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SCS 63: The Fell Compactification

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ABSTRACT:

The Fell compactification $\underline{H}(X)$ of a locally quasi-compact T_o-space X can be viewed as a compact ordered space. Then $\underline{H}(X)$ corresponds to a quasi-compact, locally quasi-compact super-sober space ψX whose open sets are all the open upper sets of $\underline{H}(X)$. There is an essential extension $X \hookrightarrow \psi X$ in the category $\underline{T}_o$ of T_o-spaces and continuous maps. We show that

$$\underline{O}(\psi X) \cong DID(L)$$

for the distributive continuous lattice $L=\underline{O}(X)$ – where $\underline{O}(Y)$ is the lattice of open sets of a space Y, D(P) is the dual of a continuous poset P, and I(P) is the continuous lattice underlying the injective hull of P (endowed with the Scott topology σ_P) in the category $\underline{T}_o$.

This result relies upon a representation of ID(L) for a continuous $1,\wedge$-semilattice L, viz.

$$ID(L) \cong Filt_{\sigma}L,$$

the (continuous) lattice of all those filters of L which are generated by Scott-open subsets of L. For a distributive continuous lattice L, the meet-prime elements of $DFilt_{\sigma}L$ in their (hull-kernel) topology are (topologically) identified with the pseudo-meet-prime (=weakly meet-prime) elements of L endowed with the Γ-topology of L^{op}.

Furthermore both $\underline{H}(?)$ and $\psi(?)$ are shown to be functorial on the category of locally quasicompact T_o-spaces and continuous perfect mappings.

I WOULD BE GLAD IF THERE IS ANYBODY WILLING TO CHECK THE PROOFS (AND TELLS ME ABOUT)

The Fell Compactification Revisited

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In [Fe_2] J.M.G.Fell considers, for a topological space X, a certain topology on the complete lattice $\underline{A}(X)$ of all closed subsets of X (ordered by the inclusion relation) for which the sets

$U(C;V_1,\dots,V_n):=\{A\in\underline{A}(X)\mid A\cap C=\emptyset, A\cap V_i\neq\emptyset \text{ for } i=1,\dots n\}$

with C quasi-compact and V_i open in X, $n\in\mathbb{N}\cup\{0\}$, form an open basis.

As noted in [C] pp.151/152, the Fell topology is, for a locally quasi-compact space X, the Lawson topology λ of the lattice $\underline{O}(X)$ of open subsets of X (ordered by inclusion) transferred to $\underline{A}(X)$ along the bijection $\underline{O}(X)\to\underline{A}(X)$, $V\mapsto X-V$ - where a space X is said to be locally quasi-compact iff every point has a neighborhood basis consisting of quasicompact (but not necessarily open) subsets. 1)

The <u>Fell compactification</u> $\underline{H}(X)$ of a locally quasi-compact space X is the closure of

$$\{cl\{x\}\mid x\in X\}$$

in $\underline{A}(X)$ with regard to the Fell topology. By [C] VI-3.4(1) and VI-1.14, $(\underline{O}(X),\lambda_{\underline{O}(X)})$ is a <u>compact p(artially) o(rdered) space</u> (in the sense of L.Nachbin [N], [C] VI-1.1), hence so is $\underline{H}(X)$ in its inclusion order (reversing the order is no problem in a p.o.space).

For a locally compact, non-compact Hausdorff space X, $\underline{H}(X)$ is the Alexandrov one-point-compactification of X with $\emptyset$ adjoined as a new point ([Fe_2] p. 475) - considered as a compact p.o.space in which $\emptyset\leq\{x\}$ for every $x\in X$ is the only non-trivial occurrence of $\leq$. Thus, in the setting of locally quasicompact spaces, the Fell compactification may be viewed as a substitute for the Alexandrov one-point-compactification. (In other contexts, of course, different substitutes can be adequate, cf. e.g. [H_2] §3, [Wi_1]).

In [Fe_1] §2, Fell has provided, in a special case, an interpretation of his construction in functional-analytic terms.

In [HL_2] K.H.Hofmann and J.D.Lawson have given, for a distributive continuous lattice L, various characterizations of the closure of the set consisting of the greatest element 1 of L and of all meet-prime elements of L with regard to the Lawson topology of L. By the celebrated theorem of K.H.Hofmann and J.D.Lawson that the distributive continuous lattices L are - up to an isomorphism - precisely the lattices $\underline{O}(X)$ of open sets (ordered by the inclusion relation) of locally quasi-compact (T_o-)spaces X, these results are - as has been observed in [H_8] - intimately related to Fell's construction $\underline{H}(X)$, since X can be chosen as a sober space: In that case, X is uniquely determined by L (up to a homeomorphism) and can be canonically represented by the set of meet-prime elements of L (endowed with a topology): The points of $\underline{H}(X)$ correspond - via the obvious anti-isomorphism $\underline{A}(X)\to\underline{O}(X)\cong L$ - to the pseudo-meet-prime elements of L, i.e. the suprema of the prime

1) It has been observed by J.Flachsmeyer [Fl_2], that for a locally compact Hausdorff space X, the Fell topology induced on $\underline{A}(X)-\{\emptyset\}$, coincides with the "lbc-topology" of S.Mrówka [Mr].

ideals (≠L) of L. [2].

Using this latter observation, we have noted in $[H_8]$ (with benefit from discussions with K.H.Hofmann) that the points of the Fell compactification $\underline{H}(X)$ of a locally quasi-compact T_o-space X are contained in an extension $X \hookrightarrow \gamma X$ of the space X studied in $[H_3]$ [3]. This extension $X \hookrightarrow \gamma X$, defined for arbitrary T_o-spaces X, is an equivalent representation of the <u>greatest essential extension</u> $X \hookrightarrow \lambda X$ of the T_o-space X discovered by B.Banaschewski $[Ba_1]$ §2. Thus, for a locally quasicompact T_o-space X, the co-restriction of the extension $X \hookrightarrow \gamma X$ to the points of the Fell compactification $\underline{H}(X)$ gives an extension.

$$X \hookrightarrow \psi X$$

(with a new topology on ψX such that X is contained in ψX as a subspace) which is an <u>essential extension</u>.

Whereas in $[H_8]$ §3, the extension $\psi X \hookrightarrow \gamma X$ for an arbitrary (T_o-)space X has been studied, we shall investigate here the extension

$$X \hookrightarrow \psi X$$

for locally quasicompact T_o-spaces X or, slightly more

2) It has to be noted, however, that $1 \in L$ is (in the definition used in $[H_8]$) <u>never</u> meet-prime and that it <u>need not</u> be contained in the closure of the set of meet-prime elements of L with regard to the Lawson topology. To this extent the definitions of $[H_8]$ differ from those of $[HL_2]$.

3) This observation can be also based opon Fell's result ($[Fe_2]$p.475) that the points of $\underline{H}(X)$ are the convergence sets of the "primitive" nets of X - cf. $[H_3]$p.419, $[H_8]$3.13.

general [4], for those T_o-spaces X for which $\underline{O}(X)$ is a continuous lattice. We show that under this hypothesis,

ψX is a sober space, and $\underline{O}(\psi X)$ is a continuous lattice, in which the greatest element is compact and the way below relation $\ll$ is multiplicative, i.e.

$$U \ll V \text{ and } U \ll W \text{ imply } U \ll V \cap W,$$

or, equivalently, (by a result of K.H.Hofmann and M.W.Mislove [HM]) X is a quasi-compact, locally quasi-compact super-sober space,

and, furthermore,

the canonical embedding $\psi X \hookrightarrow \psi(\psi X)$ is a homeomorphism.

and, finally,

the space ψX corresponds to the compact ordered space $\underline{H}(X)$ via an isomorphism between the category of compact ordered spaces and isotone continuous maps and the category of quasi-compact, locally quasi-compact, super-sober spaces and "perfect" continuous maps (where "perfect" means that the pre-image map between the lattices of open sets preserves the way below relation), which is (a slight modification of the isomorphism) described in [C]VII-3. Moreover, both $\underline{H}(?)$ and $\psi(?)$ can be extended to functors defined on the category of locally quasicompact sober spaces and continuous perfect mappings.

For the proofs, we develop a program which seems to be of interest in itself, since it exhibits an intriguing interaction between two of the basic constructions for continuous posets: the dual ($[L_1]$,$[L_2]$,$[H_7]$) and the injective hull (in the category $\underline{T_o}$ of T_o-spaces and continuous maps, cf. $[H_4]$3.14).

4) J.R.Isbell [I] and K.H.Hofmann and J.D.Lawson $[HL_2]$ have provided an example of a T_o-space X for which $\underline{O}(X)$ is a continuous lattice but X fails to be locally quasi-compact. Note that a sober space X is locally quasi-compact if and only if $\underline{O}(X)$ is a continuous lattice.

For a distributive continuous lattice L, let

$$D(L)$$

denote the "dual" of L, consisting of those filters (= down-directed upper sets) of L which are open in the Scott topology σ_L of L. This is - with regard to the inclusion order - a continuous 1,$\wedge$-semi-lattice. Then we form the injective hull

$$(D(L),\sigma_{D(L)}) \hookrightarrow (I(D(L)),\sigma_{I(D(L))})$$

in the category $\underline{T}_o$, using the result of [H_4] 3.14:

> The continuous posets in their Scott topology are precisely those T_o-spaces which are sober and have an injective hull in $\underline{T}_o$, i.e. their greatest essential extension space (in the sense of B.Banaschewski [Ba_1]) is an injective T_o-space (i.e. - by D.Scott's result [Sc_2] 2.12 - a continuous lattice in its Scott topology).

Let us motivate briefly why we did expect that ID(L) or rather DID(L) is related to our problem viz.

$$DID(L) \cong \underline{O}(\psi X)$$

if $L=\underline{O}(X)$ is a distributive continuous lattice:

Firstly note that we had seen in [H_8] (slightly extending a result of K.H.Hofmann and J.D.Lawson [HL_2]) that the canonical embedding $X \hookrightarrow \psi X$ for a locally quasi-compact sober space X is a homeomorphism iff the way below relation of $\underline{O}(X)$ is multiplicative and the greatest element of $\underline{O}(X)$ is compact.

Secondly recall that, by a result of J.D.Lawson [L_1], the dual D(S) of a continuous 1,$\wedge$-semi-lattice S is a continuous lattice iff 1 is compact in S and the way below relation $\ll$ of S is multiplicative. Thus, for a continuous lattice L, D(L) has these properties (since $DD(L)\cong L$).

Now there is some hope that these properties are preserved under the injective hull

$$D(L) \hookrightarrow ID(L),$$

since an injective hull of a continuous poset preserves arbitrary infima (to the extent they exist) and the way below relation ([H_9] 3.7). Thus the greatest element of ID(L) must be compact, and also a torso of multiplicativity of the way below relation in ID(L) is present.

As one may suspect, the proof of the full multiplicativity of $\ll$ in ID(L) requires the use of a suitable representation:

The one we need, has not been used before. The inspiration to find it came from the problem which is still left open when it is established (via the proof of multiplicativity) that DID(L) is a continuous lattice, viz:

Is DID(L) distributive?

It is not difficult to see that this [5] is equivalent to

Is ID(L) distributive?

It is well known that a lattice L is distributive iff the complete lattice FiltL (of all filters of L, ordered by inclusion) is distributive. FiltL contains DL, but it must be too big in general, since it is an algebraic lattice. Thus

$$Filt_\sigma L,$$

the complete lattice of all those filters of L which are generated by Scott-open subsets of L, ordered by inclusion, seemed to be a reasonable candidate for ID(L). Indeed, $Filt_\sigma L$ is (isomorphic to) ID(L). (The "handling" of $Filt_\sigma L$ in this paper owes much to the skilful techniques introduced by J.D.Lawson for D(L) in [L_1] (cf. [L_2]).)

I am indebted to K.H.Hofmann for discussions on some of the material in section 1 of this paper.

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5) The dual D(L) of a distributive continuous lattice L is distributive provided that it is a complete lattice. (A proof of this observation is immediate from the representation of D(L) in terms of quasicompact saturated subsets of a locally quasicompact space, cf. [HM] §2.)

§ o Basic concepts

o.1 For an arbitrary partially ordered set (= poset, for short) $(P,\leq)$, we have

$$x \ll y \quad (\text{"x is } \underline{\text{way below}} \text{ y"})$$

iff whenever $y \leq \sup D$ (the supremum of D) for some non-empty, up-directed subset D (i.e. $a,b \in D$ implies $a,b \leq c$ for some $c \in D$) of P, then $x \leq d$ for some $d \in D$ (cf. [Sc_2] p.11o).

We note the following properties of $\ll$:

$s \leq t$, $t \ll x$, $x \leq y$ imply $s \ll y$,

$x \ll y$ implies $x \leq y$,

$x \ll y$ and $y \ll z$ imply $x \ll z$

for $s,t,x,y,z \in P$.

o.2 A poset $(P,\leq)$ is said to be a continuous poset iff

i) P is "up-complete", i.e. for every up-directed subset D of $(P,\leq)$, supD exists;

ii) for every $x \in P$, $\{y \in P | y \ll x\}$ is non-empty and up-directed, and

$$\sup\{y \in P | y \ll x\} = x .$$

Note that, in a continuous poset P, the way below relation has the following interpolation property: If $x \ll y$ in P, then $x \ll z$ and $z \ll y$ for some $z \in P$ (cf. [Ma] 2.5).

A continuous lattice L is a continuous poset which is a complete lattice (or, equivalently, a o,v-semilattice).

The notion of a continuous lattice is due to D.S.Scott [Sc_1,Sc_2], that of a continuous poset is a natural generalization of it in the realm of up-complete posets. It was suggested first by G.Markowsky [Ma] (in the setting of chain-complete posets). Cf. also [H_4], [H_5], [H_7], [L_1], [L_2], [Wi_2].

o.3 For an arbitrary poset $(P,\leq)$, a subset M is said to be open in the Scott topology (= Scott-open) ([Sc_2] p.1o1), iff

i) M is an "upper set", i.e. $x \leq y$, $x \in M$, $y \in P$ imply $y \in M$;

ii) whenever $\sup D \in M$ for a non-empty, up-directed

subset D of M, then $D \cap M$ not $= \emptyset$.

The Scott topology on a poset P is designated by σ_P.

For up-complete posets P,Q, a map $f:(P,\sigma_P) \rightarrow (Q,\sigma_Q)$ is continuous iff f preserves suprema of non-empty up-directed subsets, i.e. $f(\sup D) = \sup(f[D])$ for every non-empty up-directed subset D of P (cf. [Wy] 3.5).

For a continuous poset P, the sets of the form

$$\Uparrow x := \{y \in P | x \ll y\}$$

with x ranging through P, form an open basis of the Scott topology σ_P. It results that, in a continuous poset P, $x \ll y$ iff

$$y \in U \subseteq \uparrow x := \{z \in P | x \leq z\}$$

for some Scott-open subset U of P (cf. [Ma] 3.2).

A subset U of a continuous poset P is Scott-open iff it is an upper set of P and for every $y \in U$ there is some $x \in U$ with

$$x \ll y .$$

o.4 Every topology τ on a set M induces a pre-order (=quasi-order), i.e. a transitive and reflexive relation, on this set

$$x \leq y \quad \text{iff} \quad x \in cl\{y\} \quad (x,y \in M),$$

the "specialization pre-order" ([AGV] IV,4.2.2); this pre-order is antisymmetric (i.e. a partial order) iff (M,τ) is T_o. The compatible topologies on a pre-ordered set are those which induce the given pre-order.

