drift-implicit Euler methods (1.6) with scalar implicitness $\theta_n = 1$ have mean square contractive perturbations when applied to SDEs (1.1) with mean square monotone coefficients (a, b^j) with contractivity constant

$$K_C^Y \quad = \quad \sup_{n\in I\!N} \frac{2K_a + K_b}{1 - 2h_n K_a}. \quad (2.6)$$

If additionally $2K_a + K_b \le 0$ then they are mean square non-expansive and

$$K_C^Y \quad = \quad \frac{2K_a + K_b}{1 - 2K_a \sup_{n\in I\!N} h_n}. \quad (2.7)$$

Proof. Rearrange the scheme (1.6) for the drift-implicit Theta methods with non-random scalar implicitness $(\Theta_n) = \theta_n I$ with $0 \leq\leq \theta_n \leq 1$ to separate implicit from explicit part such that

$$X_{n+1}-\theta_n h_n a(t_{n+1},X_{n+1}) \;=\; X_n+(1-\theta_n)h_n a(t_n,X_n)+\sum_{j=1}^{m} b^j(t_n,X_n)\Delta W_n^j. \quad (2.8)$$

Recall that X and Y denote the values of the same scheme (1.6) started at values X_0 and Y_0, respectively. Now, take the square of Euclidean norms on both sides. By taking the expectation on both sides we arrive at

$$\begin{aligned}
&\mathbb{E}\,||X_{n+1}-Y_{n+1}||^2-2\theta_n h_n \mathbb{E}\,\langle X_{n+1}-Y_{n+1}, a(t_{n+1}, X_{n+1})-a(t_{n+1}, Y_{n+1})\rangle_d \\
&\quad +\theta_n^2 h_n^2 \mathbb{E}\,||a(t_{n+1}, X_{n+1}) - a(t_{n+1}, Y_{n+1})||^2 \\
&= \mathbb{E}\,||X_n - Y_n||^2 + 2(1-\theta_n)h_n \mathbb{E}\,\langle X_n - Y_n, a(t_n, X_n) - a(t_n, Y_n)\rangle_d \\
&\quad +(1-\theta_n)^2 h_n^2 \mathbb{E}\,||a(t_n,X_n)-a(t_n,Y_n)||^2+h_n\sum_{j=1}^{m}\mathbb{E}\,||b^j(t_n,X_n)-b^j(t_n,Y_n)||^2.
\end{aligned}$$

Under the assumption (iii) with $\theta_n \geq 0$ we have

$$-2\theta_n h_n \langle a(t,x) - a(t,y), x-y\rangle_d \;\geq\; -2\theta_n h_n K_a ||x-y||^2 \;\geq\; 0$$

for all $x, y \in \mathbb{R}^d$ and $t \geq 0$. Consequently, under (iii) and (iv) with $0 \leq \theta_n \leq 1$, we may estimate

$$\begin{aligned}
(1-2\theta_n h_n K_a)\mathbb{E}\,||X_{n+1}-Y_{n+1}||^2 \;\leq\; & [1+(2(1-\theta_n)K_a+K_b)h_n]_+\mathbb{E}\,||X_n-Y_n||^2 \\
& +(1-\theta_n)^2 h_n^2 \mathbb{E}\,||a(t_n,X_n)-a(t_n,Y_n)||^2.
\end{aligned}$$

for all $n \in \mathbb{N}$. This leads to the estimate

$$\begin{aligned}
&\mathbb{E}\,||X_{n+1} - Y_{n+1}||^2 \\
&\leq \frac{[1+(2(1-\theta_n)K_a+K_b)h_n]_+}{1-2\theta_n h_n K_a}\mathbb{E}\,||X_n-Y_n||^2 \\
&\quad +\frac{(1-\theta_n)^2 h_n^2}{1-2\theta_n h_n K_a}\mathbb{E}\,||a(t_n,X_n)-a(t_n,Y_n)||^2 \\
&= \left(1+\frac{(2K_a+K_b)h_n}{1-2\theta_n h_n K_a}\right)\mathbb{E}\,||X_n-Y_n||^2+\frac{(1-\theta_n)^2 h_n^2}{1-2\theta_n h_n K_a}\mathbb{E}\,||a(t_n,X_n)-a(t_n,Y_n)||^2 \\
&\leq \exp\left(\frac{(2K_a+K_b)h_n}{1-2\theta_n h_n K_a}\right)\mathbb{E}\,||X_n-Y_n||^2+\frac{(1-\theta_n)^2 h_n^2}{1-2\theta_n h_n K_a}\mathbb{E}\,||a(t_n,X_n)-a(t_n,Y_n)||^2
\end{aligned}$$

since $1 + z \leq \exp(z)$ for $z \geq -1$. Now, set all parameters $\theta_n = 1$ in the above inequality. In this case one encounters

$$\mathbb{E}\,||X_{n+1} - Y_{n+1}||^2 \;\leq\; \exp\left(\frac{2K_a + K_b}{1-2h_n K_a}h_n\right)\mathbb{E}\,||X_n - Y_n||^2.$$

Therefore, the drift-implicit backward Euler methods have mean square contractive perturbations with contractivity constant

$$\begin{aligned} K_C^Y &= \sup_{n\in\mathbb{N}} \frac{2K_a+K_b}{1-2h_nK_a} \\ &= \frac{2K_a+K_b}{1-2\sup\limits_{n\in\mathbb{N}} h_nK_a} \text{ if } 2K_a+K_b\leq 0. \end{aligned}$$

Obviously, if $2K_a+K_b \leq 0$, then the perturbations are mean square non-expansive.

□

2.3. P-th mean non-contractivity and expansivity of Euler methods. Let $X_{s,x}(t)$ denote the value of the stochastic process X at time $t \geq s$, provided that it has started at the value $X_{s,x}(s) = x$ at prior time s. x and y are supposed to be adapted initial values. Let Π denote an ordered time-scale (discrete ($\Pi = \mathbb{N}$) or continuous ($\Pi = [0, +\infty)$)) and $p > 0$ be a nonrandom constant.

Definition 2.10 (P-th Mean Noncontractive SPs)**.** A stochastic process $X = (X(t))_{t\in\Pi}$ with stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\Pi}, \mathbb{P})$ is said to be *p-th mean (forward) non-contractive* (in the strict sense) on $\mathbb{R}^d$ iff $\forall t \geq s \in \Pi$ $\forall x, y \in \mathbb{R}^d$ (adapted)

$$\mathbb{E}\left[||X_{s,x}(t) - X_{s,y}(t)||^p \middle| \mathcal{F}_s\right] \quad \geq \quad ||x-y||^p. \tag{2.9}$$

X is said to be a process with *p-th mean expansive perturbations* iff $\forall t > s \in \Pi$ $\forall x, y \in \mathbb{R}^d (x \neq y)$ (adapted)

$$\mathbb{E}\left[||X_{s,x}(t) - X_{s,y}(t)||^p \middle| \mathcal{F}_s\right] \quad > \quad ||x-y||^p. \tag{2.10}$$

If $p = 2$ then we speak of *mean square non-contractivity* and *mean square expansivity*, respectively.

For non-contractive processes, perturbations in the initial data may have significant impact on its long-term dynamic behavior. Adapted perturbations of expansive processes lead to chaotic, sensitive dynamic behavior along the entire time-scale Π. These concepts are important for the long-term control of error propagation through numerical methods. They are meaningful to test numerical methods while applying to SDEs with non-contractive coefficient systems.

Let $p > 0$ be a nonrandom constant.