On every pre-ordered set $(P,\leq)$ there is a weakest compatible topology, the "weak topology" for which the sets

$$\downarrow x := \{y \in P | y \leq x\}$$

with x ranging through P, form a subbasis for the closed sets. (See [H_3] §2 for references). In [C] II-1.16, this topology is called the "upper topology" $\nu(P)$ of P. The weak topology on $(P,\leq)^{op}$ (i.e. the "opposite" $(P,\leq^*)$ of $(P,\leq)$ – where $x \leq^* y$ iff $y \leq x$) will be designated by ω_P (= the "lower topology" of $(P,\leq)$ in [C] III-1.1).

Note that the Scott topology on a poset is compatible.

The pre-order induced on a subspace is the induced pre-order, i.e. a topological embedding induces an order-embedding. (For pre-ordered sets P and Q, a map $e:P \to Q$ is an <u>order-embedding</u> (= <u>order-extension</u>) iff e is one-to-one and $x \leq y$ in P is equivalent to $e(x) \leq e(y)$ in Q.)

o.5 A subset F of a poset $(P,\leq)$ is said to be a <u>filter</u> <u>of</u>(!) P or <u>in</u> (!) P iff

i) F is an upper set;

ii) F is non-empty and down-directed, i.e. for every $x,y \in F$ there is some $z \in F$ with $z \leq x$ and $z \leq y$.

A filter F is said to be proper iff $F \neq P$, otherwise F is improper. A filter <u>F on</u> (!)a set M is a filter in the complete lattice $\underline{P}(M)$ of all subsets of M, ordered by the inclusion relation. An open filter of a topological space X is a filter in the lattice $\underline{O}(X)$ of open subsets of X.

For 1,$\wedge$-semilattices P, condition ii) can be replaced by: $1 \in F$ and $x \wedge y \in F$ whenever x and $y \in F$.

A subset J of a poset P is an <u>ideal</u> iff it is a filter in P^{op}.

o.6 The following observation, due to J.D.Lawson ([L_1], cf.[C]p.84), is crucial for the <u>duality</u> of continuous posets:

If $x \ll y$ in a continuous poset P, then, by the interpolation property,

$$x \ll \ldots \ll y_n \ll \ldots \ll y_1 \ll y_o = y$$

for some $y_1, y_2, \ldots \in P$. Thus

$$F = \bigcup \{\uparrow y_n \mid n \in \mathbb{N}\}$$

is a Scott-open filter of P with

$$y \in F \subseteq \uparrow x .$$

Thus every Scott-open subset of a continuous poset P is the union of Scott-open filters.

The <u>dual</u> D(P) of a continuous poset P is the set of all Scott-open filters of P, ordered by inclusion. The poset D(P) is a continuous poset: For $F,G \in D(P)$

$$F \ll G \text{ in } D(P) \quad \text{iff} \quad F \subseteq \uparrow x \subseteq G \text{ for some } x \in P.$$

The natural map

$$\varepsilon_P : P \longrightarrow DD(P), \quad x \mapsto \{G \in D(P) \mid x \in G\}$$

is an isomorphism. The inverse

$$\mu_P : DD(P) \longrightarrow P$$

of ε_P assigns to $\underline{F} \in DD(P)$ the supremum, in P, of the non-empty up-directed subset

$$\{x \in P \mid G \subseteq \uparrow x \text{ for some } G \in \underline{F}\}.$$

The dual of a continuous poset P is uniquely determined up to an isomorphism by the fact that it is a continuous poset whose lattice of Scott-open sets is anti-isomorphic to σ_P.

The duality theory for continuous posets has been developed in [L_2] and [H_7] with forerunners in [L_1] and [H_6].

o.7 A topological space X is called "<u>sober</u>" ([AGV]IV,4.2.1; [Br]II, condition (1) on p.17 [6]) iff every non-empty, irreducible, closed subspace A of X has a unique "generic" point x, i.e. a point x with $cl\{x\}=A$. (A subspace A is irreducible iff it is not the union of two proper closed subsets; "sober" is strictly between T_o and T_2, it does not imply, nor is it implied by T_1.)

The category <u>Sob</u> of sober spaces and continuous maps is a full reflective subcategory of the category <u>Top</u> of all topological spaces and continuous maps. For a space X, let ${}^sX$ be the space of all non-empty irreducible closed subsets of X with open sets

$${}^sO := \{C \in {}^sX \mid C \cap O \neq \emptyset\}$$

with O ranging through the lattice $\underline{O}(X)$ of all open subsets of X, then the mapping

$$\tilde{s}_X : X \to {}^sX, \; x \mapsto cl\{x\} \quad \text{(the closure of x in X)}$$

is the <u>Sob</u>-reflection morphism ([AGV], IV,4.2.1). This mapping $\tilde{s}_X$ is one-to-one iff it is an embedding iff X is a T_o-space; $\tilde{s}_X$ is bijective iff $\tilde{s}_X$ is a homeomorphism (onto) iff X is sober. Further, note that the lattice homomorphism

$$\underline{O}(\tilde{s}_X) : \underline{O}({}^sX) \to \underline{O}(X)$$

induced by $\tilde{s}_X$ is an isomorphism.

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6) Further historical information is given in [H_1] pp.365/366.

o.8 For T_o-spaces X and Y, a continuous map $f:X \to Y$ is called an <u>essential extension</u> in the category $\underline{T}_o$ of T_o-spaces and continuous maps iff

i) $f:X \to Y$ is an embedding (=extension), and

ii) whenever $gf:X \to Z$ is an embedding for some continuous map $g:Y \to Z$, then g is an embedding.

In [Ba] prop.2 (p.237), B.Banaschewski has shown that every T_o-space X has an <u>essential hull</u>, viz. a <u>unique greatest essential extension</u> $\lambda_X:X \hookrightarrow \lambda X$, i.e. whenever $f:X \hookrightarrow Y$ is an essential extension, then $hf=\lambda_X$ for some embedding $h:Y \hookrightarrow X$. Banaschewski's space λX is a subspace of the filter space $\Phi(X)$, the algebraic lattice of all (proper or improper) open filters of X (ordered by the inclusion relation) endowed with the Scott topology.

A T_o-space X is said to be <u>essentially complete</u> iff $\lambda_X:X \hookrightarrow \lambda X$ is a homeomorphism, i.e. iff X does not admit any non-trivial essential extension. Every essentially complete T_o-space is sober ([H_3] o.1). For further information see [H_3], in particular sections 1 and 2.

This theme will be pursued further in section 4 below.

o.9 One of the major insights at the root of the theory of continuous lattices is a result of D.S.Scott's [Sc_2] 2.12: The continuous lattices endowed with their Scott topology are precisely the <u>injective</u> T_o<u>-spaces</u>, i.e. the injective objects X in the category $\underline{T}_o$ of T_o-spaces and continuous maps with regard to the class of all (topological) embeddings, i.e. whenever $e:Y \hookrightarrow Z$ is an embedding for T_o-spaces Y and Z and $f:Y \to Z$ is a continuous map, then there is a continuous map $g:Z \to X$ rendering

$$\begin{array}{ccc} Z & \overset{g}{\dashrightarrow} & X \\ e \uparrow & \nearrow_{f} & \\ Y & & \end{array}$$

commutative.

Every injective T_o-space is essentially complete, hence - a fortiori - sober.

o.1o In [H_4] 3.14 it is shown that the continuous posets in their Scott topology are precisely those sober spaces X which have an <u>injective hull</u> in $\underline{T}_o$, i.e. whose greatest essential extension space λX is an injective T_o-space, i.e. - by [Sc_2] 2.12 - a continuous lattice endowed with its Scott topology. Thus, for a continuous poset P, the essential hull is of the form

$$(P,\sigma_P) \hookrightarrow (L,\sigma_L)$$

where the continuous lattice L is - up to an isomorphism - uniquely determined (via the specialization order of the space). Therefore it is a natural abuse of language to call the order-extension

$$P \hookrightarrow L,$$

thus obtained, the injective hull of the continuous poset P, viz. the injective hull in $\underline{T}_o$ induced by the Scott topology.

The continuous posets endowed with their Scott topology are also known as the <u>projective sober spaces</u>, [H_4](2.19).

o.11 The <u>Lawson topology</u> (or λ-topology or <u>CL</u>-topology) of a (continuous) poset $(P,\leq)$ is the weakest topology on P finer than both the Scott topology of $(P,\leq)$ and the weak topology of $(P,\leq)^{op}$ (cf. [C] III-1.5). It is designated by λ_P.

The Lawson topology of a continuous lattice is compact Hausdorff ([C] III-1.1o).

§ 1 Meet-prime, pseudo-meet-prime, and quasi-meet-prime elements. Locally quasi-compact strongly sober spaces.

This is a survey of known results.

After having reviewed the (classical) theory of meet-prime elements, we discuss the notion of a pseudo-meet-prime element and that of a quasi-meet-prime element. The latter notion is - in a continuous lattice - essentially an equivalent description of what has been called a "weakly prime element" by K.H.Hofmann and J.D.Lawson ([HL_1], 1.7, p.313). The notion of a pseudo-meet-prime element is essentially due to K.Keimel and M.W.Mislove [KM] and K.H.Hofmann and J.D.Lawson [HL_2]. "Essentially" indicates the following difference: Whereas these authors assume that the unit element 1 of a complete lattice L is always "prime", we insist here that 1 is never meet-prime. It is, in a sense (which will become precise later), a consequence of this modification that 1 can be, but need not be pseudo-meet-prime or quasi-meet-prime. Thus the results of these authors need a slight adaptation to the present definitions.

The results of K.H.Hofmann and J.D.Lawson ([HL_1], [HL_2]) and K.Keimel and M.W.Mislove [KM] appear in [C], in particular in [C] I-3.23 to I-3.27 and V-3. In [H_8] §3 some of the modifications, we need here, have been derived from the results in [C]. Here we give direct proofs. The ideas involved are not new, but we have reduced the number of "auxiliary concepts" introduced in [C], and we have tried to single out the precise hypotheses actually needed for the lemmata into which the proofs are decomposed.

We conclude with a theorem, largely due to K.H.Hofmann and M.W.Mislove [HM], on strongly sober locally quasi-compact spaces.

Let L be a complete lattice.

1.1 An element $p \in L$ is said to be meet-prime iff for every finite subset F of L $\inf F \leq p$ implies $x \leq p$ for some $x \in F$.

Note that $1 = \inf \emptyset$ is not meet-prime (in contrast to the definition of a "prime" element given in [C] I-3.11).

A theorem due to J.R.Büchi [Bü] and S.Papert [Pa] (and other authors) says that a complete lattice L is isomorphic to the lattice $\underline{O}(X)$ of open subsets of a topological space X iff every element x of L is the infimum of a family of meet-prime elements.

The set of meet-prime elements of a complete lattice L will be endowed with the trace of the weak topology of L^{op}: The resulting space will be designated by

$$\mathrm{Spec}^{*}L.$$

This notation is a compromise between the notation used in [C], [HL_2] and that of [H_3], [H_8].[7] The specialization partial order of $\mathrm{Spec}^{*}L$ is inverse to the order induced by L.

A subset M of $\mathrm{Spec}^{*}L$ is closed iff

$$M = \mathrm{Spec}^{*}L \cap \uparrow x$$

for some $x \in L$.

For every complete lattice L, $\mathrm{Spec}^{*}L$ is a sober space ([H_3] 3.5, [C] V-4.4), and we have

$$\underline{O}(\mathrm{Spec}^{*}L) \cong L$$

iff every element of L is a meet of meet-prime elements.

Note that, for a space X, there is a homeomorphism

$$^{s}X \to \mathrm{Spec}^{*}\underline{O}(X)$$

since the elements of ^{s}X are the join-prime elements of the lattice $\underline{A}(X)$ of all closed subsets of X ordered by inclusion which correspond - via the anti-isomorphism $\underline{A}(X) \to \underline{O}(X)$, $A \mapsto X-A$ - to the meet-prime elements of $\underline{O}(X)$.

---

7) In [C] and [HL_2], "SpecL" is used to designate $\mathrm{Spec}^{*}L$, whereas in [H_3], [H_8] "SpecL" (or "v-SpecL") designates the set of join-prime elements of L, endowed with the trace of the weak topology of L.

1.2 An ideal J in a 0,v-semi-lattice is said to be a "prime" ideal iff J is a meet-prime element in the complete lattice IdL (ordered by inclusion) consisting of all ideals of L.

An ideal J in a lattice L with 0 and 1 is a prime ideal iff

(i) $x \wedge y \in J$ always implies $x \in J$ or $y \in J$,

(ii) 1 not$\in J$, i.e. J not $\neq$ L .

A prime filter in a lattice L with 0 and 1 is a meet-prime element of $\mathrm{FiltL}:=\mathrm{Id}(L^{op})$. Note that, by the preceding criterion (and its dual), $M \subseteq L$ is a prime filter iff L-M is a prime ideal (cf. [C] I-3.16).

Now, an element x of a lattice L with 0 and 1 is meet-prime iff $\downarrow x$ is a prime ideal.

1.3(i) An element a of a complete lattice is said to be pseudo-meet-prime iff there exists a prime ideal J such that

$$a = \sup J \ .$$

(ii) An element $b \in L$ is quasi-meet-prime iff, whenever $\inf F \ll b$ for a finite subset F of L, then there is some $x \in F$ with $x \leq b$.

(Note that $F=\emptyset$ is not excluded).

The set of pseudo-meet-prime elements of L and the set of quasi-meet-prime elements of L will be designated by $\psi^{*}L$ and $\kappa^{*}L$, respectively.

Later (in section 4), we will endow $\psi^{*}L$ with a topology: The resulting space will be also designated by $\psi^{*}L$. The notation ψL will be reserved to designate both the set and the space $\psi^{*}(L^{op})$ of all pseudo-join-prime elements of L. Note that in this section $\psi^{*}L$ is always the set of pseudo-meet-prime elements of L without any topology.

1.4 LEMMA:

a) Every meet-prime element is pseudo-meet-prime.

b) Every pseudo-meet-prime element is quasi-meet-prime.

Proof:

a) $x \in L$ is meet-prime iff $\downarrow x$ is a prime ideal. Clearly $x=\sup\downarrow x$. (Cf. [C]p.75).

b) Cf. [C] I-3.24 "(1) implies (2)". (Referring to the continuity of L is clearly unnecessary).

We now have the following inclusions

$$\mathrm{Spec}^{*}L \subseteq \psi^{*}L \subseteq \kappa^{*}L \ .$$

of sets.

1.5 REMARKS:

Let L be a complete lattice.

(1) An element p is quasi-meet-prime in L iff the filter generated by L-$\downarrow$p does not meet $\Downarrow$p.

(2) In the unit square $[0,1]\times[0,1]$ (a distributive continuous lattice) the element $(1,\frac{1}{2})$ is meet-prime, hence quasi-meet-prime and $(1,0)\wedge(0,1)\ll(1,\frac{1}{2})$, but neither $(1,0)\ll(1,\frac{1}{2})$ nor $(0,1)\ll(1,\frac{1}{2})$, since (1,1) is the supremum of the ideal $[0,1)\times[0,1)$. Thus the definition of a quasi-meet-prime element b cannot be strengthened to $\inf F \ll b$ with $F \subseteq L$ finite implies $x \ll b$ for some $x \in F$. - Cf. [C] V-(remark after) 3.4 (p.248).

(3) If the greatest element 1 is compact in L, i.e. $1 \ll 1$, then 1 fails to be quasi-meet-prime.

(4) If, in addition, L is distributive, and if 1 is not pseudo-meet-prime, then 1 is compact.

Proof:

(1) The filter generated by L-$\downarrow$p is

$$\{y \in L \mid \inf F \leq y \quad \text{for some finite} \quad F \subseteq L-\downarrow p\}.$$

Now our assertion is immediate from (the contraposition) of the definition of "quasi-meet-prime".

(3) If 1 is compact, then $\inf\emptyset \ll 1$, hence 1 fails to be quasi-meet-prime.

(4) If 1 not $\ll 1$, then there exists an ideal J of L such that supJ=1, but 1 not $\in$ J. By a standard argument for distributive lattices (cf. [C] I-3.19), there is a prime ideal P with $J \subseteq P$, but 1 not $\in$ P. Clearly, $1 = \sup J \leq \sup P \leq 1$, hence 1=supP is pseudo-meet-prime.

Note that 1.5(1) corrects a slight inaccuracy of [C] I-3.24 (where it is overlooked that a filter F - by definition [C] O-1.3 - contains $\inf\emptyset = 1$).

Parts (2) and (3) of the following lemma 1.6 are modifications of [C] I-3.27 "(2) iff (3)" and [C] I-3.24 "(1) iff (2)", respectively. Part (1) extends the additional remark in [C] I-3.27 set free from the hypothesis of continuity for L.

1.6 LEMMA:

Let L be a complete lattice;

(1) Suppose, in addition, that L is distributive. If $\mathrm{Spec}^*L = \psi^*L$, then 1 is compact in L, i.e. $1 \ll 1$, and the way below relation $\ll$ in L is multiplicative, i.e. $x \ll y$ and $x \ll z$ imply $x \ll y \wedge z$ for all $x,y,z \in L$.

(2) Suppose L is a continuous lattice, $1 \in L$ is compact, and $\ll$ is multiplicative. Then $\mathrm{Spec}^*L = \varkappa^*L$, i.e. every quasi-meet-prime element p of L is meet-prime.

(3) If L is a distributive continuous lattice, then $\psi^*L = \varkappa^*L$, i.e. every quasi-meet-prime element of L is pseudo-meet-prime.

Proof:

(1) a) Assume $\ll$ is not multiplicative, hence there are elements $a,x,y \in L$ with $a \ll x$ and $a \ll y$, but a not $\ll x \wedge y$. Thus there is an ideal J with $x \wedge y \leq \sup J$, but a not $\in$ J. By a standard argument for distributive lattices (cf. [C] I-3.19), there is a prime ideal P with $J \subseteq P$, but a not $\in$ P. Now let p=supP. Clearly, $x \wedge y \leq \sup J \leq \sup P = p$. However, $x \leq p$ would give $a \ll x \leq \sup P$, hence $a \in P$ - contradicting the choice of P. Thus x not $\leq$ p and, analogously, y not $\leq$ p. As a consequence, the pseudo-meet-prime element p fails to be meet-prime. A contradiction.

b) If 1 not $\ll 1$, then 1 is pseudo-meet-prime by 1.5(4), but not meet-prime. A contradiction.

(2) Suppose $x \wedge y \leq p$ with $x,y \in L$. Then, by continuity of L, $x \leq p$ or $y \leq p$ - as in [C] I-3.27 "(2) implies (3)". Since 1 is compact, it fails to be quasi-meet-prime (by 1.5(3)), hence $1 \neq p$. Thus p is meet-prime.

(3) See [C] I-3.24 "(3) implies (1)".

Recall that λ denotes the Lawson- or λ-topology of a complete lattice L (cf. o.11 above).

1.7 LEMMA:

Let L be a complete lattices.

(1) If every element of L is a meet (=infimum) of meet-prime elements, then ψ^*L is contained in the λ-closure of Spec^*L in L.

(2) Suppose that the way below relation $\ll$ has the interpolation property, i.e. $x \ll y$ implies $x \ll z$ and $z \ll y$ for some $z \in L$, then $\varkappa^*L$ is λ-closed in L.

Proof:

(1) Suppose $a \in L$ is pseudo-meet-prime, i.e. a=supP for a prime ideal P of L and $a \in U - (\uparrow x_1 \cup \ldots \cup \uparrow x_n)$ for some Scott-open subset U of L and $x_1, \ldots, x_n \in L$ $(n \in \mathbb{N} \cup \{o\})$; note that these sets are the standard basic Lawson-open neighborhoods of a in L. Since U is Scott-open and P is an ideal with $\sup P \in U$, there is some $q \in P$ with $q \in U$. Assume now that $\inf\{x_1, \ldots, x_n\} \leq q$, then $x_i \in P$ for some $i \in \{1, \ldots, n\}$, since P is prime. Thus $x_i \leq \sup P = a$ - contradicting our hypothesis

that $a \in U-(\uparrow x_1 \cup \ldots \cup \uparrow x_n)$. Thus we have $x:=\inf\{x_1,\ldots,x_n\}$ not $\leq q$.

By hypothesis, $q=\inf\{p_k \mid k \in K\}$ for meet-prime elements p_k of L. There is some $k_o \in K$ such that x not $\leq p_{k_o}$ (otherwise $x \leq \inf\{p_k \mid k \in K\}=q$, contradicting the above). It results that p_{k_o} not $\in \uparrow x_1 \cup \ldots \cup \uparrow x_n$. Since $q \in U$ and $q \leq p_k$, we have $p_k \in U$. Thus, as required,

$$p_{k_o} \in U-(\uparrow x_1 \cup \ldots \cup \uparrow x_n).$$

(2) Let $p \in L$ denote a limit of a net $(p_j)_{j \in J}$ of quasi-meet-prime elements of L in the λ-topology, and let

$$x:=\inf\{x_1,\ldots,x_n\} \ll p$$

for some $x_1,\ldots,x_n \in L$. We have to show that $x_i \leq p$ for some $i \in \{1,\ldots,n\}$. Suppose, on the contrary, that x_i not $\leq p$ for every $i \in \{1,\ldots,n\}$. Then $U:=\Uparrow x-(\uparrow x_1 \cup \ldots \cup \uparrow x_n)$ contains p. Since $\ll$ interpolates, $\Uparrow x$ is Scott-open, hence U is Lawson-open. Thus there is some $j \in J$ with $p_j \in U$. However, $\inf\{x_1,\ldots,x_n\} \ll p_j$ implies $x_i \leq p_j$ for some $i \in \{1,\ldots,n\}$, i.e. $p_j \in \uparrow x_i$ - contradicting $p_j \in U$. Thus $x_i \leq p$ for some $i \in \{1,\ldots,n\}$. This shows that p is quasi-meet-prime.

1.8 <u>REMARK</u>:
It is immediate from 1.6(2) and 1.7(2) that in a continuous lattice L, not necessarily distributive, with 1 compact in L and $\ll$ multiplicative, $Spec^*L$ is closed with regard to the λ-topology of L. (Cf. [HL_2] 6.8).

1.9 <u>PROPOSITION</u>:
For a distributive continuous lattice L,

$$\psi^*L = \kappa^*L$$

is the closure of $Spec^*L$ with regard to the Lawson topology λ on L.

<u>Proof</u>:
By 1.6(3), $\psi^*L=\kappa^*L$. Since, by [C] I-3.7, every element of a distributive continuous lattice L is the meet of meet-prime elements, we have

$$Spec^*L \subseteq \kappa^*L = \psi^*L \subseteq cl_\lambda(Spec^*L)$$

by 1.7(1). Since in a continuous lattice L the way below relation interpolates ([C] I-1.18), κ^*L is λ-closed by 1.7(2). Thus $\kappa^*L=cl_\lambda(Spec^*L)$.

1.1o <u>COROLLARY</u>:
For a distributive continuous lattice L, $Spec^*L$ is λ-closed iff 1 is compact in L and the way below relation $\ll$ is multiplicative.

<u>Proof</u>:
The first implication is established in 1.8. Suppose $Spec^*L$ is λ-closed, then - by 1.9 - $Spec^*L=\psi^*L$. Now 1.6(1) applies.

The preceding results modify (and sharpen) analogous results in [C] V-3.

1.11 By a celebrated theorem of K.H.Hofmann and J.D.Lawson [HL_2], every distributive continuous lattice is isomorphic to the lattice $\underline{O}(X)$ of open subsets, ordered by inclusion, of a topological space X, and X can be chosen as a locally quasicompact sober space, namely $X=Spec^*L$. Furthermore, if X is a locally quasicompact space, then $\underline{O}(X)$ is a distributive continuous lattice. Thus every result on distributive continuous lattices may be viewed as a result on locally quasicompact (sober) spaces, and - for sober spaces - conversely.

For a space X, let

$$\psi^*X:=\psi^*\underline{O}(X)$$

and

$$\psi X:=\psi\underline{A}(X).$$

Note that there is a canonical mapping

$$\psi_X: X \to \psi X, \quad x \mapsto cl\{x\}$$

the composite of $\tilde{s}_X: X \to {}^sX$ (1.1) and the inclusion ${}^sX \to \psi X$. This mapping is one-to-one iff X is a T_o-space.

Note that, for a locally quasicompact (sober) space X, ψX consists - by 1.9 - of the same points as the <u>Fell compactification</u> $\underline{H}(X)$ of X. (As noted in [H_8] 3.13, this observation extends to non-sober locally quasicompact spaces).

For a filter $\underline{F}$ on a space X, let

$$\text{conv}\underline{F} = \{x \in X \mid \underline{O}(x) \subseteq \underline{F}\}$$

denote the "convergence set" of $\underline{F}$ – where $\underline{O}(x)$ denotes the open neighborhood filter of x in X. Note that the improper filter $\underline{P}(X) := \{M \mid M \subseteq X\}$ is here not excluded.

1.12 DEFINITION:

A space X is said to be strongly sober iff for every ultrafilter $\underline{U}$ on X, conv$\underline{U}$ has a unique generic point, i.e.

$$\text{conv}\underline{U} = \text{cl}\{x\}$$

for a unique element x of X.

The notion of "strong sobriety" is a slight modification of the notion of super-sobriety ([C] VII-1.1o, p.31o; strongly sober = super-sober + quasicompact).

Every essentially complete T_o-space is strongly sober, since – by $[H_3]$3.11 – a space X is an essentially complete T_o-space iff, for every filter $\underline{F}$ on X, conv$\underline{F}$ has a unique generic point. Every strongly sober space is sober ([C] VII-1.11), since – by $[H_1]$ 1.9 – a space X is sober iff conv$\underline{U}$ has a unique genericpoint for all those ultrafilters $\underline{U}$ which enjoy the property that conv$\underline{U} \in \underline{U}$ ("irreducible" ultrafilters, $[H_1]$ 1.4). Further, note that the strongly sober T_1-spaces are precisely the compact Hausdorff spaces.

The following result is (up to slight modifications of the statement) due to K.H.Hofmann and M.W.Mislove ([HM] 4.8).

1.13 THEOREM:

Let X be a locally quasicompact sober space, then the following conditions are equivalent:

(1) The way below relation of the (continuous) lattice $\underline{O}(X)$ is multiplicative, and the unit element of $\underline{O}(X)$ is compact (i.e, X is a quasicompact space).

(2) $\text{Spec}^{*}\underline{O}(X)$ is λ-closed in $\underline{O}(X)$.

(3) X is strongly sober.

(4) X is quasi-compact and the intersection of two quasi-compact saturated subsets is quasi-compact.

(A subset of a space is saturated iff it is the intersection of its open neighborhoods – cf. [C] V-5.2, p.258).

The equivalence of (1) and (2) is established in 1.1o. In view of 1.9 we may add the following equivalent condition:

(5) The canonical mapping $\psi_X : X \to \psi X$ is bijective.

It has been observed in $[H_8]$ 3.12 that this is equivalent to:

(6) For every open prime filter $\underline{F}$ on X, conv$\underline{F}$ has a unique generic point.

1.14 It may be worth pointing out that strong sobriety per se does not imply local quasicompactness: The essential hull X^{*} of a Hausdorff space X (i.e. $X \cup \{0,1\}$, where 0 and 1 are adjoined as a smallest and a largest point, respectively – cf. $[H_3]$ §5) is, of course, essentially complete, hence strongly sober, but X is an intersection of an open and a closed set of X^{*}, hence X^{*} cannot be locally quasicompact, unless X is locally compact.

1.15 REMARK:

In [Si], H.Simmons has shown that the strongly sober locally quasicompact spaces form the Eilenberg-Moore algebras for a "triple" (or "monad") $\Psi = \langle P, \eta, \mu \rangle$ on the category $\underline{T}_o$ of T_o-spaces and continuous maps. The functor part P of this monad assigns to a T_o-space X its space of open prime filters (i.e. meet-prime elements in the complete lattice, ordered by inclusion, of all filters of the lattice $\underline{O}(X)$ of open sets of X) – with the topology inherited from the space of all open filters of X (cf. [Ba_1] §1). This space P(X) is - via a result of J.Schröder [Sch_1] - homeomorphic to the extension space of X induced by the open finite decomposition spectrum ("Zerlegungsspektrum") of X introduced by J.Flachsmeyer [Fl_1] p.264). The Ψ-homomorphisms are those continuous maps $f:X \to Y$ which enjoy the property that the induced map $\underline{O}(f):\underline{O}(Y) \to \underline{O}(X)$ (assigning to an open subset of Y its inverse image) preserves the way below relation.

It may be noted that - as a consequence of this result - the category of locally quasicompact strongly sober spaces and those continuous maps whose inverse image map preserves the way below relation is complete (i.e. has (projective) limits for all diagrams indexed over categories which are "small" with regard to the given universe) and that these limits can be constructed in the category $\underline{T}_o$. (An Eilenberg-Moore category over a complete category is complete and limits are "constructed" in the underlying category - cf. [ML] VI-2, exercise 2; [Sch_2] 21.3.9.) A different argument results from the isomorphism explained in section 6 below.

1.16 REMARK:
Suppose X and Y are locally quasicompact sober spaces. In [C] V-5.14, 5.15 it is shown that for a continuous map $f:X \to Y$ the following conditions are equivalent:

(i) $\underline{O}(f):\underline{O}(Y) \to \underline{O}(X)$ preserves the way below relation.

(ii) The inverse image of every saturated quasicompact subset of Y is quasicompact (and saturated) in X.

For locally compact Hausdorff spaces X and Y the mappings enjoying property (ii) are known as the perfect maps (note that in a T_1-space every subset is saturated). This name may be extended to the setting of locally quasicompact sober spaces.

1.17 REMARK:
It has been observed in [H_8] 3.8 that the very definition of $\psi X := \psi(\underline{A}(X))$ implies that the points of ψX are the convergence sets of the open prime filters of the space X. For a locally quasicompact space X, the points of ψX also can be characterized as the convergence sets of the primitive nets; this results - via the observation in 1.11 above - from [Fe_2] p.475. (A net is said to be primitive iff every adherence point of the net is also a limit point. Via the topological equivalence between nets and filters - observed in [BrS] - primitive nets can be replaced by primitive filters). I do not know whether every member of ψX is a convergence set of an ultrafilter (under the proviso that X is a locally quasicompact sober space).

1.18 REMARK:
It has been observed in [H_8] §3 (in the notes added) that the greatest essential extension of a strongly sober (not necessarily locally quasicompact) space coincides - on the level of the specialization orders - with the MacNeille completion. Thus a space X is essentially complete if (and only if) X is strongly sober and a complete lattice in its specialization order.

Also note that every spectral space (i.e. every prime spectrum of a commutative, associative ring with 1, cf. [Ho]) is strongly sober and locally quasicompact. Indeed the strongly sober locally quasicompact spaces are precisely the retracts, in $\underline{T}_o$, of the spectral spaces ([J] §2, [Si]; cf. also [Ba_2] prop.2).

§ 2 The injective hull of D(L) for a continuous 1,∧-semilattice L

In this section, L denotes a continuous 1,∧-semilattice L. However, some of the results are based on the assumption that L is a <u>distributive continuous lattice</u>.