Definition 2.11 (P-th Mean Non-decreasing Coefficient Systems)**.** A coefficient system (a, b^j) of SDEs (1.1) and its SDE are said to be *strictly uniformly p-th mean non-decreasing* on $\mathbb{R}^d$ iff $\exists K_{UC} \geq 0 \in \mathbb{R}$ $\forall t \in \mathbb{R}$ $\forall x, y \in \mathbb{R}^d$

$$\begin{aligned} \langle a(t,x) - a(t,y), x-y\rangle_d + \frac{1}{2}\sum_{j=1}^m ||b^j(t,x) - b^j(t,y)||^2 \\ + \frac{p-2}{2}\sum_{j=1}^m \frac{\langle b^j(t,x) - b^j(t,y), x-y\rangle_d^2}{||x-y||^2} \geq \quad K_{UC}\, ||x-y||^2. \end{aligned} \tag{2.11}$$

If $K_{UC} > 0$ in (2.11) then the coefficient system (a, b^j) is said to be *p-th mean expansive* and its SDE has p-th mean expansive perturbations. Moreover, if $p = 2$

then we speak of *mean square non-decreasing* and *mean square expansive* perturbations and systems, respectively.

Lemma 2.12. *Assume that X satisfies SDE (1.1) with p-th mean non-decreasing coefficient system (a, b^j).*
Then X has p-th mean non-decreasing perturbations for all $p \geq 2$. If additionally $K_{UC} > 0$ in (2.11) then X possesses p-th mean expansive perturbations.

This lemma can be proved by Dynkin's formula (averaged Itô formula). Let us discuss the possible "worst case effects" on perturbations of numerical methods under condition (2.11) with $p = 2$. It turns out that the drift-implicit forward Euler methods are mean square non-contractive under this condition and may have even mean square expansive perturbations.

Theorem 2.13. *Assume that*

(i) $\forall n \in I\!N : \theta_n = 0$,
(ii) $0 < \inf_{n \in I\!N} h_n \leq \sup_{n \in I\!N} h_n < +\infty$, *all h_n nonrandom (i.e. only admissible step sizes),*
(iii) $\exists \underline{K}_a \, \forall x, y \in R^d \, \forall t \geq 0 \; : \; \langle a(t,x) - a(t,y), x - y \rangle_d \; \geq \; \underline{K}_a ||x - y||^2$,
(iv) $\exists \underline{K}_b \, \forall x, y \in R^d \, \forall t \geq 0 \; : \; \sum_{j=1}^m ||b^j(t,x) - b^j(t,y)||^2 \; \geq \; \underline{K}_b ||x - y||^2$.

Then, the (forward) Euler methods (1.6) with scalar implicitness $\theta_n = 0$ have mean square non-contractive perturbations when applied to SDEs (1.1) with coefficients (a, b^j) satisfying $2\underline{K}_a + \underline{K}_b \geq 0$. If additionally $2\underline{K}_a + \underline{K}_b > 0$ then they are mean square expansive.

Proof. Consider the scheme (1.6) for the drift-implicit Theta methods with nonrandom scalar implicitness $(\Theta_n) = \theta_n I$ and separate implicit from explicit part such that

$$X_{n+1} - \theta_n h_n a(t_{n+1}, X_{n+1}) \; = \; X_n + (1-\theta_n) h_n a(t_n, X_n) + \sum_{j=1}^m b^j(t_n, X_n) \Delta W_n^j \quad (2.12)$$

Recall that X and Y denote the values of the same scheme (1.6) started at values X_0 and Y_0, respectively. Now, take the square of Euclidean norms on both sides. By taking the expectation on both sides we arrive at

$$\begin{aligned} & I\!E \, ||X_{n+1} - Y_{n+1}||^2 - 2\theta_n h_n I\!E \, \langle X_{n+1} - Y_{n+1}, a(t_{n+1}, X_{n+1}) - a(t_{n+1}, Y_{n+1}) \rangle_d \\ & \qquad + \theta_n^2 h_n^2 I\!E \, ||a(t_{n+1}, X_{n+1}) - a(t_{n+1}, Y_{n+1})||^2 \\ = & \; I\!E \, ||X_n - Y_n||^2 + 2(1-\theta_n) h_n I\!E \, \langle X_n - Y_n, a(t_n, X_n) - a(t_n, Y_n) \rangle_d \\ & \quad + (1-\theta_n)^2 h_n^2 I\!E \, ||a(t_n, X_n) - a(t_n, Y_n)||^2 + h_n \sum_{j=1}^m I\!E \, ||b^j(t_n, X_n) - b^j(t_n, Y_n)||^2. \end{aligned}$$

Under the assumption (iii) and $\theta_n \leq 1$ we have

$$\begin{aligned} 2(1-\theta_n) h_n \langle a(t,x) - a(t,y), x - y \rangle_d \quad + \quad & h_n \sum_{j=1}^m ||b^j(t,x) - b^j(t,y)||^2 \\ \geq \quad & [2(1-\theta_n)\underline{K}_a + \underline{K}_b] h_n ||x - y||^2 \end{aligned}$$

for all $x, y \in \mathbb{R}^d$ and $t \geq 0$. Consequently, under (ii) - (iv), $\theta_n \leq 1$ and $2\underline{K}_a + \underline{K}_b \geq 0$, we may estimate

$$(1-2\theta_n h_n \underline{K}_a)\mathbb{E}||X_{n+1}-Y_{n+1}||^2+\theta_n^2 h_n^2 \mathbb{E}||a(t_{n+1},X_{n+1})-a(t_{n+1},Y_{n+1})||^2$$
$$\geq \ [1+(2(1-\theta_n)\underline{K}_a+\underline{K}_b)h_n]\mathbb{E}||X_n-Y_n||^2+(1-\theta_n)^2 h_n^2 \mathbb{E}||a(t_n,X_n)-a(t_n,Y_n)||^2$$

for all $n \in \mathbb{N}$. Now, set $\theta_n = 0$. This leads to the estimate

$$\mathbb{E}\,||X_{n+1}-Y_{n+1}||^2 \quad \geq \quad [1+(2\underline{K}_a+\underline{K}_b)h_n]\mathbb{E}\,||X_n-Y_n||^2 \ \geq \ \mathbb{E}\,||X_n-Y_n||^2.$$

Therefore, the forward Euler methods have mean square non-contractive perturbations under the condition $2\underline{K}_a + \underline{K}_b \geq 0$. After returning to the latter inequality above, one clearly recognizes that, if additionally $2\underline{K}_a + \underline{K}_b > 0$, then the perturbations are mean square expansive. □

2.4. Dynamic mean square BN- and B-stability of backward Euler methods. It is natural to ask for transferring the deterministic concept of B-stability to stochastic dynamical systems (random sequences) in an adequate dynamic fashion. This can be done in the p-th mean moment sense fairly straight–forward, and it has been started by [20] and [21]. A more adequate definition of moment B-stability is given as follows.

Definition 2.14 (Dynamic B-Stability)**.** A numerical sequence $Z = (Z_n)_{n\in\mathbb{N}}$ (method, scheme, approximation, etc.) is called *dynamically p-th mean B-stable* iff it is p-th mean contractive for all autonomous SDEs (1.1) with p-th mean monotone coefficient systems (a, b^j) and for all admissible step sizes. It is said to be *dynamically p-th mean BN-stable* iff it is p-th mean contractive for all non-autonomous SDEs (1.1) with p-th mean monotone coefficient systems (a, b^j) for all admissible step sizes. If $p = 2$ then we also speak of *dynamic mean square B-* and *dynamic BN-stability.*

This definition is consistent with definition 1.1. Indeed, the drift-implicit backward Euler methods are appropriate to control the growth of its perturbations as long as the underlying SDE does. This fact is documented by the dynamic mean square B-stability of these methods in the following theorem.

Theorem 2.15 (Dynamic M.S. BN-, B-Stability of Backward Euler Methods)**.** *The drift-implicit backward Euler method (1.2) applied to Itô SDEs (1.1) with scalar implicitness parameters $\theta_n = 1$ and nonrandom step sizes h_n is (dynamically) mean square BN-stable and B-stable.*

Proof. Combine the main assertions of Lemma 2.8 and Theorem 2.9 □

3. Exponential Mean Square B-Stability of Stochastic Theta Methods

Under further restriction one may even gain exponential B-stability of drift-implicit Theta methods applied to contractive SDEs.

3.1. Exponential mean square B-Stability of backward Euler methods. As an immediate consequence of Theorem 2.9 we gain exponential B-stability of drift-implicit Euler methods (i.e. with all $\theta_n = 1$).

Theorem 3.1 (Exponential M.s. B-stability of Backward Euler Methods)**.** *Assume that all parameters $\theta_n = 1$ and step sizes h_n are nonrandom,*

$$\sup_{n \in I\!N} h_n < +\infty,$$

the same assumptions (i)-(iv) as in Theorem 2.9 are required, and additionally

$$2K_a + K_b < 0.$$

Then the drift-implicit Euler method (1.2) is exponential B-stable with exponents (i.e. generalized rates of exponential convergence)

$$\alpha_n := -\frac{2K_a + K_b}{1 - 2K_a h_n} \geq -\frac{2K_a + K_b}{1 - 2K_a \sup_{k \in I\!N} h_k} > 0. \tag{3.1}$$

Proof. Return to Theorem 2.9 and extract the inequality

$$\begin{aligned} &\mathbb{E}\,||X_{n+1} - Y_{n+1}||^2 \\ &\leq\ \exp\left(\frac{(2K_a + K_b)h_n}{1 - 2\theta_n h_n K_a}\right) \mathbb{E}\,||X_n - Y_n||^2 + \frac{(1-\theta_n)^2 h_n^2}{1 - 2\theta_n h_n K_a} \mathbb{E}\,||a(t_n, X_n) - a(t_n, Y_n)||^2 \end{aligned}$$

from its proof. For the drift-implicit backward Euler method (1.2) with $\theta_n = 1$, this inequality obviously reduces to

$$\mathbb{E}\,||X_{n+1} - Y_{n+1}||^2 \ \leq\ \exp\left(\frac{(2K_a + K_b)h_n}{1 - 2h_n K_a}\right) \mathbb{E}\,||X_n - Y_n||^2$$

which can be further estimated by

$$\begin{aligned} &\mathbb{E}\,||X_{n+1} - Y_{n+1}||^2 \\ &\leq\ \exp\left((2K_a + K_b) \inf_{k \in \mathbb{N}} \frac{1}{1 - 2h_k K_a} h_n\right) \mathbb{E}\,||X_n - Y_n||^2 \\ &\leq\ \exp\left((2K_a + K_b) \inf_{k \in \mathbb{N}} \frac{1}{1 - 2h_k K_a} (t_{n+1} - t_0)\right) \mathbb{E}\,||X_0 - Y_0||^2 \\ &\leq\ \exp\left((2K_a + K_b) \frac{1}{1 - 2K_a \sup_{k \in \mathbb{N}} h_k} (t_{n+1} - t_0)\right) \mathbb{E}\,||X_0 - Y_0||^2 \end{aligned}$$

under $2K_a + K_b < 0$. Hence, we conclude that

$$\lim_{n \to +\infty} ||X_n - Y_n||_{L^2} = 0$$

for all $L^2(\Omega, \mathcal{F}_0, \mathbb{P})$-integrable initial data X_0 and Y_0. Moreover, the estimation (3.1) of B-stability exponents α_n can be extracted from the above inequality chain. $\square$

3.2. Exponential mean square B-Stability of general theta methods. We can indeed establish numerical exponential B-stability of drift-implicit Theta methods (1.6) with $\theta_n \neq 1$ under some specific circumstances such as Lipschitz continuous drift a and diffusion coefficients b^j.

Theorem 3.2 (Exponential M.s. B-stability of Theta Methods). *Assume that all parameters $\theta_n \in [0,1]$ and step sizes h_n are nonrandom,*

(i) $\sup_{n\in\mathbb{N}} \theta_n h_n < +\infty$,

the same assumptions (ii)-(iv) as in Theorem 2.9 are required, and additionally there is a real constant $K_L > 0$ such that

$$2K_a + K_b + K_L^2 \sup_{n\in\mathbb{N}} (1-\theta_n)^2 h_n < 0$$

with

(v) $\forall (t,x,y) \in \mathbb{R}^{1+2d} : ||a(t,x) - a(t,y)|| \le K_L ||x-y||$.

Then the drift-implicit Theta method (1.6) is numerically exponential B-stable with exponents (i.e. generalized rates of exponential convergence)

$$\alpha_n^\theta := -\frac{2K_a + K_b + K_L^2(1-\theta_n)^2 h_n}{1-2K_a\theta_n h_n} \ge -\frac{2K_a + K_b + K_L^2 \sup_{k\in\mathbb{N}}(1-\theta_k)^2 h_k}{1-2K_a \sup_{k\in\mathbb{N}} \theta_k h_k} > 0. \tag{3.2}$$

Proof. Return to the proof of Theorem 2.9. Recall that we derived the inequality

$$\mathbb{E}\,||X_{n+1}-Y_{n+1}||^2 \quad \le \quad \frac{[1+(2(1-\theta_n)K_a+K_b)h_n]_+}{1-2\theta_n h_n K_a}\,\mathbb{E}\,||X_n - Y_n||^2 + \frac{(1-\theta_n)^2 h_n^2}{1-2\theta_n h_n K_a}\,\mathbb{E}\,||a(t_n,X_n)-a(t_n,Y_n)||^2$$

for all $\theta_n \ge 0$. A further estimation yields that

$$\begin{aligned}
&\mathbb{E}\,||X_{n+1}-Y_{n+1}||^2\\
&\le \left(1+\frac{(2K_a+K_b)h_n}{1-2\theta_n h_n K_a}\right)\mathbb{E}\,||X_n-Y_n||^2 + \frac{(1-\theta_n)^2 h_n^2}{1-2\theta_n h_n K_a}\mathbb{E}\,||a(t_n,X_n)-a(t_n,Y_n)||^2\\
&\le \left(1+\frac{(2K_a+K_b)h_n}{1-2\theta_n h_n K_a}\right)\mathbb{E}\,||X_n-Y_n||^2 + K_L^2\frac{(1-\theta_n)^2 h_n^2}{1-2\theta_n h_n K_a}\mathbb{E}\,||X_n-Y_n||^2\\
&= \left(1+\frac{2K_a+K_b+K_L^2(1-\theta_n)^2 h_n}{1-2\theta_n h_n K_a}h_n\right)\mathbb{E}\,||X_n-Y_n||^2\\
&\le \left(1+\frac{2K_a+K_b+K_L^2\sup_{k\in\mathbb{N}}(1-\theta_k)^2 h_k}{1-2\theta_n h_n K_a}h_n\right)\mathbb{E}\,||X_n-Y_n||^2\\
&\le \exp\left(\frac{2K_a+K_b+K_L^2\sup_{k\in\mathbb{N}}(1-\theta_k)^2 h_k}{1-2\theta_n h_n K_a}h_n\right)\mathbb{E}\,||X_n-Y_n||^2\\
&\le \exp\left(\frac{2K_a+K_b+K_L^2\sup_{k\in\mathbb{N}}(1-\theta_k)^2 h_k}{1-2K_a\sup_{k\in\mathbb{N}}\theta_k h_k}(t_{n+1}-t_0)\right)\mathbb{E}\,||X_0-Y_0||^2 \to 0
\end{aligned}$$

as $n \to +\infty$, under assumptions (i) - (v). Thus, from here, one obviously obtains numerical exponential mean square B-stability of Theta methods for sufficiently small step sizes h_n under $2K_a + K_b < 0$. Moreover, its exponential decay rates α_n^θ can be estimated as in (3.2) (i.e. directly extract that from the latter inequality chain). This completes the proof. □