2.o For a continuous 1,∧-semilattice L, let

$\mathrm{Filt}_\sigma L = \{$all filters in L which are generated by a Scott-open subset of L$\}$

Recall that, in a 1,∧-semilattice L, the filter generated by a subset M of L (i.e. the smallest filter of L containing M) is the set

$$\varphi(M) = \{x \in L \mid \inf F \leq x \text{ for some finite set } F \subseteq M\}$$

If L is a distributive lattice and M is an upper set, then

(*) $\varphi(M) = \{x \in L \mid \inf F = x$ for some finite set $F \subseteq M\}$

2.1 For a continuous 1,∧-semilattice L, we have

(i) $$D(L) \subseteq \mathrm{Filt}_\sigma L$$

where D(L) denotes the dual of L (consisting of all Scott-open filters of L) - cf.o.6.

(ii) $\mathrm{Filt}_\sigma L$ is a complete lattice, since

$$\varphi(\bigcup_i M_i) = \varphi(\bigcup_i \varphi(M_i))$$

is the supremum of $(\varphi(M_i))_{i \in I}$ in $\mathrm{Filt}_\sigma L$ for every family $(M_i)_{i \in I}$ of Scott-open sets M_i of L.

(iii) Since every Scott-open set is the union of Scott-open filters (cf. o.6), D(L) is join-dense in $\mathrm{Filt}_\sigma L$, i.e. every member of $\mathrm{Filt}_\sigma L$ is a supremum of a family of members of D(L) in $\mathrm{Filt}_\sigma L$.

2.2 <u>LEMMA</u>:

For a continuous 1,∧-semilattice L, and $F, G \in \mathrm{Filt}_\sigma L$, the following are equivalent:

(i) $F \ll G$ in $\mathrm{Filt}_\sigma L$

(ii) $F \subseteq \uparrow x \subseteq G$ for some $x \in L$.

<u>Proof</u>:

It is readily clear that, for $F_i \in \mathrm{Filt}_\sigma L$,

$$\bigcup_i F_i$$

is the supremum of $\{F_i \mid i \in I\}$ in $\mathrm{Filt}_\sigma L$ if this family is non-empty and up-directed.

(a) Thus "(ii) implies (i)" is evident.

(b) In order to prove that (i) implies (ii), let $F = \varphi(K)$, $G = \varphi(M)$ for $K, M \in \sigma_L$ (=the Scott topology of L). For every $x \in M$ choose some $x' \in M$ with

$$x' \ll x$$

in L (by o.3), and some Scott-open filter F_x in L such that

$$x \in F_x \subseteq \uparrow x'$$

(by J.D.Lawson's argument, cf. o.6).

Then

$$M = \bigcup\{F_x \mid x \in M\},$$

hence

$$\varphi(M) = \sup\{F_x \mid x \in M\},$$

where "sup" denotes the supremum in $\mathrm{Filt}_\sigma L$.

Since

$$F_x \subseteq \uparrow x' \subseteq M \subseteq \varphi(M),$$

we have

$$F_x \ll \varphi(M)$$

by part (a), hence, by hypothesis (i), there are $x_1, \ldots, x_n \in M$ $(n \geq o)$ with

$$\varphi(K) \subseteq \sup\{F_{x_1}, \ldots, F_{x_n}\}.$$

Clearly

$$\sup\{F_{x_1}, \ldots, F_{x_n}\} = \varphi(F_{x_1} \cup \ldots \cup F_{x_n}) \subseteq \varphi(\uparrow x_1' \cup \ldots \cup \uparrow x_n') \subseteq \uparrow \inf\{x_1', \ldots, x_n'\}.$$

Since $x_1', \ldots, x_n' \in M$, we infer that

$$y := \inf\{x_1', \ldots, x_n'\} \in \varphi(M).$$

Thus

$$\varphi(K) \subseteq \uparrow y \subseteq \varphi(M),$$

as claimed.

2.3 PROPOSITION:
For a continuous $1,\wedge$-semilattice L, $\mathrm{Filt}_\sigma L$ is a continuous lattice.

Proof:
For $M \in \sigma_L$ we have

$$\varphi(M) = \sup\{F_x \mid x \in M\}$$

where (as in the proof of 2.2) $x \in F_x \subseteq \uparrow x'$ for some $F_x \in D(L)$ and some $x' \in M$, hence, by 2.2,

$$F_x \ll \varphi(M)$$

in $\mathrm{Filt}_\sigma L$ for all $x \in M$.

2.4 The dual $D(L)$ of a continuous $1,\wedge$-semilattice L is a continuous $1,\wedge$-semilattice.

In order to show that

$$D(L) \hookrightarrow \mathrm{Filt}_\sigma L$$

is the (an) injective hull in $\underline{T}_o$ (with regard to the respective Scott topologies), the following characterization of the injective hull of a continuous $1,\wedge$-semilattice will be used:

Suppose S is a continuous $1,\wedge$-semilattice. A map $f:S \to K$ into a continuous lattice K is an injective hull iff the following conditions are satisfied

(1) $f:(S,\lambda_S) \to (K,\lambda_K)$ is a (topological) embedding with regard to the Lawson topologies λ_S and λ_K of S and K, respectively;

(2) $f[S]$ is dense in (K,λ_K);

(3) $f\ S \to K$ is an order-embedding;

(4) $f[S]$ is join-dense in K, i.e. every member of K is a supremum, in K, of elements of $f[S]$.

This has been established in $[H_9]$ 3.8. It is also shown there ($[H_9]$ 3.9, 3.1o, 3.11) that, in the presence of (3) and (4), condition (1) may be replaced by the conjunction of (a) and (b):

(a) $f:S \to K$ preserves suprema of non-empty up-directed lower sets,

(b) $f:S \to K$ preserves the way below relation.

2.5 THEOREM:
For a continuous $1,\wedge$-semilattice L,

$$D(L) \hookrightarrow \mathrm{Filt}_\sigma L$$

is an injective hull.

Proof:
We have already seen that $\mathrm{Filt}_\sigma L$ is a continuous lattice (2.3). Condition (3) in the preceding remarks is evident. For condition (4) see 2.1(iii). Now (a) and (b) are immediate consequences of the explicit description of suprema of non-empty up-directed subsets (=set-theoretic unions) and of the way below relation in $D(L)$ and $\mathrm{Filt}_\sigma L$, respectively (cf.o.6 and 2.2).

Thus it remains to show that $D(L)$ is (topologically) dense in $\mathrm{Filt}_\sigma L$ with regard to the Lawson topology:

Suppose

$$V = \Uparrow F - (\uparrow G_1 \cup \ldots \cup \uparrow G_n)$$

(where, for once, $\Uparrow$ and $\uparrow$ have to be interpreted in $\mathrm{Filt}_\sigma L$) is non-empty for $F, G_1, \ldots, G_n \in \mathrm{Filt}_\sigma L$ $(n \geq o)$, i.e.

$$F \subseteq \uparrow x \subseteq H$$

and

$$G_i \text{ not} \subseteq H \qquad (i=1,\ldots,n)$$

for some $H \in \mathrm{Filt}_\sigma L$ and some $x \in L$.

For every $i=1,\ldots,n$, there is some $x_i \in L$ with $x_i \ll x$ in L and

$$G_i \text{ not} \subseteq \uparrow x_i$$

(otherwise $G_i \subseteq \bigcap\{\uparrow y \mid y \in L,\ y \ll x\} = \uparrow x \subseteq H$).

Thus, for $z = \sup\{x_1, \ldots, x_n\}$, we have

$$z \ll x,$$

and

$$G_i \text{ not} \subseteq \uparrow z .$$

There is a Scott-open filter M of L such that

$$x \in M \subseteq \Uparrow z .$$

Clearly, $F \subseteq \uparrow x \subseteq M$ and G_i not $\subseteq M$ $(i=1,\ldots,n)$, hence

$$M \in V,$$

as claimed.

Since these sets V form an open basis of the Lawson topology of $Filt_\sigma L$, this proves that $D(L)$ is dense in $Filt_\sigma L$ with regard to the Lawson topology.

2.6 For a $1,\wedge$-semilattice L, let $FiltL$ denote the complete (algebraic) lattice of all filters of L. Note that the meet (=infimum) in $FiltL$ is the set-theoretic intersection.

For a continuous $1,\wedge$-semilattice L, we have:

(a) $Filt_\sigma L$ is stable in $FiltL$ under arbitrary joins.

(b) The order-embedding $Filt_\sigma L \hookrightarrow FiltL$ preserves and reflects the way below relation, i.e., for $F,G \in Filt_\sigma L$, $F \ll G$ in $Filt_\sigma L$ iff $F \ll G$ in $FiltL$.

<u>Proof</u>:

(a) is clear from 2.1(ii), since the formula given there describes the suprema in $FiltL$ - when M_i is interpreted as an arbitrary subset of L.

(b) In the algebraic lattice $FiltL$ we have

$$G \ll F$$

(for $G,F \in FiltL$) iff

$$G \subseteq \uparrow x \subseteq F$$

for some $x \in L$ (since the principal filters $\uparrow x$ are the compact elements of $FiltL$). Now 2.2 applies.

2.7 <u>LEMMA</u>:

For a distributive (!) continuous lattice L, $Filt_\sigma L$ is stable in $FiltL$ under finite meets.

<u>Proof</u>:

Clearly, L is the greatest element of both $Filt_\sigma L$ and $FiltL$.

Suppose K,M are Scott-open subsets of L.

We prove that

$$\varphi(K) \cap \varphi(M) \subseteq \varphi(K \cap M).$$

(The other inclusion is evident).

If $x \in \varphi(K) \cap \varphi(M)$, then - by 2.o(*) - there are $k_1,\dots,k_l \in K$, and $m_1,\dots,m_n \in M$ $(l,n \in \mathbb{N} \cup \{o\})$ with

$$x = \inf\{k_1,\dots,k_l\} = \inf\{m_1,\dots,m_n\}.$$

It results that

$$k_i \vee m_j \in K \cap M$$

and, by <u>distributivity</u> of L,

$$x = \inf\{k_i \vee m_j \mid i \in \{1,\dots,l\} \text{ and } j \in \{1,\dots,n\}\}$$

This implies that

$$x \in \varphi(K \cap M) ,$$

as claimed.

2.8 The dual $D(P)$ of a continuous poset P is a continuous lattice if and only if P is a $1,\wedge$-semilattice, the way below relation $\ll$ of P is multiplicative, i.e. $x \ll y$ and $x \ll z$ for $x,y,z \in P$ imply $x \wedge y \ll z$, and the greatest element 1 of P is compact, i.e. $1 \ll 1$ ([L_1], [L_2] 9.6, [H_7] 3.13).

2.9 <u>LEMMA</u>:

For a continuous lattice L, we have:

(a) The greatest element L of $Filt_\sigma L$ is compact.

(b) If, in addition, L is distributive, then the way below relation of $Filt_\sigma L$ is multiplicative.

<u>Proof</u>:

(a) Evidently, $L = \uparrow o$, hence $L \ll L$, by 2.2.

(b) If $G \ll F_1$ and $G \ll F_2$ in $Filt_\sigma L$, then

$$G \subseteq \uparrow x_1 \subseteq F_1 \text{ and } G \subseteq \uparrow x_2 \subseteq F_2,$$

hence

$$G \subseteq \uparrow(x_1 \vee x_2) \subseteq F_1 \cap F_2 ,$$

hence

$$G \ll F_1 \cap F_2 .$$

(Recall that, by 2.7, $F_1 \cap F_2$ is the meet of F_1 and F_2 in $Filt_\sigma L$).

It is well known that a lattice L is distributive iff the lattice FiltL of all filters of L is distributive. The following observation is now immediate from 2.3, 2.6(a) and 2.7.

2.1o LEMMA:

For a distributive continuous lattice L, $\mathrm{Filt}_\sigma L$ is a distributive continuous lattice.

2.11 LEMMA:

Let K be a distributive continuous lattice such that $1 \in K$ is compact and the way below relation is multiplicative. Then D(K) is a distributive continuous lattice.

Proof:

For $F,G,H \in D(K)$ we clearly have

$$(F \wedge G) \vee (F \wedge H) \subseteq F \wedge (G \vee H)$$

where $\wedge$ and $\vee$ denotes the binary infimum (=intersection, by 2.7) and the binary supremum in D(K), respectively.

For the proof of distributivity of D(K), suppose $x \in F \cap (G \vee H) = F \cap (G \vee H)$, hence $x \in F$ and $x \in G \vee H$. Since

$$G \vee H = \{y \in L \mid g \wedge h = y \text{ for some } g \in G \text{ and some } h \in H\}$$

(by distributivity of K, multiplicativity of $\ll$ and compactness of 1),

$$x = g \wedge h$$

for some $g \in G$, $h \in H$, hence

$$g \in F \cap G \text{ and } h \in F \cap H,$$

hence

$$x = g \wedge h \in (F \wedge G) \vee (F \wedge H),$$

as we want.

2.12 THEOREM:

For a distributive continuous lattice L, $D(\mathrm{Filt}_\sigma L)$ is a distributive continuous lattice.

Proof:

Immediate from 2.1o and 2.11.

2.13 REMARK:

For a continuous $1,\wedge$-semilattice L, the order-embedding

$$DL \hookrightarrow \mathrm{Filt}_\sigma L$$

preserves finite meets (as every join-dense order-embedding does) and non-empty up-directed joins. Thus it induces a Scott-continuous $1,\wedge$-homomorphism

$$D\mathrm{Filt}_\sigma L \to DDL$$

assigning to a Scott-open filter Φ of $\mathrm{Filt}_\sigma L$ the Scott-open filter

$$\Phi \cap DL$$

of DL. Via the canonical isomorphism $\mu_L : DDL \to L$ (cf.o.6) this induces a morphism

$$D\mathrm{Filt}_\sigma L \to L.$$

2.14 REMARKS:

a) For a continuous $1,\wedge$-semilattice L,

$$D\mathrm{Filt}_\sigma(L)$$

is an "idempotent" construction in the sense that the induced morphism (cf.2.13)

$$D\mathrm{Filt}_\sigma(D\mathrm{Filt}_\sigma L) \to D\mathrm{Filt}_\sigma L$$

is an isomorphism.

This is readily clear from the observation that, for a continuous $1,\wedge$-semilattice K, such that DK is complete, i.e. 1 is compact and $\ll$ is multiplicative in K, $\mathrm{Filt}_\sigma K$ coincides with DK (in other words: the filter generated by a Scott-open subset of K is itself Scott-open).

The idempotency of $D\mathrm{Filt}_\sigma(?)$ can also be based upon 2.5. This argument may be sketched in the following way:

$$(D\mathrm{Filt}_\sigma)^2(L) \cong (DID)^2(L) \cong DID^2ID(L)$$
$$\cong DI^2D(L) \cong DID(L) \cong D\mathrm{Filt}_\sigma L$$

Since $D^2(P) \cong P$ und $I^2(P) \cong I(P)$ for every continuous poset P (where $P \hookrightarrow I(P)$ denotes an injective hull of P).

b) For a distributive continuous lattice L,

$$(\mathrm{Filt}_\sigma)^3 L \cong \mathrm{Filt}_\sigma L$$

since, by 2.12, $D\mathrm{Filt}_\sigma L$ is complete (i.e. $IDIDL \cong DIDL$), hence

$$(\mathrm{Filt}_\sigma)^3 L \cong (ID)^3 L \cong IDIDIDL \cong ID^2IDL$$
$$\cong I^2DL \cong IDL \cong \mathrm{Filt}_\sigma L .$$

2.15 REMARK:

For a distributive continuous lattice L

$$\psi^* D\mathrm{Filt}_\sigma L = \mathrm{Spec}^* D\mathrm{Filt}_\sigma L$$

by 2.8, 2.12 and 1.6(2) .

§ 3 The meet-prime elements of $DFilt_\sigma L$ correspond to the quasi-meet-prime elements of L, for a distributive continuous lattice L

In this section, L denotes a continuous lattice, D(L) its dual (o.6), and $Filt_\sigma L$ denotes the continuous lattice defined in 2.o. The hypothesis of distributivity for L can be deferred until the final step in the proof of proposition 3.9.