4. V-stability of Two-Point Process of Theta Methods

This section aims at a slight relaxation of the notion of B-stability. A first generalization of B-stability is given by V-stability of the two-point motion process belonging to Theta methods along Lyapunov-type functionals V. Define

$$\begin{aligned} V_n(X,Y) \quad &:= \quad ||X_n - Y_n||^2 - 2\theta_n h_n \langle X_n - Y_n, a(t_n,X_n) - a(t_n,Y_n)\rangle_d \\ &\quad +\theta_n^2 h_n^2 ||a(t_n,X_n) - a(t_n,Y_n)||^2 \end{aligned}$$

for $n \in \mathbb{N}$. Set $\theta_{-1} = \theta_0$, $h_{-1} = h_0$.

4.1. V-stability and invariance principle for $\theta_n \geq 0.5$.

Theorem 4.1 (Moment V-stability and Invariance Principle of Theta Methods)**.** *Assume that*

(i) $\theta_n h_n$ *is nonincreasing in* $n \in I\!N$, $\theta_n \geq 0.5$,

(ii) $\forall (t,x,y) \in I\!R^{1+2d} : 2\langle x-y, a(t,x)-a(t,y)\rangle_d + \sum_{j=1}^{m} ||b^j(t,x)-b^j(t,y)||^2 \leq 0$,

(iii) $I\!E\,[V_0(X_0,Y_0)] < +\infty$.

Then $v_n := I\!E\,[V_n(X,Y)]$ *is nonincreasing in* $n \in I\!N$ *and* $\exists \lim_{n\to+\infty} v_n$, *more precisely*

$$\forall n \in I\!N\colon v_n := I\!E\,[V_n(X,Y)] \leq v_0 - \sum_{k=0}^{n} (2\theta_n - 1) h_n^2 I\!E ||a(t_n,X_n) - a(t_n,Y_n)||^2$$

$$v_\infty := \lim_{n\to+\infty} I\!E\,[V_n(X,Y)] \leq v_0 - \sum_{k=0}^{+\infty} (2\theta_n - 1) h_n^2 I\!E\,[||a(t_n,X_n) - a(t_n,Y_n)||^2]$$

$$\lim_{n\to+\infty} (2\theta_n - 1) h_n^2 I\!E\,||a(t_n,X_n) - a(t_n,Y_n)||^2] = 0.$$

Moreover, if additionally

(iv) $\inf_{n\in I\!N}(2\theta_n - 1)h_n^2 > 0$ *(as in autonomous case with* $\theta > 0.5$*)*

then

$$\sum_{n=0}^{+\infty} I\!E\,[||a(t_n,X_n) - a(t_n,Y_n)||^2 < \infty$$

$$\lim_{n\to\infty} I\!E\,[||a(t_n,X_n) - a(t_n,Y_n)||^2 = 0$$

$$\forall X_0, Y_0 \in L^2(\Omega, \mathcal{F}_0, I\!P)\colon \lim_{n\to\infty} ||a(t_n,X_n)||_{L^2} = \lim_{n\to\infty} ||a(t_n,Y_n)||_{L^2}$$

$$\forall X_0, Y_0 \in L^2(\Omega, \mathcal{F}_0, I\!P)\colon \lim_{n\to\infty} ||a(t_n,X_n)||_d = \lim_{n\to\infty} ||a(t_n,Y_n)||_d \ (I\!P-a.s.).$$

Proof. From the proof of Theorem 2.13, we extract the identity

$$\begin{aligned}
&\mathbb{E}\,||X_{n+1}-Y_{n+1}||^2-2\theta_n h_n \mathbb{E}\,\langle X_{n+1}-Y_{n+1}, a(t_{n+1},X_{n+1})-a(t_{n+1},Y_{n+1})\rangle_d \\
&\quad +\theta_n^2 h_n^2 \mathbb{E}\,||a(t_{n+1},X_{n+1})-a(t_{n+1},Y_{n+1})||^2 \\
&= \mathbb{E}\,||X_n-Y_n||^2+2(1-\theta_n)h_n \mathbb{E}\,\langle X_n-Y_n, a(t_n,X_n)-a(t_n,Y_n)\rangle_d \\
&\quad +(1-\theta_n)^2 h_n^2 \mathbb{E}\,||a(t_n,X_n)-a(t_n,Y_n)||^2 \\
&\quad +h_n \sum_{j=1}^m \mathbb{E}\,||b^j(t_n,X_n)-b^j(t_n,Y_n)||^2.
\end{aligned}$$

Define

$$v_n := \mathbb{E}\,[V_n(X,Y)]$$

for $n \in \mathbb{N}$. Thus, the above mentioned identity is equivalent to

$$\begin{aligned}
v_{n+1} = &\\
& v_n+h_n\mathbb{E}\Big[2\langle X_n-Y_n,a(t_n,X_n)-a(t_n,Y_n)\rangle_d+\sum_{j=1}^m||b^j(t_n,X_n)-b^j(t_n,Y_n)||^2\Big] \\
&+[(1-2\theta_n)h_n^2+\theta_n^2h_n^2-\theta_{n-1}^2h_{n-1}^2]\mathbb{E}\left[||a(t_n,X_n)-a(t_n,Y_n)||^2\right] \\
&+2[\theta_{n-1}h_{n-1}-\theta_n h_n]\mathbb{E}\left[\langle X_n-Y_n,a(t_n,X_n)-a(t_n,Y_n)\rangle_d\right] \qquad (4.1)
\end{aligned}$$

which can be estimated by

$$0 \le v_{n+1} \le v_n-(2\theta_n-1)h_n\mathbb{E}\,[||a(t_n,X_n)-a(t_n,Y_n)||^2]$$

under (i) - (iii). Now, it remains to apply the discrete invariance principle as stated by Lemma 7.1 to $(v_n)_{n\in\mathbb{N}}$ with

$$w_n := (2\theta_n-1)h_n\mathbb{E}\,[||a(t_n,X_n)-a(t_n,Y_n)||^2]$$

in order to conclude all implications of Theorem 4.1 on L^2-convergence under (iv). Finally, note that fast L^2-convergence (i.e. $\sum_{n=1}^\infty ||Z_n||_{L^2}^2 < +\infty$) implies a.s. convergence by Borel-Cantelli Lemma (i.e. $Z_n \to 0$ (a.s.) as $n \to +\infty$). Therefore, the proof is complete. □

4.2. Asymptotic B-stability of theta methods. Consider the following notion of asymptotic B-stability.

Definition 4.2 (Asymptotic P-th Moment B-Stability)**.** A numerical method with scheme $Y_{n+1} = Y_n + \Phi_n(Y)$ is called (globally) *asymptotically p-th moment B-stable* iff

$$\lim_{n\to+\infty} \mathbb{E}\,||X_n-Y_n||_d^p = 0 \qquad (4.2)$$

for $\mathcal{F}_0$-adapted initial data with $\mathbb{E}\,||X_0-Y_0||_d^p < +\infty$. If $p=2$, we additionally speak of *asymptotic mean square B-stability.* A numerical method with scheme $Y_{n+1} = Y_n + \Phi_n(Y)$ is called (globally) *a.s. asymptotically B-stable* iff

$$\lim_{n\to+\infty} ||X_n-Y_n||_d = 0 \quad (\mathbb{P}-a.s.) \qquad (4.3)$$

for $\mathcal{F}_0$-adapted initial data with finite moments.