3.1 LEMMA:

Suppose L is a continuous lattice. An element $F \in DL$ is meet-prime in DL iff

(1) $o \text{ not} \in F$, and

(2) whenever $x \vee y \in F$ for some $x,y \in L$, then $x \in F$ or $y \in F$.

Proof:

(a) Suppose (1) and (2) are satisfied. Then $F \neq L$, by (1). Suppose $G \cap H \subseteq F$ for some $G,H \in DL$. Assume, on the contrary, $G \text{ not} \subseteq F$ and $H \text{ not} \subseteq F$, i.e. $x \in G$, $y \in H$ and $x,y \text{ not} \in F$ for some $x,y \in L$. Then $x \vee y \in G \cap H \subseteq F$, hence, by (2), $x \in F$ or $y \in F$ - a contradiction. Thus F is meet-prime in DL.

(b) Suppose F is meet-prime in DL. Then $F \neq L$, hence $o \text{ not} \in F$. Suppose $x \vee y \in F$ for some $x,y \in L$. Since

$$x = \sup \Downarrow x \quad \text{and} \quad y = \sup \Downarrow y,$$

in the continuous lattice L, we have

$$x \vee y = \sup D$$

for $D := \{s \vee t \mid s,t \in L,\ s \ll x, t \ll y\}$. Since D is non-empty and up-directed, and since F is Scott-open, there are $s \ll x$ and $t \ll y$ with $s \vee t \in F$. Since $s \ll x$ and $t \ll y$, there are Scott-open filters F_x and F_y with

$$x \in F_x \subseteq \Uparrow s \quad \text{and} \quad y \in F_y \subseteq \Uparrow t,$$

hence

$$F_x \cap F_y \subseteq \Uparrow s \cap \Uparrow t = \Uparrow (s \vee t) \subseteq F.$$

As a consequence,

$$F_x \subseteq F \quad \text{or} \quad F_y \subseteq F,$$

since F is meet-prime (by hypothesis), hence $x \in F$ or $y \in F$, as claimed.

3.2 Suppose $\underline{G} \in \underline{P}Filt_\sigma L$, the power set of $Filt_\sigma L$, for a continuous lattice L. Let

$$\Delta(\underline{G}) := \{x \in L \mid x \text{ is a lower bound of some } F \in \underline{G}\}$$

and

$$\chi(\underline{G}) := \sup \Delta(\underline{G}).$$

It is easy to see that $\chi : \underline{P}Filt_\sigma L \to L$ is an isotone map.

3.3 LEMMA:

For a continuous lattice L and $\underline{G} \in DFilt_\sigma L$, $\Delta(\underline{G})$ is a non-empty and up-directed lower set of L.

Proof:

We shall write Δ instead of $\Delta(\underline{G})$, for brevity.

If $a,b \in \Delta$, then $F \subseteq \uparrow a$ and $G \subseteq \uparrow b$ for some $F,G \in \underline{G}$. Since $\underline{G}$ is a filter, there is some $H \in \underline{G}$ with $H \subseteq F$ and $H \subseteq G$, hence

$$H \subseteq \uparrow a \cap \uparrow b = \uparrow (a \vee b),$$

hence $a \vee b \in \Delta$. Consequently, Δ is a non-empty $(o \in \Delta)$ and up-directed lower set of L.

For a continuous lattice L and $x \in L$, let

$$H_x := \{F \in Filt_\sigma L \mid x \in F\},$$
$$K_x := \{F \in Filt_\sigma L \mid x \in intF\}$$
$$= \{F \in Filt_\sigma L \mid x' \ll x \text{ for some } x' \in F\}.$$

(where int denotes the interior operator of the Scott topology).

We also write H(x) instead of H_x and K(x) instead of K_x.

3.4 LEMMA:

For a continuous lattice L, and $x \in L$,

$$K_x := \{F \in Filt_\sigma L \mid x \in intF\}$$

is an element of $DFilt_\sigma L$.

Proof:

(a) Clearly, K_x is an upper set of $\mathrm{Filt}_\sigma L$.

(b) If $\{F_i\}_{i\in I}$ is a non-empty up-directed family of members of $\mathrm{Filt}_\sigma L$ and

$$\sup\{F_i \mid i\in I\} = \bigcup\{F_i \mid i\in I\} \in K_x,$$

then there is some $x' \in \bigcup\{F_i \mid i \in I\}$ with $x' \ll x$. Consequently,

$$x' \in F_i$$

for some $i \in I$, hence

$$x \in \mathrm{int}F_i = \{y \in F_i \mid y' \ll y \text{ for some } y' \in F_i\}.$$

It results that $F_i \in K_x$ for some $i \in I$.

By (a) and (b), K_x is a Scott-open subset of $\mathrm{Filt}_\sigma L$.

(c) In order to see that K_x is a filter in the lattice $\mathrm{Filt}_\sigma L$, note first that $L \in K_x$. If $F, G \in K_x$, then $\mathrm{int}F \cap \mathrm{int}G$ is a Scott-open neighborhood of x in L, hence there is a Scott-open filter H of L with

$$x \in H \subseteq \mathrm{int}F \cap \mathrm{int}G,$$

since the Scott-open filters form a basis for the open sets of the Scott topology of L. Clearly, we have $H \in K_x$ and $H \subseteq F, G$.

3.5 LEMMA:

For a distributive (!) continuous lattice L and $x \in L$,

$$H_x = \{F \in \mathrm{Filt}_\sigma L \mid x \in F\}$$

is an element of $\mathrm{DFilt}_\sigma L$.

Proof:

Similar reasoning as in the proof of 3.4 yields that H_x is a Scott-open subset of Filt L. Part (c) of this proof can be substituted by the observation that, for a distributive continuous lattice L, the meet of a finite number of members of $\mathrm{Filt}_\sigma L$ is, by 2.7, their set-theoretic intersection.

3.6 REMARK:

For a meet-prime element p of a continuous lattice L we have

$$H_p = K_p.$$

3.7 LEMMA:

For every element x of a continuous lattice L,

$$x = \chi(H_x) = \chi(K_x)$$

Proof:

Since $K_x \subseteq H_x$, we have $\Delta(K_x) \subseteq \Delta(H_x)$, hence

$$\chi(K_x) = \sup\Delta(K_x) \le \sup\Delta(H_x) = \chi(H_x).$$

Since $y \le x$ for every $y \in \Delta(H_x)$, we can infer

$$\chi(H_x) \le x.$$

Suppose $z \in L$ with $z \ll x$, then there is some Scott-open filter F in L with

$$x \in F \subseteq \uparrow z,$$

hence z is a lower bound of $F \in K_x$. Thus

$$z \le \chi(K_x).$$

for every $z \in L$ with $z \ll x$. As a consequence,

$$x \le \chi(K_x),$$

since $x = \sup\{z \in L \mid z \ll x\}$.

3.8 LEMMA:

Suppose $\underline{G} \in \mathrm{DFilt}_\sigma L$ for a continuous lattice L. Then

$$K_x \subseteq \underline{G} \subseteq H_x$$

for $x := \chi(\underline{G})$.

Proof:

We write Δ instead of $\Delta(\underline{G})$.

(i) Suppose $x \in \mathrm{int}F$ for some $F \in \mathrm{Filt}_\sigma L$, i.e. $x' \ll x$ for some $x' \in F$. Since $x = \sup\Delta$, and Δ is a non-empty up-directed lower set of L' (by 3.3), we infer that $x' \in \Delta$, hence there is some $G \in \underline{G}$ such that

$$G \subseteq \uparrow x' \subseteq F.$$

Consequently, $F \in \underline{G}$, as claimed.

(ii) Suppose $H \in \underline{G}$. Since $\underline{G}$ is a Scott open subset of $\mathrm{Filt}_\sigma L$, $H' \ll H$ in $\mathrm{Filt}_\sigma L$ for some $H' \in \underline{G}$, i.e., by 2.2,

$$H' \subseteq \uparrow y \subseteq H,$$

for some $y \in L$, hence $y \in \Delta$. Consequently, $y \le x$, hence $x \in H$, as claimed.

3.9 PROPOSITION:

Suppose L is a distributive continuous lattice. A member $\underline{P}$ of $DFilt_\sigma L$ is meet-prime in $DFilt_\sigma L$ iff

$$\underline{P} = K_x = \{S \in Filt_\sigma L \mid x \in int_\sigma S\}$$

for a quasi-meet-prime element x of L. This element x is uniquely determined:

$$x = \chi(\underline{P})$$

Proof:

(a) Suppose $x \in L$ is quasi-meet-prime. Let

$$K_x := \{S \in Filt_\sigma L \mid x' \in S \text{ for some } x' \ll x\}.$$

We verify the conditions (i) and (ii) of 3.1 for K_x: Suppose $F \vee G \in K_x$. Then there is some $x' \in F \vee G$ with $x' \ll x$, hence

$$\inf\{f_1,\dots,f_n,g_1,\dots,g_m\} \leq x'$$

for some $f_1,\dots,f_n \in intF$ and some $g_1,\dots,g_m \in intG$ with natural numbers $n,m \geq o$.

Since x is quasi-meet-prime in L, there is some natural number i with $1 \leq i \leq n$ or $1 \leq i \leq m$ such that

$$f_i \leq x \quad \text{or} \quad g_i \leq x,$$

hence $x \in intF$ or $x \in intG$, hence $F \in K_x$ or $G \in K_x$.

It remains to show that $\{1\} = \varphi(\emptyset)$ is not an element of K_x. Suppose, on the contrary, that $\{1\} \in K_x$, hence $x \in \{1\}$ and there is some $x' \ll x$ with $x' \in \{1\}$, i.e. $1 \ll 1$. However, 1 fails to be quasi-meet-prime if it is compact (by 1.5(3)).

(b) Now suppose that $\underline{P}$ is a meet-prime element of $DFilt_\sigma L$. Let

$$x := \chi(\underline{P})$$

(1) Since we already know from 3.8 that

$$K_x \subseteq \underline{P},$$

suppose that $F \in \underline{P}$. Since $\underline{P}$ is a Scott-open subset of $Filt_\sigma L$, there is some $F' \in \underline{P}$ with $F' \ll F$ in $Filt_\sigma L$, i.e. (by 3.1)

$$F' \subseteq \uparrow y \subseteq F$$

for some $y \in L$. Consequently, $y \in \Delta(\underline{P})$, hence $y \leq x$. Since $y \in F$, there are $y_i \in intF$ $(i=1,\dots,n)$ with $n \geq o$ such that

$$\inf\{y_1,\dots,y_n\} \leq y.$$

Since $y_i \in intF$, there are $y_i', y_i'' \in F$ with

$y_i'' \ll y_i' \ll y_i$ $(i=1,\dots,n)$. (The existence of the y_i's is guaranteed by the interpolation property of $\ll$). Thus there are Scott-open filters G_i in L such that

$$y_i \in G_i \subseteq \uparrow y_i' \subseteq F.$$

Since $\inf\{y_1,\dots,y_n\} \leq y$, we conclude that

$$y \in G_1 \vee \dots \vee G_n,$$

where "$\vee$" denotes the join (=supremum) in $Filt_\sigma L$.

Since $F' \subseteq \uparrow y$ and $F' \in \underline{P}$, we can infer that $F' \subseteq \uparrow y \subseteq G_1 \vee \dots \vee G_n$, hence

$$G_1 \vee \dots \vee G_n \in \underline{P}.$$

Since $\underline{P}$ is meet-prime in $DFilt_\sigma L$, we conclude, by 3.1, that

$$G_i \in \underline{P}$$

for some $i \in \{1,\dots,n\}$. Consequently,

$$y_i' \leq x,$$

since y_i' is a lower bound of $G_i \in \underline{P}$. It results that $y_i'' \ll x$. Since $y_i'' \in F$, we infer that $x \in intF$, as claimed.

In all this says that

$$K_x = \underline{P}.$$

(2) We infer from $\underline{P} = K_y$ that

$$y = \chi(K_y) = \chi(\underline{P}) = \chi(K_x) = x,$$

hence x is uniquely determined.

(3) In order to show that $x = \chi(\underline{P})$ is quasi-meet-prime, suppose that

$$\inf\{y_1,\dots,y_n\} \ll x$$

for some $y_1,\dots,y_n \in L$ and a natural number $n \geq o$. Then

$$H(y_1) \cap \dots \cap H(y_n) = H(\inf\{y_1,\dots,y_n\}) \subseteq K_x \subseteq \underline{P},$$

since every $F \in Filt_\sigma L$ containing $\inf\{y_1,\dots,y_n\}$ contains x as an inner point.

Since L is, by hypothesis, a <u>distributive</u> lattice,

$$H(z) \in DFilt_\sigma L$$

for every $z \in L$, by 3.5.
Consequently, (in view of 2.7)

$$H(y_i) \subseteq \underline{P}$$

for some $i \in \{1,\dots,n\}$, since $\underline{P}$ is meet-prime in $DFilt_\sigma L$. Since χ is an isotone map $\underline{P}Filt_\sigma L \to L$, we infer

$$y_i = \chi(H(y_i)) \leq \chi(\underline{P}) = x,$$

as we want.

§ 4 The trace of the Γ^*-topology on $\psi^* L$

In $[H_3]$ §3 I have given another representation $\gamma_X : X \hookrightarrow \gamma X$ of the essential hull $\lambda_X : X \hookrightarrow \lambda X$ of a T_o-space X, discovered by B.Banaschewski $[Ba_1]$. The elements of γX are the <u>convergence</u> <u>sets</u> of X, i.e. those (closed) subsets M of X such that either M=X or there exists an ordinary proper filter (or, equivalently, a net) on X which converges precisely to the points of M.

Also, in $[H_3]$ 3.4(a) the notion of a γ-element of a lattice L has been defined and it has been shown there that the γ-elements of the (complete) lattice $\underline{A}(X)$ of all closed subsets (ordered by inclusion) of a T_o-space X are precisely the convergence sets of X ($[H_3]$ 3.9, $[H_8]$ 2.8). Furthermore, the Γ-topology is introduced on L such that the set of γ-elements of L endowed with the trace of this topology constitutes a space

$$\gamma L$$

which, for the lattice $L = \underline{A}(X)$ of closed subsets of a T_o-space X, coincides with γX.

In $[H_8]$ an extensive study of the space γL has been made and it is observed there that every pseudo-join-prime element (1.3) of a complete lattice L is a γ-element ($[H_8]$ 3.4(2)).

Since in the present paper the basic notion is that of a distributive continuous lattice, i.e. a continuous lattice which is - by a celebrated theorem of K.H.Hofmann and J.D.Lawson $[HL_2]$ - representable as the lattice $\underline{O}(X)$ of open subsets (ordered by inclusion) of some T_o-space X, it is convenient here to adapt the definition of the Γ-topology and of a γ-element so as to apply to the lattice $\underline{O}(X)$ rather than $\underline{A}(X)$, i.e., in a sense, to dualize:

4.1 For a complete lattice L and $a, b \in L$ we write

$$a \vdash b$$

iff, whenever $\inf F \leq a$ for a finite subset F of L (where $F = \emptyset$ is not excluded) then $x \leq b$ for some $x \in F$.

This is the dual of the relation $\dashv$ of $[H_3]$, §3, $[H_8]$. It "relativizes" the notion of a meet-prime element p of a complete lattice L in the same way as the way below relation relativizes the notion of a compact element, viz. $p \in L$ is meet-prime in L iff

$$p \vdash p .$$

Thus $\vdash$ may be read as "<u>relatively</u> <u>meet-prime</u> <u>below</u>".

4.2 It is observed in $[H_8]$ 1.1 that $\inf F \vdash a$ for a finite subset F of a complete lattice L implies $x \vdash a$ for some $x \in F$. It results that the sets

$$\Gamma^*(a) := \{x \in L \mid a \vdash x\}$$

(with a ranging through L) form a <u>basis</u> for the <u>closed</u> sets for a topology on (the underlying set of) L which will be referred to as the Γ^*-<u>topology</u> of L (i.e. the Γ-topology of L^{op}-cf. $[H_3]$ 3.2, $[H_8]$ 1.2).