Of course, every exponentially B-stable numerical method is asymptotically B-stable. However, the B-stability according to definition 1.1 does not imply asymptotic B-stability. So this notion of asymptotic B-stability lies somewhere in-between exponential and classical B-stability.

Theorem 4.3 (Asymptotic Mean Square B-stability of Theta Methods (1.6))**.** *Assume that*

(i) $\theta_n h_n$ *is nonincreasing in* $n \in I\!N$, $\theta_n \geq 0.5$,

(ii) $\exists K_{OL} > 0 \; \forall (t,x,y) \in I\!R^{1+2d}$

$$2\langle x-y, a(t,x)-a(t,y)\rangle_d + \sum_{j=1}^{m} ||b^j(t,x)-b^j(t,y)||^2 \leq -K_{OL}||x-y||_d^2 \leq 0,$$

(iii) $v_0 = I\!E\,[V_0(X_0,Y_0)] < +\infty$,

(iv) $\inf_{n\in I\!N} h_n > 0$.

Then Theta methods (1.6) are asymptotically mean square B-stable. Moreover, we have a.s. asymptotic B-stability, i.e. more precisely

$$\lim_{n\to+\infty} ||X_n - Y_n||_d = 0 \quad (I\!P - a.s.)$$

for all $X_0, Y_0 \in L^2(\Omega, \mathcal{F}_0, I\!P)$ *with* $v_0 < +\infty$.

Proof. Return to the identity (4.1) from previous proof of Theorem 4.1. Using the requirements (i)-(iv) with $K_{OL} < 0$ and all $\theta_n \geq 0.5$, we find that

$$\begin{aligned} v_{n+1} \leq\; & v_n \\ & + h_n \mathbb{E}\left[2\langle\!\langle X_n - Y_n, a(t_n,X_n) - a(t_n,X_n)\rangle\!\rangle_d + \sum_{j=1}^{m} ||b^j(t_n,X_n) - b^j(t_n,Y_n)||_d^2\right] \\ \leq\; & v_n - h_n K_{OL}||X_n - Y_n||_{L^2}^2 \leq v_0 - K_{OL}\sum_{k=0}^{n} ||X_k - Y_k||_{L^2}^2 h_k \\ \leq\; & v_0 - K_{OL} \inf_{i\in\mathbb{N}} h_i \sum_{k=0}^{n} ||X_k - Y_k||_{L^2}^2 \end{aligned}$$

whenever $v_0 < +\infty$. Now, apply DIP from appendix (Lemma 7.1) to get to

$$0 \leq \lim_{n\to+\infty} \leq v_0 - K_{OL} \inf_{i\in\mathbb{N}} h_i \sum_{n=0}^{\infty} ||X_n - Y_n||_{L^2}^2,$$

hence

$$\sum_{n=0}^{\infty} ||X_n - Y_n||_{L^2}^2 \leq \frac{v_0}{K_{OL} \inf_{n\in\mathbb{N}} h_n} < +\infty.$$

The finite square summability of L^2-norms $||X_n - Y_n||_{L^2}$ implies that

$$\lim_{n\to+\infty} ||X_n - Y_n||_{L^2} = 0.$$

The principle of fast L^2-convergence (due to the Borel-Cantelli Lemma) guarantees that we may also establish a.s. convergence of $||X_n - Y_n||_d$ to 0 as $n \to +\infty$. Therefore, the proof is complete. □

Theorem 4.4 (Asymptotic Mean Square B-stability of Theta Methods (1.6)). *Assume that*

(i) $\theta_n h_n$ *is nonincreasing in* $n \in I\!N$, $\theta_n > 0.5$,

(ii) $\forall (t,x,y) \in I\!R^{1+2d} : 2\langle x-y, a(t,x)-a(t,y)\rangle_d + \sum_{j=1}^{m} ||b^j(t,x)-b^j(t,y)||^2 \leq 0$,

(iii) $v_0 = I\!E\,[V_0(X_0,Y_0)] < +\infty$,

(iv) $\inf_{n\in I\!N}(2\theta_n - 1)h_n^2 > 0$ *(as in autonomous case with* $\theta > 0.5$*)*,

(v) $\exists \underline{K}_a > 0\ \forall (t,x,y) \in I\!R^{1+2d} : ||a(t,x)-a(t,y)|| \geq \underline{K}_a ||x-y||_d$.

Then Theta methods (1.6) are asymptotically mean square B-stable. Moreover, we have a.s. asymptotic B-stability, i.e. more precisely

$$\lim_{n\to+\infty} ||X_n - Y_n||_d = 0 \quad (\mathbb{P} - a.s.)$$

for all $X_0, Y_0 \in L^2(\Omega, \mathcal{F}_0, \mathbb{P})$ *with* $v_0 < +\infty$.

Proof. Return to the identity (4.1) from previous proof of Theorem 4.1. Using the requirements (i)-(v) with $\underline{K}_a < 0$ and all $\theta_n > 0.5$, we find that

$$\begin{aligned} v_{n+1} &\leq v_n + (1-2\theta_n)h_n^2 \mathbb{E}\left[||a(t_n,X_n)-a(t_n,Y_n)||_d^2\right] \\ &\leq v_n - (2\theta_n - 1)h_n^2 \underline{K}_a^2 ||X_n - Y_n||_{L^2}^2 \leq v_0 - \underline{K}_a^2 \sum_{k=0}^{n} ||X_k - Y_k||_{L^2}^2 (2\theta_k - 1)h_k^2 \\ &\leq v_0 - \underline{K}_a^2 \inf_{i\in\mathbb{N}} (2\theta_i - 1)h_i \sum_{k=0}^{n} ||X_k - Y_k||_{L^2}^2 \end{aligned}$$

whenever $v_0 < +\infty$. Now, apply DIP from appendix (Lemma 7.1) to get to

$$0 \leq \lim_{n\to+\infty} \leq v_0 - \underline{K}_a^2 \inf_{i\in\mathbb{N}} (2\theta_i - 1)h_i \sum_{n=0}^{\infty} ||X_n - Y_n||_{L^2}^2,$$

hence

$$\sum_{n=0}^{\infty} ||X_n - Y_n||_{L^2}^2 \leq \frac{v_0}{\underline{K}_a^2 \inf_{n\in\mathbb{N}}(2\theta_n - 1)h_n} < +\infty.$$

The finite square summability of L^2-norms $||X_n - Y_n||_{L^2}$ implies that

$$\lim_{n\to+\infty} ||X_n - Y_n||_{L^2} = 0.$$

The principle of fast L^2-convergence (due to the Borel-Cantelli Lemma) guarantees that we may also establish a.s. convergence of $||X_n - Y_n||_d$ to 0 as $n \to +\infty$. Therefore, the proof is complete. □