4.3 An element p of a complete lattice L is said to be a γ^*-<u>element</u> (i.e. γ-element of L^{op}) iff it enjoys one of the following conditions (1), (2) and (3) which are pairwise equivalent ($[H_8]$ 1.5, 2.7):

(1) $p = \sup\{x \in L \mid x \vdash p\}$,

(2) $p = \sup(L - F)$ for some filter (i.e. non-empty, down-directed lower set) F of L,

(3) $\uparrow p = \{y \in L \mid p \leq y\}$ is closed in the Γ^*-topology of L.

It results from (3) that the trace of the Γ^*-topology of L on the set of all γ^*-elements defines a topological T_o-space

$$\gamma^* L$$

whose associated specialization partial order is <u>inverse</u> to (the trace of) the order of L.

4.4 For a T_o-space X there is an embedding

$$\gamma^*_X : X \hookrightarrow \gamma^* X$$

into the space $\gamma^* X := \gamma^*(\underline{O}(X))$ given by

$$x \mapsto X - cl\{x\}.$$

Obviously $\gamma^*_X : X \hookrightarrow \gamma^* X$ is (an equivalent representation of) the greatest essential extension of the T_o-space X.

4.5 DEFINITION:

a) For a complete lattice L, let ψ^*L denote (both the set and) the space of a all pseudo-meet-prime elements of L endowed with the trace of the Γ^*-topology of L.

b) For a T_o-space X let $\psi^*X := \psi^*(O(X))$, and let

$$\psi^*_X : X \hookrightarrow \psi^*X$$

denote the embedding

$$x \mapsto X\text{-cl}\{x\}$$

to be referred to as the "ψ^*-extension" of X.

4.6 It is immediate from 4.3(2) that every pseudo-meet-prime element of a complete lattice is a γ^*-element (cf. [H_8] 3.4(2)), since the complement of a prime ideal is a filter, hence ψ^*X is a subspace of γ^*X. Thus we can infer (from [Ba_1], lemma 2, p.235).

4.7 PROPOSITION:

For a T_o-space X, $\psi^*_X : X \hookrightarrow \psi^*X$ is an essential extension.

4.8 REMARK:

It is convenient to topologize also the set ψL of all join-prime elements of a complete lattice L with the trace of the Γ-topology of L.

For a T_o-space X, we obtain an essential extension, the "ψ-extension" of X,

$$\psi_X : X \hookrightarrow \psi X := \psi \underline{A}(X)$$

co-restricting the extension $\gamma_X : X \hookrightarrow \gamma X$.

The points of ψX are - as observed in 1.11 - precisely the points of the Fell compactification $\underline{H}(X)$ of X if X is a locally quasicompact space.

The elements of ψX are the complements, in X, of the members of ψ^*X. Obviously, every result on the ψ^*-extension has an analogue for the ψ-extension.

4.9 THEOREM:

For a distributive continuous lattice L, the mapping

$$K : \psi^*L \to \text{Spec}^*\text{DFilt}_\sigma L$$

with

$$K(x) := \{F \in \text{Filt}_\sigma L \mid x \in \text{int}_\sigma F\}$$

is a homeomorphism.

Proof:

In 3.9, it is shown that, for a distributive continuous lattice L, K(x) is a member of $\text{Spec}^*\text{DFilt}_\sigma L$, i.e. a meet-prime element of $\text{DFilt}_\sigma L$, if and only if x is a quasi-meet-prime element of L. Moreover, this element $x \in L$ is uniquely determined. By 1.6(3), an element $x \in L$ is quasi-meet-prime iff x is pseudo-meet-prime. Thus

$$K : \psi^*L \to \text{Spec}^*\text{DFilt}_\sigma L$$

is a bijection.

It remains to show that this mapping K is continuous and that every (basic) closed subset of ψ^*L is the inverse image of a closed subset of $\text{Spec}^*\text{DFilt}_\sigma L$.

(a) We first show that the inverse image of a closed set of $\text{Spec}^*\text{DFilt}_\sigma L$ under the mapping

$$K : \psi^*L \to \text{Spec}^*\text{DFilt}_\sigma L$$

is closed in ψ^*L, i.e. the trace, on ψ^*L, of a Γ^*-closed subset of L. Since a closed subset of $\text{Spec}^*\text{DFilt}_\sigma L$ can be (uniquely) represented in the form

$$\{\underline{G} \in \text{Spec}^*\text{DFilt}_\sigma L \mid \underline{F} \subseteq \underline{G}\}$$

for some $\underline{F} \in \text{DFilt}_\sigma L$, it suffices to establish the following:

$$\{x \in \psi^*L \mid \underline{F} \subseteq K(x)\} = \bigcap\{\Gamma^*(y) \mid y \in \Delta_{\underline{F}}\},$$

where

$$\Delta_{\underline{F}} = \{y \in L \mid \text{there is some } G \in \underline{F} \text{ with } G \subseteq \uparrow y\},$$

i.e. $\Delta_{\underline{F}}$ consists of those $y \in L$ which are the lower bounds of some member of $\underline{F}$.

Suppose first that $\underline{F} \subseteq K(x)$ for some $\underline{F} \in \text{DFilt}_\sigma L$ and some $x \in \psi^*L$. Let $y \in L$ be a lower bound of some member G of $\underline{F}$. Assume that

$$\inf\{u_1, \dots, u_n\} \leq y$$

for some $u_1, \dots, u_n \in L$ and some $n \in \mathbb{N}$, $n \geq o$.

Since $\underline{F} \subseteq K(x)$, x is an inner point of G with regard to the Scott topology of L , hence

$$z \ll x$$

for some $z \in G$. It results that $\inf\{u_1,\dots,u_n\} \leq y \leq z \ll x$, hence

$$u_k \leq x$$

for some $k \in \{1,\dots,n\}$, since x is quasi-meet-prime in L. In all this says that

$$y \vdash x$$

for every $y \in \Delta_{\underline{F}}$, whence

$$\{x \in \psi^{*}L \mid \underline{F} \subseteq K(x)\} \subseteq \bigcap\{\Gamma^{*}(y) \mid y \in \Delta_{\underline{F}}\}.$$

In order to prove the inverse inclusion, suppose $\underline{F} \in DFilt_{\delta}L$ and

$$y \vdash x$$

for every $y \in \Delta_{\underline{F}}$. Assume that $G \in \underline{F}$. We have to show that x is an inner point of G with regard to the Scott topology of L, in order to prove that $\underline{F} \subseteq K(x)$. Since $\underline{F}$ is a Scott-open subset of $Filt_{\delta}L$, there is some $F \in \underline{F}$ with

$$F \ll G,$$

where $\ll$ denotes the way below relation in $Filt_{\delta}L$, hence

$$F \subseteq \Uparrow z \subseteq G$$

for some $z \in L$, by 2.2. We infer from $F \subseteq \Uparrow z$ that $z \in \Delta_{\underline{F}}$, hence $z \vdash x$ by hypothesis. Since $z \in G$, there are $u_1,\dots,u_n \in int_{\delta}G$ $(n \in \mathbb{N} \cup \{o\})$ with

$$z = \inf\{u_1,\dots,u_n\},$$

by the very definition of $Filt_{\delta}L$. Since $z \vdash x$, we can infer that

$$u_i \leq x$$

for some $i \in \{1,\dots,n\}$, hence $x \in int_{\delta}G$, as claimed.

(b) It remains to show that every basic closed subset of $\psi^{*}L$, i.e. the trace, on $\psi^{*}L$, of a basic Γ^{*}-closed set of L, is an inverse image of some closed subset of $Spec^{*}DFilt_{\delta}L$ under $K:\psi^{*}L \to Spec^{*}DFilt_{\delta}L$.

A basic closed subset of the Γ^{*}-topology of L is of the form

$$\Gamma^{*}(x) = \{y \in L \mid x \vdash y\}$$

for some $x \in L$. Using the member

$$H(x) = \{G \in Filt_{\delta}L \mid x \in G\}$$

of $DFilt_{\delta}L$ (cf.3.5), we shall establish that

$$\{y \in \psi^{*}L \mid H(x) \subseteq K(y)\} = \psi^{*}L \cap \Gamma^{*}(x).$$

First assume that $x \vdash y$ for some $x \in L$ and some $y \in \psi^{*}L$. Suppose that $G \in H(x)$. We want to show that $G \in K(y)$, i.e. y is an inner point of G with regard to the Scott topology of L. Since $x \in G$, there are $u_1,\dots,u_n \in int_{\delta}G$ $(n \in \mathbb{N} \cup \{o\})$ with

$$x \geq \inf\{u_1,\dots,u_n\},$$

by the very definition of $Filt_{\delta}L$ - cf.2.o .

Since $x \vdash y$, we can infer that

$$u_i \leq y$$

for some $i \in \{1,\dots,n\}$, hence $y \in int_{\delta}G$.

This proves

$$\psi^{*}L \cap \Gamma^{*}(x) \subseteq \{y \in \psi^{*}L \mid H(x) \subseteq K(y)\}.$$

In order to prove the inverse inclusion, we assume that $H(x) \subseteq K(y)$ for some $x \in L$ and some $y \in \psi^{*}L$. Let us assume, to the contrary, that

$$x \text{ not} \vdash y .$$

Then there are $u_1,\dots,u_n \in L$ with $n \in \mathbb{N} \cup \{o\}$ such that

$$\inf\{u_1,\dots,u_n\} \leq x,$$

but

$$u_k \text{ not } \leq y$$

for every $k \in \{1,\dots,n\}$. Since L is a continuous lattice, we have

$$u = \sup\{v \in L \mid v \ll u\} ,$$

for every $u \in L$, hence there are $v_1,\dots,v_n \in L$ with

$$v_k \ll u_k \text{ and } v_k \text{ not} \leq y$$

for every $k \in \{1,\dots,n\}$. We consider the filter G generated by the Scott open set

$$\Uparrow v_1 \cup \dots \cup \Uparrow v_n,$$

i.e. $G = \varphi(\Uparrow v_1 \cup \dots \cup \Uparrow v_n)$. Since

$$\inf\{u_1,\dots,u_n\} \leq x ,$$

we have $x \in G$, hence - by hypothesis - $y \in int_{\delta}G$. Thus there is some $z \in G$ with

$$z \ll y .$$

As a consequence, there are $z_1,\dots,z_m \in L$

$$z = \inf\{z_1,\dots,z_m\},$$

such that for every $j \in \{1,\dots,m\}$ there is some

$k(j) \in \{1,\dots,n\}$ with

$$v_{k(j)} \ll z_j \ ,$$

by the very definition of G.

Since y is quasi-meet-prime, we can infer that

$$z_j \leq y$$

for some $j \in \{1,\dots,m\}$, hence

$$v_{k(j)} \ll z_j \leq y \ ,$$

contradicting the assumption that $v_k \not\leq y$ for every $k=1,\dots,n$. Thus we have

$$x \vdash y \ ,$$

as we want.

Since for a distributive continuous lattice L, $DFilt_\sigma L$ is a distributive continuous lattice (by 2.12), we can infer from the result of K.H.Hofmann and J.D.Lawson $[HL_2]$ that

$$\underline{O}Spec^*DFilt_\sigma L \cong DFilt_\sigma L.$$

Thus we have

4.1o COROLLARY:

For a distributive continuous lattice L,

$$\underline{O}(\psi^* L) \cong DFilt_\sigma L \cong DID(L) \ .$$

Here, as before, $P \hookrightarrow I(P)$ denotes a representation of the injective hull of a continuous poset P.

4.11 PROPOSITION:

For a T_o-space X whose lattice $\underline{O}(X)$ of open subsets is a continuous lattice, $\psi^* X$ is a strongly sober locally quasicompact space.

Proof:

(1) Since $\psi^*\underline{O}(X)$ is homeomorphic to $Spec^*DFilt_\sigma\underline{O}(X)$ (by 4.9), and since $Spec^*K$ is is sober for every complete lattice K (cf.1.1), $\psi^*\underline{O}(X)$ is sober.

(2) By 4.1o,

$$\underline{O}(\psi^* X) = DFilt_\sigma \underline{O}(X)$$

hence $\underline{O}(\psi^* X)$ is a continuous lattice in which the greatest element is compact and the way below relation is multiplicative. Now, 1.13 "(1) iff (3)" applies.

4.12 COROLLARY:

For a T_o-space X whose lattice $\underline{O}(X)$ of open subsets is a continuous lattice, the canonical embedding

$$\psi_{\psi X} : \psi X \hookrightarrow \psi(\psi X)$$

is a homeomorphism.

Proof:

By 1.13(5), $\psi_Y : Y \hookrightarrow \psi Y$ is bijective for every strongly sober locally quasicompact space. By 4.11 this applies to $Y := \psi X$.

The above result says that the ψ-extension is "idempotent" (up to an isomorphism) for spaces X with $\underline{O}(X)$ continuous.

4.13 REMARK:

From the proof of 4.11 we extract that, in $\underline{O}(\psi^* L)$ (for a distributive continuous lattice L), the pseudo-meet-prime elements are exactly the meet-prime elements:

$$\psi^*\underline{O}(\psi^* L) = Spec^*\underline{O}(\psi^* L).$$

4.14 REMARK:

We shall need (in the following section) a little bit more information about the relationship between the homeomorphisms

$$K : \psi^* L \to Spec^*DFilt_\sigma L$$

and

$$\psi^*_{\psi^* L} : \psi^* X \to \psi^*(\psi^* X) = \psi^*(\underline{O}_\psi{}^* L)$$

for a space X with $L := \underline{O}(X)$ continuous.

Clearly, there is an induced homeomorphism $K':\psi^*(\underline{O}(\psi^*L)) \to \mathrm{Spec}^*\mathrm{DFilt}_\sigma L$ such that

$$\begin{array}{ccc} \psi^*L & \xrightarrow{\gamma^*_{\psi^*L}} & \psi^*(\underline{O}\psi^*L) \\ & \searrow^{K} & \downarrow K' \\ & & \mathrm{Spec}^*\mathrm{DFilt}_\sigma L \end{array}$$

commutes.

Since $\psi^*(\underline{O}\psi^*L)=\mathrm{Spec}^*(\underline{O}\psi^*L)$, K' is induced by an isomorphism

$$K'':\underline{O}\psi^*L \to \mathrm{DFilt}_\sigma L$$

(which assigns to $V \in \underline{O}\psi^*L$

$$\inf\{K'(p) \mid V \le p \in \mathrm{Spec}^*(\underline{O}\psi^*L)\}$$

where the infimum is taken in $\mathrm{DFilt}_\sigma L$),
such that

$$\begin{array}{ccc} \psi^*(\underline{O}\psi^*L) & \hookrightarrow & \underline{O}\psi^*L \\ K'\downarrow & & \downarrow K'' \\ \mathrm{Spec}^*\mathrm{DFilt}_\sigma L & \hookrightarrow & \mathrm{DFilt}_\sigma L \end{array}$$

commutes – where the horizontal maps are the inclusions.

§ 5 The relationship between the Fell compactification $\underline{H}(\psi X)$ of ψX and $\underline{H}(X)$ when $\underline{O}(X)$ is a continuous lattice

For a space X whose lattice $\underline{O}(X)$ of open subsets is a continuous lattice, we have seen in 4.12 that

$$\gamma_{\psi X}:\psi X \to \psi(\psi X)$$

is a bijection, hence a homeomorphism. Since, for a locally quasicompact space X, ψX has the same points as the Fell compactification $\underline{H}(X)$ of X and $\psi(\psi X)$ has the same points as the Fell compactification of $\underline{H}(\psi X)$ of ψX, it is a natural question whether the given bijection

$$\underline{H}(X) \to \underline{H}(\psi X)$$

is a homeomorphism.