5. A Discussion on Applications by Examples

A series of linear and nonlinear examples will illustrate some of our findings by this section, and above all, the practical applicability of our results to nontrivial situations. All examples are real-valued, however one can also extend them to complex-valued ones (this is left to the reader). In what follows, for the mean square analysis, we can take any independent random variables ΔW_n with moments

$$\mathbb{E}\,[\Delta W_n] = 0, \quad Var(\Delta W_n) = \mathbb{E}\,[\Delta W_n]^2 = h_n.$$

Of course, Gaussian variables $\Delta W_n \in \mathcal{N}(0, h_n)$ would satisfy these moment relations. So, note that the results of our paper apply to both so-called weak and strong convergent approximations of SDEs. Moreover, any additive martingale noise such as $\sigma \Delta W_n$ would not change our findings due to the nature of criterion of B-stability canceling state-independent additive terms both in deterministic and in stochastic settings. We will refer to **admissible step sizes** h_k only here, i.e. the requirement

$$0 < \inf_{n \in \mathbb{N}} h_n \leq h_k \leq \sup_{n \in \mathbb{N}} h_n < +\infty$$

is fulfilled throughout this section.

5.1. A linear SDE (geometric Brownian motion). In mathematical finance they often refer to the geometric Brownian motion which satisfies the Itô SDE

$$dX(t) \;=\; \lambda X(t)dt + \sigma X(t) dW(t). \tag{5.1}$$

This equation is also used as test equation for A-stability of numerical methods applied to SDEs with commutative noise. Obviously, for this SDE (5.1), we exactly know the constants

$$K_a = \lambda, K_L = |\lambda|, K_b = |\sigma|, K_{OL} = -2\lambda - \sigma^2(p-1)$$

occurring in diverse theorems in our paper. The drift-implicit Theta method (1.6) reduces here to

$$Y_{n+1} \;=\; Y_n + \lambda[\theta_n Y_{n+1} + (1-\theta_n) Y_n] h_n + \sigma Y_n \Delta W_n$$

with independent random variables ΔW_n with finite moments $\mathbb{E}\,[\Delta W_n] = 0$ and $Var(\Delta W_n) = \mathbb{E}\,[\Delta W_n]^2 = h_n$. Due to its linearity, this scheme is equivalent to

$$Y_{n+1} \;=\; \frac{1 + \lambda(1-\theta_n)h_n + \sigma \Delta W_n}{1 - \lambda \theta_n h_n} Y_n$$

for sufficiently small $\theta_n h_n$ such that

$$[\lambda]_+ \sup_{n \in \mathbb{N}} \theta_n h_n < 1$$

(just needed for existence, i.e. absence of any numerical explosions). Obviously, SDE (5.1) has a p-th mean nonexpansive coefficient system for $p > 1$ iff

$$2\lambda + \sigma^2(p-1) \leq 0$$

and a strictly p-th mean contractive coefficient system for $p > 1$ iff

$$2\lambda + \sigma^2(p-1) < 0$$

which represents the appropriate condition for an adequate investigation with respect to p-th mean B-stability. Let this condition be satisfied for $p = 2$. Then, Theorem 4.3 with

$$K_{OL} = -2\lambda - \sigma^2 > 0$$

confirms the asymptotic mean square B-stability of all Theta methods with $\theta = \theta_n > 0.5$ and admissible step sizes h_n. Theorem 3.1 implies that the drift-implicit backward Euler method with all $\theta_n = 1$ and admissible step sizes h_n with

$$0 < \inf_{k\in\mathbb{N}} h_k \le h_n \le \sup_{k\in\mathbb{N}} h_k < +\infty$$

is even exponentially mean square B-stable since $2K_a + K_b = 2\lambda + \sigma^2 < 0$. The same is true for Theta methods with $0 \le \theta_n \le 1$ and sufficiently small step sizes by Theorem 3.2. More precisely their (numerical) exponential mean square B-stability is concluded from Theorem 3.2 if

$$2\lambda + \sigma^2 + \lambda^2 \sup_{n\in\mathbb{N}} (1-\theta_n)^2 h_n < 0.$$

More general, V-stability for Theta methods (1.6) with $\theta_n \ge 0.5$ and admissible variable step sizes h_n can be established (see Theorem 4.1). Similarly, asymptotic B-stability of Theta methods by Theorems 4.3 and 4.4 is confirmed.

5.2. A SDE with nonLipschitz drift. Consider the Itô SDE

$$dX(t) \quad = \quad [\lambda X(t) - \gamma |X(t)|^{\alpha} X(t)]dt + \sigma X(t) dW(t) \tag{5.2}$$

where $\gamma > 0, \alpha > 0, \lambda, \sigma$ are real constants. This SDE has a strictly p-th mean contractive coefficient system (a, b) iff

$$K_{OL} := -2\lambda - \sigma^2(p-1) \quad > \quad 0$$

since, for all $x, y \in \mathbb{R}^1$, we have

$$\begin{aligned} &2(x-y)(a(x)-a(y)) + (p-1)|b(x)-b(y)|^2 \\ &= \quad [2\lambda + (p-1)\sigma^2](x-y)^2 - \gamma(x-y)[|x|^{\alpha}x - |y|^{\alpha}y] \\ &\le \quad [2\lambda + (p-1)\sigma^2](x-y)^2 \end{aligned}$$

with $p \ge 1$ and $\gamma \ge 0$. Let this equation be discretized by Theta methods which reduce to the scheme

$$Y_{n+1} \;=\; Y_n + \Big[\theta_n \Big(\lambda - \gamma |Y_{n+1}|^{\alpha}\Big) Y_{n+1} + (1-\theta_n)\Big(\lambda - \gamma |Y_n|^{\alpha}\Big) Y_n\Big] h_n + \sigma Y_n \Delta W_n$$

when applied to (5.2). Thanks to Theorem 4.3, we know about their asymptotic mean square B-stability under $\theta_n \ge 0.5$ and $K_{OL} > 0$. Theorem 4.1 says that they are V-stable under same assumptions. Since $\lambda < 0$, $K_{OL} > 0$, $\underline{K}_a = |\lambda| > 0$, we may even conclude exponential mean square stability for sufficiently small step sizes h_n whenever all $\theta_n > 0.5$. More practically, we suggest to implement the **linear-implicit Theta methods**

$$\begin{aligned} Y_{n+1} \;&=\; Y_n + \Big[\theta_n \Big(\lambda - \gamma |Y_n|^{\alpha}\Big) Y_{n+1} + (1-\theta_n)\Big(\lambda - \gamma |Y_n|^{\alpha}\Big) Y_n\Big] h_n + \sigma Y_n \Delta W_n \\ &=\; \frac{1 + (1-\theta_n)(\lambda - \gamma|Y_n|^{\alpha})h_n + \sigma \Delta W_n}{1 - \theta_n(\lambda - \gamma |Y_n|^{\alpha})h_n} Y_n \end{aligned}$$

which are "asymptotically equivalent" to "standard" Theta methods from above. In particular, among them, the linear-implicit midpoint method with all $\theta_n = 0.5$ is a computationally efficient variant (in order to avoid costly local resolution of implicit equations by numerical root-finding methods resulting into additional numerical errors).