It is convenient for the proofs to use the representation ψ^*X of ψX by open subsets of X, i.e. we shall show that the bijection

$$\gamma^*_{(\psi^*X)}:\psi^*X \hookrightarrow \psi^*(\psi^*X)$$

gives a homeomorphism

$$\underline{H}^*(X) \to \underline{H}^*(\psi^*X)$$

where $\underline{H}^*(Y)$ denotes the space of pseudo-meet-prime elements of $\underline{O}(Y)$ with the topology inherited from the Lawson topology of $\underline{O}(Y)$ – under the proviso that $\underline{O}(Y)$ is a continuous lattice. (Note that $\underline{H}^*(Y)$ has the same points as ψ^*Y and that – by the remarks in the introduction of this paper – passing to complements relative to Y gives a homeomorphism $\underline{H}^*(Y) \to \underline{H}(Y)$.)

5.1 Recall that the inclusion

$$e:DL \hookrightarrow \mathrm{Filt}_\sigma L ,$$

for a continuous lattice L, preserves $1,\wedge$ and suprema of non-empty up-directed subsets and – as noted in 2.13 – e induces a map

$$D(e):\mathrm{DFilt}_\sigma L \to DDL$$

which takes $\Phi \in \mathrm{DFilt}_\sigma L$ into

$$\Phi \cap DL .$$

This map D(e) also preserves $1,\wedge$ and suprema of non-empty up-directed subsets, hence it is continuous with regard to the respective Scott topologies.

5.2 In $[L_2]$ 9.7, J.D.Lawson establishes a necessary and sufficient criterion in order that, for continuous posets S and T, a map $f:S \to T$ with the property that an inverse image of a Scott-open filter of T is a Scott-open filter of S be right adjoint [8], viz. that the induced map $D(f):D(T) \to D(S)$ preserves the way below relation $\ll$. (Note that a $1,\wedge$-preserving Scott-continuous map $f:S \to T$ between continuous $1,\wedge$-semi-lattices S and T automatically has the property that inverse images of Scott-open filters are Scott-open).

5.3 We have already observed in 2.5 that, for a continuous lattice L, $e:DL \hookrightarrow Filt_\sigma L$ preserves the way below relation. Thus we can infer, by 5.2, from the commutativity of

$$\begin{array}{ccc} DL & \xrightarrow{\ e\ } & Filt_\sigma L \\ \varepsilon_{DL}\downarrow & & \downarrow \varepsilon_{Filt_\sigma L} \\ D^3L & \xrightarrow{\ D^2(e)\ } & D^2 Filt_\sigma L \end{array}$$

(where the vertical arrows are isomorphisms) that

$$D(e) : DFilt_\sigma L \to DDL$$

is right adjoint, hence so is the composite

$$g:=\mu_L \circ D(e):DFilt_\sigma L \to DDL \to L$$

where $\mu_L:DDL \to L$ denotes the isomorphism, inverse to $\varepsilon_L:L \to DDL$.

Since a right adjoint preserves arbitrary infima, it results that $g:DFilt_\sigma L \to L$ is continuous with regard to the respective Lawson topologies ([C] III-1.8).

---

8) The terminology of $[L_2]$ is in conflict with the one used above which is generally accepted among categorists: An isotone map $f:S \to T$ is right adjoint to $g:T \to S$ iff $g(y) \le x$ is equivalent to $y \le f(x)$ for $x \in S$, $y \in T$ (cf. [ML] I.2, p.11).

5.4 <u>LEMMA</u>:

For a distributive continuous lattice L, the mapping $g:DFilt_\sigma L \to L$ defines, by restriction and co-restriction, a bijection

$$d:Spec^* DFilt_\sigma L \to \psi^* L$$

inverse to K(?) (defined in 3.4).

<u>Proof</u>:

Every meet-prime element F of $DFilt_\sigma L$ is of the form

$$K(x):=\{F \in Filt_\sigma L \mid x \in intF\}$$

for a unique quasi-meet-prime element x of L, by 3.9. Evidently

$$D(e)(K(x)) = \{F \in DL \mid x \in F\} = \varepsilon_L(x) ,$$

where $\varepsilon_L:L \to DDL$ denotes the canonical isomorphism. Thus

$$d(K(x))=g(K(x))=(\mu_L \circ D(e))(K(x))=\mu_L \varepsilon_L(x)=x .$$

Since, by 3.9, K(?) is a bijective map

$$\psi^* L \to Spec^* DFilt_\sigma L,$$

we can infer that d is inverse to K.

Now we have

5.5 <u>PROPOSITION</u>:

For a distributive continuous lattice L, the mapping $K: \psi^* L \to Spec^* DFilt_\sigma L$ with

$$K(x)=\{F \in Filt_\sigma L \mid x \in int_\sigma F\}$$

is a homeomorphism with regard to the topologies inherited from the Lawson topologies of L and $DFilt_\sigma L$, respectively.

<u>Proof</u>:

We observe first that

$$d:Spec^* DFilt_\sigma L \to \psi^* L$$

is continuous with regard to the traces of the Lawson topologies of $DFilt_\sigma L$ and L, respectively (since it restricts and co-restricts the Lawson-continuous map $g:DFilt_\sigma L \to L$). By 1.9, both the domain and the co-domain of this mapping are compact Hausdorff spaces. (Recall from 2.15 that $\mathrm{Spec}^*\mathrm{DFilt}_\delta L = \psi^*\mathrm{DFilt}_\delta L$.) Thus the inverse K of d is also continuous and, in fact, a homeomorphism.

5.6(i) It has been observed in 4.14 that, for a distributive continuous lattice L, the bijection K' making

$$\begin{array}{ccc} \psi^* L & \xrightarrow{\psi^*_{\psi^* L}} & \psi^* \underline{O}(\psi^* L) \\ & {\scriptstyle K} \searrow & \downarrow {\scriptstyle K'} \\ & & \mathrm{Spec}^*\mathrm{DFilt}_\delta L \end{array}$$

commutative is induced - by restriction and corestriction - from an isomorphism $K'' : \underline{O}(\psi^* L) \longrightarrow \mathrm{DFilt}_\delta L$. This isomophism K" is, of course, a homeomorphism for the respective Lawson topologies, hence K' is a homeomorphism with regard to the traces of these topologies.

Combining this observation with 5.5 we obtain that, for a distributive continuous lattice L,

$$\psi^*_{\psi^* L} : \psi^* L \longrightarrow \psi^* \underline{O}(\psi^* L)$$

is a homeomorphism with regard to the traces of the Lawson topologies of L and $\underline{O}(\psi^* L)$ respectively, i.e. (by the remarks in 5.o) a homeomorphism

$$\underline{H}^*(X) \longrightarrow \underline{H}^*(\psi^* L)$$

provided that X is a space with $L = \underline{O}(X)$.

(ii) Since, by 4.12, the mapping

$$\psi_{\psi X} : \psi X \longrightarrow \psi(\psi X),$$

for a space X with $\underline{O}(X)$ a continuous lattice, is a homeomorphism (with regard to the genuine topologies of these spaces), it is an order-isomorphism with regard to the respective specialization orders. These are the partial orders induced from the lattices $\underline{A}(X)$ and $\underline{A}(\psi X)$, respectively, i.e. the (restricted) inclusion relations.

In all this gives

5.7 THEOREM:

For a space X whose lattice $\underline{O}(X)$ of open subsets (ordered by inclusion) is a continuous lattice, the canonical embedding

$$\psi X \hookrightarrow \psi(\psi X)$$

"is" (i.e. determines) a homeomorphism and an order-isomorphism

$$\underline{H}(X) \longrightarrow \underline{H}(\psi X),$$

i.e. an isomorphism in the category of compact ordered spaces.

5.8 REMARK:

The mapping $K(?) : \psi^* L \longrightarrow \mathrm{Spec}^*\mathrm{DFilt}_\delta L$ extends (with the same definition) to a map

$$L \longrightarrow \mathrm{DFilt}_\delta L.$$

which is easily shown to be Scott-continuous. I do not know whether it is Lawson-continuous. (This would give an alternative proof of 5.7).

§ 6 The relationship between $\underline{H}(X)$ and ψX for a space X whose lattice $\underline{O}(X)$ of open subsets is continuous. Functoriality of $\underline{H}(?)$ and $\psi(?)$.

For a space X whose lattice $\underline{O}(X)$ of open subsets is a continuous lattice we want to show that the open sets of ψX are precisely the open upper sets of $\underline{H}(X)$.

Due to the homeomorphisms

$$\psi X \to \psi\psi X \quad \text{and} \quad \underline{H}(X) \to \underline{H}(\psi X)$$

established in 4.12 and 5.7, respectively, this question can be reduced to the study of those spaces X for which $\underline{O}(X)$ is a continuous lattice and the canonical mapping

$$X \to \psi X$$

is a homeomorphism, i.e. - by 1.13 - the strongly sober locally quasi-compact spaces X.

Thus the result, we claim, is a consequence of an isomorphism between the category

whose objects are the strongly sober, locally quasi-compact spaces and whose morphisms are the continuous perfect maps (i.e. those continuous maps which enjoy the property that the inverse image of every saturated quasi-compact subset of the co-domain is quasi-compact)

and the category

of compact ordered spaces and continuous isotone maps.

This isomorphism is implicit in the construction of an isomorphism between the category of compact ordered spaces and isotone continuous maps and the category of distributive continuous lattices with 1 compact and $\ll$ multiplicative and those mappings preserving $\ll$, finite infima and arbitrary suprema, described in [C]VII-3, cf. in particular VII-3.7(iii). (The result is - on the object level - due to G.Gierz and K.Keimel [GK].)

We reformulate the key results of [C]VII-3 in order to make visible the ingredients of the desired isomorphism.

For the subset $|\psi^{*}L|$ of a (distributive) continuous lattice L, let λ' denote the trace of the Lawson topology λ_L of L,

6.1 LEMMA:

Let L be a distributive continuous lattice in which $Spec^{*}L$ is closed with respect to the Lawson topology λ_L of L. A subset U of $Spec^{*}L$ is an open lower set of $(\psi^{*}L, \lambda')$ iff

$$U = Spec^{*}L - \uparrow a$$

for some $a \in L$, i.e. iff U is open in the space $Spec^{*}L$.

Proof:

If $Spec^{*}L$ is closed in (L, λ_L), then so is $\{1\} \cup Spec^{*}L$. Now [C]VII-3.1 applies.

6.2 PROPOSITION:

Let X be a compact p(artially) o(rdered) space. The system $\Omega(X)$ of all open lower sets of X is a topology on $|X|$ which is a (distributive) continuous lattice with the property that $Spec^{*}\Omega(X)$ is closed with regard to the Lawson topology on $\Omega(X)$. The specialization order of $(|X|, \Omega(X))$ is the inverse of the order the p.o. space X. The mapping

$$x \mapsto X - \uparrow x \quad (x \in X)$$

is a bijection $|X| \to |Spec^{*}\Omega(X)|$ which is a homeomorphism

$$X \to (\psi^{*}\Omega(X), \lambda')$$

where λ' is the trace, on $Spec^{*}\Omega(X) = \psi^{*}\Omega(X)$, of the Lawson topology of $\Omega(X)$.

Proof:

This is an obvious modification of [C]VII-3.3 (cf.also [C]VII-3.7).

6.3 The above results 6.1 and 6.2 establish (in view of 1.13) a one-to-one correspondence between compact partially ordered spaces and strongly sober locally quasi-compact spaces.

It is not difficult to prove - along the lines of [C]pp.323/324 - that a continuous isotone map $f:X \to Y$ between compact partially ordered spaces X and Y induces a continuous map

$$f:(X,\Omega(X)) \longrightarrow (Y,\Omega(Y))$$

with the property that the induced map

$$\Omega(f) : \Omega(Y) \to \Omega(X)$$

preserves the way below relation.
Also, the arguments given in [C]p.324 suffice to show that every continuous perfect map between strongly sober, locally quasi-compact spaces is induced by a (unique) continuous map between the associated compact ordered spaces.

In all this gives

6.4 THEOREM:
There is an isomorphism J^* between the category of compact partially ordered spaces and continuous isotone maps and the category of locally quasicompact strongly sober spaces and continuous perfect mappings. The functor J^* assignes to a compact ordered space X the space with the same points as X whose open sets are the open lower sets of X, and leaves the morphisms unchanged.

6.5 REMARK:
Reversing the order defines an automorphism (of order 2) of the category of compact p(artially) o(rdered) spaces and isotone continuous maps. Thus, by composition with J^* (of 6.4), we obtain an isomorphism J from the category of compact p.o.spaces and isotone continuous map to the category of locally quasicompact strongly sober spaces and continuous perfect maps. The functor J assigns to a compact ordered space X the space with the same points as X whose open sets are the open upper sets of X, and leaves the morphisms unchanged.

In view of 4.12 and 5.7, lemma 6.1 gives

6.6 THEOREM:
Let X be a T_o-space with $\underline{O}(X)$ a continuous lattice:
a) The open sets of $\psi^* X$ (in its genuine topology) are precisely the open lower sets of the compact ordered space $\underline{H}^*(X)$.
b) The open sets of ψX (in its genuine topology) are precisely the open upper sets of the Fell compactification $\underline{H}(X)$.

6.7 For locally compact (non-compact) Hausdorff spaces X and Y, a map $u:X \to Y$ extends (uniquely) to a continuous map $u^+:X^+ \to Y^+$ of the Alexandrov-one-point-quasi-compactifications X^+ and Y^+ of X and Y, respectively, such that

$$\begin{array}{ccc} X & \xrightarrow{u} & Y \\ \downarrow & & \downarrow \\ X^+ & \xrightarrow{u^+} & Y^+ \end{array}$$

commutes and $u^+(\omega_X)=\omega_Y$ (where ω denotes the adjoined point) if and only if $u:X \to Y$ is continuous and perfect (i.e. the inverse image of a compact subset of Y is compact in X).

Note that u^+ is perfect iff it is continuous, and that a non-constant map $v:\underline{H}(X) \to \underline{H}(Y)$ is isotone iff $v(\omega_X)=\omega_Y$. (Recall the definition of the partial order of X^+ given in the introduction.)

The following result partially extends these facts to spaces X and Y with $\underline{O}(X)$, $\underline{O}(Y)$ a continuous lattice.

Recall that for spaces X,Y with $\underline{O}(X)$, $\underline{O}(Y)$ a continuous lattice, a continuous map $u:X \to Y$ is called perfect iff the inverse image map $\underline{O}(u):\underline{O}(Y) \to \underline{O}(X)$ preserves the way below relation.

6.8 **THEOREM:**
For (T_o-)spaces X and Y for which $\underline{O}(X)$ and $\underline{O}(Y)$ are continuous lattices, a continuous perfect map $u:X \to Y$ uniquely extends to a continuous perfect map $\psi(u):\psi X \to \psi Y$ such that

$$\begin{array}{ccc} X & \xrightarrow{u} & Y \\ \psi_X \downarrow & & \downarrow \psi_Y \\ \psi X & \xrightarrow[\psi(u)]{} & \psi Y \end{array}$$

commutes.

Both $\psi(?)$ and $\underline{H}(?)$ extend to functors defined on the category of those (T_o-)spaces whose lattice of open sets is a (distributive) continuous lattice and the continuous perfect mappings. The codomain of ψ is the category of locally quasicompact strongly sober spaces and continuous perfect mappings; whereas the codomain of $\underline{H}$ is the category of compact ordered spaces and continuous isotone maps. The functors ψ and $\underline{H}$ are related by the isomorphism J of 6.5:

$$J \circ \underline{H} = \psi.$$

Both ψ and $\underline{H}$ are retractions.