5.3. A SDE with nonlinear diffusion. Consider the Itô SDE

$$dX(t) \quad = \quad [\lambda X(t)]dt + \sigma \sin(X(t))dW(t) \tag{5.3}$$

where $-K_{OL} := 2\lambda + \sigma^2 < 0$. In this case, it is easy to recognize that this SDE has mean square contractive perturbations since

$$|b(x) - b(y)|^2 \leq |\sigma \sin(x) - \sigma \sin(y)|^2 \leq \sigma^2 |x - y|^2$$

by Lipschitz continuity of sin(.) function (i.e. apply MVT = mean value theorem). The drift-implicit Theta methods (1.6) applied to SDE (5.3) take the explicit form

$$Y_{n+1} \quad = \quad \frac{Y_n(1 + (1-\theta_n)\lambda h_n) + \sigma \sin(Y_n)\Delta W_n}{1 - \theta_n \lambda h_n}$$

which is well-defined whenever

$$\forall n \in \mathbb{N} \quad : \quad \theta_n \lambda h_n \neq 1.$$

Thanks to Theorems 4.1, 4.3, 4.4, we may conclude that these methods have V-stable, asymptotic mean square B-stable and, by Theorem 3.1 under

$$K_{OL} - \lambda^2 \sup_{n \in \mathbb{N}} (1 - \theta_n)^2 h_n > 0,$$

exponentially mean square B-stable perturbations whenever $\theta_n > 0.5$ (i.e. with rate $K_{OL} > 0$ for sufficiently small step sizes). Their asymptotic (numerical) mean square B-stability is even maintained for the special case of $\theta_n = 0.5$ since $|a(x) - a(y)| = |\lambda||x - y|$ with $|\lambda| > 0$ under the imposed setting.

5.4. A nonlinear SDE with Lipschitz diffusion. Consider the Itô SDE

$$dX(t) \quad = \quad [\lambda X(t) - \gamma (X(t))^{2n+1}]dt + \sigma g(X(t))dW(t) \tag{5.4}$$

where g is Lipschitz continuous with Lipschitz constant L_g, $n \in \mathbb{N}$, $n > 0$, $\gamma > 0$. Suppose that

$$2\lambda + (p-1)\sigma^2 L_g^2 < 0.$$

Here, we find that

$$\begin{aligned} & 2(x-y)(a(x) - a(y)) + (p-1)|b(x) - b(y)|^2 \\ = \;& 2\lambda(x-y)^2 - \gamma(x-y)(x^{2n+1} - y^{2n+1}) + (p-1)\sigma^2 |g(x) - g(y)|^2 \\ \leq \;& (2\lambda + (p-1)\sigma^2 L_g^2)(x-y)^2 \leq 0, \end{aligned}$$

i.e. p-th mean B-stable perturbations for $p \geq 1$. The drift-implicit Theta methods (1.6) applied to SDE (5.4) take the explicit form

$$Y_{k+1} \quad = \quad \frac{Y_k(1 + (1-\theta_k)[\lambda - \gamma Y_k^{2n}]h_k) - \theta_k \gamma Y_{k+1}^{2n+1} h_k + \sigma g(Y_k)\Delta W_k}{1 - \theta_k \lambda h_k}$$

which is well-defined when

$$\forall n \in \mathbb{N} \quad : \quad \theta_n [\lambda]_+ h_n < 1.$$

Thanks to Theorems 4.1, 4.3, 4.4, we may conclude that these methods have V-stable, asymptotic mean square B-stable and exponentially mean square B-stable perturbations under $\theta_n > 0.5$ (i.e. with rate

$$K_{OL} := -2\lambda - \sigma^2 L_g^2 - \lambda^2 \sup_{n \in \mathbb{N}} (1-\theta_n)^2 h_n > 0$$

for sufficiently small step sizes). Their asymptotic (numerical) mean square B-stability is even maintained for the special case of $\theta_n = 0.5$ since $|a(x) - a(y)| = |\lambda||x-y|$ with $|\lambda| > 0$ under the imposed setting. A more practical implementation avoiding local resolution of implicit equations by numerical root-finding methods is given by the "asymptotically equivalent" **linear-implicit Theta methods**

$$Y_{k+1} \quad = \quad \frac{Y_k(1+(1-\theta_k)[\lambda - \gamma Y_k^{2n}]h_k) - (1-\theta_k)\gamma Y_k^{2n+1} h_k + \sigma g(Y_k)\Delta W_k}{1-\theta_k(\lambda - \gamma |Y_k|^{2n})h_k}.$$

5.5. A nonlinear SDE with nonLipschitz drift and diffusion. Consider the Itô SDE

$$dX(t) \quad = \quad [\lambda X(t) - \gamma (X(t))^{2n+1}]dt + \sigma [X(t)]^{n+1} dW(t) \tag{5.5}$$

where $n \in \mathbb{N}$, $n > 0$, $(p-1)\sigma^2 \leq 2\gamma$. Suppose that

$$\lambda < 0.$$

Then one can establish V-stability of perturbations and asymptotic mean square B-stability of Theta methods with $\theta_n > 0.5$ (apply Theorems 4.1, 4.3, 4.4) under $\sigma^2 \leq 2\gamma$ and $\lambda < 0$. An easy implementation of them is practically done by the linear-implicit Theta methods

$$Y_{k+1} \quad = \quad \frac{1+(1-\theta_k)[\lambda - \gamma |Y_k|^{2n}]h_k + \sigma Y_k^n \sigma W_k}{1-\theta_k[\lambda - \gamma |Y_k|^{2n}]h_k} Y_k$$

which is "asymptotically equivalent" to the "standard" Theta method, but easily implementable and computationally more efficient without solving nonlinear algebraic equations at each integration step.

5.6. A N-Dimensional nonlinear SDE. Let $\lambda_i \leq 0$ be real constants and $\gamma \geq 0$. $\langle ., . \rangle$. Here $||.||$ denote the Euclidean scalar product and norm in $\mathbb{R}^N$, respectively. Define the $N \times N$ diagonal matrices

$$\Lambda := diag(\lambda_1, \lambda_2, ..., \lambda_N) \quad \text{and} \quad \Sigma := diag(\sigma_1, \sigma_2, ..., \sigma_N)$$

Consider the N-dimensional system of Itô SDEs (occurring in discretizations of nonlinear SPDEs, e.g. [23]).

$$dX_i(t) \quad = \quad \lambda_i X_i(t)(1 + [\sum_{k=1}^{N} [X_k(t)]^2]^{\gamma/2})dt + \sigma_i X_i(t) dW(t) \tag{5.6}$$

for $i = 1, 2, ..., N$. First, we check p-th mean contractivity (for $p \geq 1$) by applying Lemma A.1 on the monotonicity of $||u||^\gamma u$ from [24] and get to the conclusion

$$\begin{aligned}
&2\langle x-y, a(t,x)-a(t,y)\rangle_d + (p-1)||b(t,x)-b(t,y)||^2 \\
&= 2\langle x-y, \Lambda x(1+||x||^\gamma) - \Lambda y(1+||y||^\gamma)\rangle_d + (p-1)||\Sigma x - \Sigma y||_N^2 \\
&= 2\langle x-y, \Lambda(x-y)\rangle_d + 2\langle x-y, \Lambda(x||x||^\gamma - y||y||^\gamma)\rangle_d + (p-1)||\Sigma(x-y)||_N^2 \\
&= 2\sum_{i=1}^N \lambda_i (x_i-y_i)^2 + 2\langle x-y, \Lambda(x||x||^\gamma - y||y||^\gamma)\rangle_d + (p-1)\sum_{i=1}^N \sigma_i^2 (x_i-y_i)^2 \\
&\leq \sum_{i=1}^N (2\lambda_i + (p-1)\sigma_i^2)(x_i-y_i)^2 \;\leq\; \max_{k=1,...,N}(2\lambda_k + (p-1)\sigma_k^2)||x-y||_N^2
\end{aligned}$$

under $\lambda_i \leq 0$, since

$$\langle x-y, \Lambda(x||x||^\gamma - y||y||^\gamma)\rangle_d \quad \leq \quad 0$$

for all $p \geq 0$. Hence, the SDE (5.6) is mean square nonexpansive if

$$\max_{k=1,...,N}(2\lambda_k + (p-1)\sigma_k^2) \quad \leq \quad 0$$

and has a p-th mean contractive coefficient system (a, b) if

$$\max_{k=1,...,N}(2\lambda_k + (p-1)\sigma_k^2) \quad < \quad 0. \tag{5.7}$$

Now, consider the backward Euler method applied to SDE (5.6) governed by the scheme

$$Y_{n+1} \quad = \quad Y_n + \Lambda Y_{n+1}(1+||Y_{n+1}||^\gamma)h_n + \Sigma Y_n \Delta W_n.$$

Note that, for sufficiently small step sizes h_n, one may solve this implicit algebraic equation by Newton-Raphson-type methods. Then, Theorem 2.9 implies that this method is (numerically) exponentially mean square B-stable for system (5.6) under condition (5.7) with $p = 2$. Moreover, under (5.7) with $p = 2$, the drift-implicit Theta methods with $\theta_n \geq 0.5$ are (numerically) asymptotic mean square B-stable according to Theorem 4.3 (also V-stable). As an alternative, we suggest the "asymptotically equivalent" discretizations by **linear-implicit Euler methods**

$$Y_{n+1} \quad = \quad Y_n + \Lambda(Y_{n+1}(1+||Y_n||^\gamma)h_n + \Sigma Y_n \Delta W_n$$

in order to avoid the time-consuming local resolution of implicit algebraic equations in N dimensions. We can still observe asymptotic stabilization of L^2-perturbations by these linear-implicit methods. Similarly, we may investigate the **linear-implicit Theta methods**

$$Y_{n+1} \quad = \quad Y_n + \Lambda(\theta_n Y_{n+1} + (1-\theta_n)Y_n)(1+||Y_n||^\gamma)h_n + \Sigma Y_n \Delta W_n$$

and establish asymptotic mean square B-stability for $\theta_n \geq 0.5$ (apply Theorem 4.3 under condition (5.7)). However, among them with all $\theta_n = 0.5$, we prefer the **linear-implicit midpoint methods**

$$Y_{n+1} \quad = \quad Y_n + \frac{1}{2}\Lambda(Y_{n+1} + Y_n)(1+||Y_n||^\gamma)h_n + \Sigma Y_n \Delta W_n.$$

to discretize SDE (5.6) with numerically B-stable perturbations as long as $\gamma \geq 0$.

6. Summary of Major Results

This list summarizes our major findings documented by a series of Theorems:

- The concepts of numeric and dynamic B-stability for stochastic-numerical methods have been introduced. Numeric B-stability of numerical methods relates to a non-increasing propagation of perturbations of initial data. Dynamic B-stability of numerical methods requires numeric B-stability of numerical methods for all underlying SDEs with monotone coefficient systems (a, b^j) and all admissible, nonrandom step sizes h_n.
- Theorem 2.1 establishes a general a.s. identity on the two-point motion process (X, Y) for numerical methods. This identity highlights the importance of adequately constructed methods with monotone increments.
- Theorem 2.4 states all midpoint-type methods are exact-monotone (i.e. possess a contraction-exact two-point motion process). Therefore, the existence of exact-monotone methods for random ODEs is constructively verified by that theorem.
- Drift-implicit backward Euler-Maruyama method is both numerically and dynamically mean square B-stable (see Theorems 2.9 and 2.15).
- Explicit forward Euler-Maruyama method is mean square expansive for SDEs with expansive coefficient systems (see Theorem 2.13).
- Theorems 3.1 and 3.2 establish exponential B-stability for Theta methods under appropriate conditions.
- The even more general concepts of V-stability along positive functionals V and asymptotic B-stability for the two-point motion process are introduced in Section 4.
- Theorem 4.1 reports on V-stability for variable step sizes with nonincreasing parameters $\theta_n h_n$ and an asymptotic invariance principle of equilibria.
- Theorems 4.3 and 4.4 provide us criteria for asymptotic B-stability of Theta methods applied to (nonlinear) SDEs with mean square nonexpansive coefficient systems with $\theta_n \geq 0.5$.
- Section 5 documents a series of examples showing several applications of our theorems to linear and nonlinear SDEs, even in multi-dimensional case.
- In the appendix we verify a more general discrete invariance principle (DIP).

This paper represents a humble trial to supplement some studies on qualitative behavior of stochastic-numerical methods for SDEs. There are numerous extensions possible, e.g., to carry on with B-stability analysis of compensated stochastic theta methods (see Wang and Gan [27]) applied to jump diffusions. An alternative to our analysis is also given by Beyn, Isaak and Kruse [4] while referring to monotone systems. There, C-stability and B-consistency of the split-step backward Euler (SSBE) method and the projected Euler-Maruyama (PEM) method are studied for SDEs with monotone coefficients.

7. Appendix: Discrete Invariance Principle (DIP)

In the previous sections we have exploited the following discrete version of invariance principle (i.e. asymptotic convergence with wedge-type functions). For convenience of readership, we add its elementary proof below.

Lemma 7.1 (Discrete Invariance Principle (DIP))**.** *Assume that the sequence* $(v_n)_{n\in I\!N}$ *of finite nonnegative real numbers* $v_n \geq 0$ *satisfies*

$$\forall n \in I\!N : \; v_{n+1} \quad \leq \quad v_n - w_n \tag{7.1}$$

with nonnegative real numbers $w_n \geq 0$*. Then*

(i) $(v_n)_{n\in I\!N}$ *is nonincreasing in* $n \in I\!N$*,*

(ii) $\sum_{n=0}^{\infty} w_n \leq v_0 < +\infty$*,*

(iii) $\lim_{n\to\infty} w_n = 0$*,*

(iv) $\exists \lim_{n\to+\infty} v_n$ *and* $0 \leq \lim_{n\to+\infty} v_n \leq v_0 - \sum_{k=0}^{\infty} w_k < +\infty$*.*

Proof. Monotonicity (i) of v_n is seen from the equivalence (7.1) with

$$v_{n+1} - v_n \leq -w_n \leq 0$$

since $w_n \geq 0$ for all $n \in \mathbb{N}$. Recall the Monotone Bounded Sequence Theorem from elementary calculus. Note that $(v_n)_{n\in\mathbb{N}}$ is bounded from below by 0 and decreasing (i.e. nonincreasing). Therefore, the finite limit $\lim_{n\to+\infty} v_n$ must exist. By induction on n, from inequality (7.1), we obtain that

$$v_{n+1} \leq v_0 - \sum_{k=0}^{n} w_k. \tag{7.2}$$

Thus, by pulling over the limit as $n \to +\infty$, the limit of v_n can be estimated by

$$0 \leq \lim_{n\to+\infty} v_n \leq v_0 - \sum_{k=0}^{\infty} w_k.$$

Consequently, (iv) is verified. This also implies that

$$0 \leq \sum_{n=0}^{\infty} w_n < +\infty.$$

From Abel's series theory, we know that this fact allows us to conclude that (iii) is true. Finally, (ii) is clear from the positivity of v_{n+1} and direct rearrangement of (7.2). This completes the proof. □

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