Proof:

(I) Uniqueness of $\psi(u)$:

If there exists a continuous perfect map $\hat{u}:\psi X \to \psi Y$ rendering

$$\begin{array}{ccc} X & \xrightarrow{u} & Y \\ \psi_X \downarrow & & \downarrow \psi_Y \\ \psi X & \xrightarrow{\hat{u}} & \psi Y \end{array}$$

commutative, then - by 6.5 and 6.6(b) - $\hat{u}:\underline{H}(X) \to \underline{H}(Y)$ is a continuous isotone map. Since $\underline{H}(Y)$ is (compact) Hausdorff and $\psi_X[X]$ is dense in $\underline{H}(X)$, $\hat{u}$ is uniquely determined by u, i.e. there is at most one such morphism $\hat{u}$.

By a standard argument, it results from the uniqueness of $\hat{u}$ that ψ and $\underline{H}$ are functors provided that the induced morphism always exists.

When the functoriality of $\psi(?)$ is established, we may infer from the idempotency of ψX for those spaces X with $\underline{O}(X)$ a continuous lattice (4.12) that ψ is a retraction. Consequently, $\underline{H} = J^{-1} \circ \psi$ (by 6.6(b)) is - under the proviso that it is functorial - also a retraction.

(II) The proof of existence of $\psi(u):\psi X \to \psi Y$ is more subtle.

We reduce the problem to a lattice-theoretic question and transfer the solution back.

(1) First note that, since Sob is a full reflective subcategory of $\underline{T_o}$ (and Top, the category of all topological spaces and continuous maps), there is a unique map ${}^s u$ rendering

$$\begin{array}{ccc} X & \xrightarrow{u} & Y \\ \tilde{s}_X \downarrow & & \downarrow \tilde{s}_Y \\ {}^s X & \xrightarrow{{}^s u} & {}^s Y \end{array}$$

commutative. Furthermore, by the same argument, there is a splitting

$$\begin{array}{ccccc} & & {}^s X & & \\ & \overset{\tilde{s}_X}{\nearrow} & & \overset{\psi'_X}{\searrow} & \\ X & & \xrightarrow[\psi_X]{} & & \psi X \end{array}$$

(and an analogous splitting for $\psi_Y:Y \hookrightarrow \psi Y$), since ψX is sober (by 4.11).

Since $\underline{O}(\tilde{s}_X):\underline{O}({}^s X) \to \underline{O}(X)$, the inverse image map, is an isomorphism, it results (via the functoriality of $\underline{O}(?)$) that

$$\psi'_X : {}^s X \hookrightarrow \psi X$$

is equivalent to $\psi_{({}^s X)}:{}^s X \to \psi({}^s X)$, since the definition of ψX depends only on the lattice $\underline{O}(X)$.

In all this gives that we may assume without loss of generality that both X and Y are sober. Now the isomorphism $\underline{O}(?)$ between the category of locally quasicompact sober spaces and continuous maps and the category of distributive continuous lattices and those maps preserving 1, $\wedge$ and arbitrary suprema ([HL$_2$]; [C]V-5.16) "reduces" the problem to the question whether there is a map $\underline{O}(\psi Y) \to \underline{O}(\psi X)$ preserving $1, \wedge, \ll$ and arbitrary suprema rendering

$$\begin{array}{ccc} \underline{O}(\psi Y) & \dashrightarrow & \underline{O}(\psi X) \\ {\scriptstyle \underline{O}(\psi_Y)}\downarrow & & \downarrow {\scriptstyle \underline{O}(\psi_X)} \\ \underline{O}(Y) & \xrightarrow[\underline{O}(u)]{} & \underline{O}(X) \end{array}$$

commutative.

(2) It is convenient, in the following, to substitute

$$L := \underline{O}(Y),\ M := \underline{O}(X),\ f := \underline{O}(u)\ ,$$

i.e. L and M are distributive continuous lattices and $f: L \to M$ is a map preserving $1, \wedge$, arbitrary suprema and - since u is perfect - the way below relation $\ll$.

Since

$$\underline{O}(\psi Y) \cong \underline{O}(\psi^* L) \cong \mathrm{DFilt}_\sigma L$$

(the last $\cong$ noted in 4.1o) and, analogously,

$$\underline{O}(\psi X) \cong \mathrm{DFilt}_\sigma M,$$

what we actually need is a map

$$h' : \mathrm{DFilt}_\sigma L \to \mathrm{DFilt}_\sigma M$$

preserving $1, \wedge, \ll$ and arbitrary suprema and rendering a certain diagram (to be specified in (6), (7) below) commutative. Such a map is the image under D(?) of a map

$$h : \mathrm{Filt}_\sigma M \to \mathrm{Filt}_\sigma L$$

preserving $1, \wedge$ and suprema of non-empty up-directed subsets, since h' has these properties (cf. [L$_2$]§7). By [L$_2$](9.7), h'=D(h) preserves $\ll$ iff h has a left adjoint, i.e. (since the domain of h is a complete lattice) iff h preserves arbitrary infima.

(3) For distributive continuous lattices L, M and a map $f: L \to M$ preserving $\ll$, 1, $\wedge$ and arbitrary suprema, we define a mapping

$$h : \mathrm{Filt}_\sigma M \to \mathrm{Filt}_\sigma L$$

by

$$h(G) := \varphi(f^{-1}[\mathrm{int}_\sigma G])$$

for every $G \in \mathrm{Filt}_\sigma M$ - where $\mathrm{int}_\sigma G$ (or intG) denotes the interior of G with regard to the Scott topology of M and $\varphi(?)$ assigns to a subset of L the smallest filter containing it.

Recall that, for a family $(G_i)_{i \in I}$ of members of $\mathrm{Filt}_\sigma M$, we have

$$\bigwedge\{G_i \mid i \in I\} = \varphi\, \mathrm{int}_\sigma(\bigcap\{G_i \mid i \in I\})$$

and, if $(G_i)_{i \in I}$ is a non-empty and up-directed family,

$$\bigvee\{G_i \mid i \in I\} = \bigcup\{G_i \mid i \in I\}$$

(3a) For a non-empty and up-directed family $(G_i)_{i \in I}$ of members of $\mathrm{Filt}_\sigma M$ we thus have

$$\begin{aligned} h(\bigvee\{G_i \mid i \in I\}) &= h(\bigcup\{G_i \mid i \in I\}) \\ &= \varphi(f^{-1}[\mathrm{int}_\sigma(\bigcup\{G_i \mid i \in I\})]) \\ &= \varphi(f^{-1}[\bigcup\{\mathrm{int}_\sigma G_i \mid i \in I\}]) \\ &= \varphi(\bigcup\{f^{-1}[\mathrm{int}_\sigma G_i] \mid i \in I\}) \\ &= \bigvee\{\varphi(f^{-1}[\mathrm{int}_\sigma G_i]) \mid i \in I\} \\ &= \bigvee\{h(G_i) \mid i \in I\}. \end{aligned}$$

(All the occurring $\bigvee$ are suprema of non-empty up-directed subsets.)

(3b) We shall prove that $h: \mathrm{Filt}_\sigma M \to \mathrm{Filt}_\sigma L$ preserves arbitrary infima.

Note first that for a family $(G_i)_{i \in I}$ of members of $\mathrm{Filt}_\sigma K$

$$\begin{aligned} h(\bigwedge\{G_i \mid i \in I\}) &= h(\varphi(\mathrm{int}(\bigcap\{G_i \mid i \in I\}))) \\ &= \varphi(f^{-1}[\mathrm{int}(\varphi(\mathrm{int}(\bigcap\{G_i \mid i \in I\})))]) \\ &= \varphi(f^{-1}[\mathrm{int}(\bigcap\{G_i \mid i \in I\})]) \end{aligned}$$

(the last "=", since $\mathrm{int}\varphi\mathrm{int}F = \mathrm{int}F$ for every filter F of M), and

$$\begin{aligned} \bigwedge\{h(G_i) \mid i \in I\} &= \bigwedge\{\varphi(f^{-1}[\mathrm{int}G_i]) \mid i \in I\} \\ &= \varphi(\mathrm{int}(\bigcap\{\varphi(f^{-1}[\mathrm{int}G_i]) \mid i \in I\})). \end{aligned}$$

The non-trivial implication is that the latter set is contained in the first.

Suppose

$$x \in \varphi(\mathrm{int}(\bigcap\{\varphi(f^{-1}[\mathrm{int}G_i]) \mid i \in I\})).$$

Then

$$x = x_1 \wedge \ldots \wedge x_n$$

for some $x_1,\ldots,x_n \in L$ with $n \in \mathbb{N} \cup \{o\}$, such that there are $y_1,\ldots,y_n \in L$ with

$$y_k \ll x_k$$

and

$$y_k \in \bigcap\{\varphi(f^{-1}[\mathrm{int}G_i]) \mid i \in I\}$$

for every $k \in \{1,\ldots,n\}$. Since

$$\varphi(f^{-1}[\mathrm{int}G_i]) \subseteq f^{-1}[\varphi(\mathrm{int}G_i)] = f^{-1}[G_i],$$

we infer that

$$f(y_k),\ f(x_k) \in \bigcap\{G_i \mid i \in I\}$$

for every $k=1,\ldots,n$. Since f preserves the way below relation, we have

$$f(y_k) \ll f(x_k),$$

hence

$$f(x_k) \in \mathrm{int}(\bigcap\{G_i \mid i \in I\})$$

for every $k=1,\ldots,n$. Consequently,

$$x_k \in f^{-1}[\mathrm{int}(\bigcap\{G_i \mid i \in I\})].$$

It results that

$$x \in \varphi(f^{-1}[\mathrm{int}(\bigcap\{G_i \mid i \in I\})],$$

since $x = x_1 \wedge \ldots \wedge x_n$.

(4) Since

$$h : \mathrm{Filt}_\sigma M \to \mathrm{Filt}_\sigma L$$

preserves non-empty up-directed suprema and finite infima, there is an induced map

$$Dh : \mathrm{DFilt}_\sigma L \to \mathrm{DFilt}_\sigma M$$

assigning to every Scott-open filter $\underline{F}$ of $\mathrm{Filt}_\sigma L$ the set of those members of $\mathrm{Filt}_\sigma M$ which are mapped by h into $\underline{F}$. Clearly, Dh preserves non-empty up-directed suprema and finite infima. Since $h:\mathrm{Filt}_\sigma M \to \mathrm{Filt}_\sigma L$ preserves arbitrary infima, h has a left adjoint, hence - by J.D.Lawson's criterion $[L_2]$9.7 - Dh preserves the way below relation.

We want to show that Dh preserves arbitrary suprema.

We observe first that the smallest element of $\mathrm{DFilt}_\sigma L$, viz. $\{L\}$ (since L is a compact element of $\mathrm{Filt}_\sigma L$), is mapped by Dh into

$$\{M\}.$$

(If, for some $F \in \mathrm{Filt}_\sigma M$, $L=h(F)=\varphi f^{-1}\mathrm{int}F$, then $o = x_1 \wedge \ldots \wedge x_n$ and $f(x_i) \in \mathrm{int}F$ for some $x_i \in L$, $i=1,\ldots,n$. Thus $o=f(o)=f(x_1) \wedge \ldots \wedge f(x_n) \in \varphi\mathrm{int}F=F$, hence $F=M$.)

Since Dh preserves suprema of non-empty up-directed subsets, it suffices now to consider binary suprema in $\mathrm{DFilt}_\sigma L$.

(4a) Let $\underline{F},\underline{G} \in \mathrm{DFilt}_\sigma L$. We show first that

$$\underline{F} \vee \underline{G} = \{F \cap G \mid F \in \underline{F} \text{ and } G \in \underline{G}\}$$

is the supremum of $\underline{F}$ and $\underline{G}$ in $\mathrm{DFilt}_\sigma L$.

Clearly, $\{F \cap G \mid F \in \underline{F}$ and $G \in \underline{G}\}$ is stable under finite intersections in $\mathrm{Filt}_\sigma L$. If $F \in \underline{F}, G \in \underline{F}$ and $V \in \mathrm{Filt}_\sigma L$ such that $F \cap G \subseteq V$, then

$$V = V \dot{\vee} (F \cap G) = (V \dot{\vee} F) \cap (V \dot{\vee} G)$$

(where $\dot{\vee}$ denotes the binary supremum in $\mathrm{Filt}_\sigma L$), since $\mathrm{Filt}_\sigma L$ is a distributive lattice (by 2.12). Since $V \dot{\vee} F \in \underline{F}$ and $V \dot{\vee} G \in \underline{G}$, this shows that

$$\{F \cap G \mid F \in \underline{F} \text{ and } G \in \underline{G}\}$$

is a filter of $\mathrm{Filt}_\sigma L$.

It remains to show that this set is Scott-open in $\mathrm{Filt}_\sigma L$:

Since $\underline{F}$ and $\underline{G}$ are Scott-open in $\mathrm{Filt}_\sigma L$, there are $F' \in \underline{F}$ and $G' \in \underline{G}$ with

$$F' \ll F \text{ and } G' \ll G$$

in $\mathrm{Filt}_\sigma L$, i.e. (by 2.2)

$$F' \subseteq \uparrow x \subseteq F \text{ and } G' \subseteq \uparrow y \subseteq G$$

for some $x,y \in L$. It results that

$$F' \cap G' \subseteq \uparrow x \cap \uparrow y = \uparrow(x \vee y) \subseteq F \cap G,$$

i.e. $F' \cap G' \ll F \cap G$ in $\mathrm{Filt}_\sigma L$. This shows that

$$\{F \cap G \mid F \in \underline{F} \text{ and } G \in \underline{G}\}$$

is Scott-open in $\mathrm{Filt}_\sigma L$, hence it is a member of $\mathrm{DFilt}_\sigma L$. Clearly, it is the smallest member of $\mathrm{DFilt}_\sigma L$ containing both $\underline{F}$ and $\underline{G}$.

(4b) In order to show that $Dh:\mathrm{DFilt}_\sigma L \to \mathrm{DFilt}_\sigma M$ preserves binary suprema it suffices to verify the inclusion

$$Dh(\underline{F} \vee \underline{G}) \subseteq Dh(\underline{F}) \vee Dh(\underline{G}).$$

Suppose $W \in Dh(\underline{F} \vee \underline{G})$, i.e. $h(W) \in \underline{F} \vee \underline{G}$.

Then

$$h(W) = F \cap G$$

for some $F \in \underline{F}$ and some $G \in \underline{G}$.

We shall consider

$$S = \uparrow f[F] = \{z \in M \mid f(x) \le z \text{ for some } x \in F\}$$

$$T = \uparrow f[G] = \{z \in M \mid f(x) \le z \text{ for some } x \in G\}$$

We observe first that $S, T \in \mathrm{Filt}_\sigma M$, since f preserves both $\wedge$ and $\ll$. Since

$$h(S) = \varphi \operatorname{int} f^{-1} \uparrow f[F] \supseteq \varphi \operatorname{int} F = F$$

we conclude that $h(S) \in \underline{F}$, hence $S \in Dh(\underline{F})$. Analogously, we obtain $T \in Dh(\underline{G})$.

Suppose now $p \in S \cap T$. Then we have

$$f(u) \le p \quad \text{and} \quad f(v) \le p$$

for some $u \in F$ and some $v \in G$.

Thus we have

$$f(u \vee v) \le p,$$

since f preserves (finite) suprema.

Since

$$\begin{aligned} u \vee v \in F \cap G &= h(W) \\ &= \varphi \operatorname{int} f^{-1}[W] \\ &\subseteq f^{-1}[W], \end{aligned}$$

we can infer

$$f(u \vee v) \in W,$$

hence

$$p \in W.$$

In all this says that $S \cap T \subseteq W$, hence

$$W \in Dh(\underline{F}) \vee Dh(\underline{G}) .$$

(5) <u>To the mapping $f: L \to M$ which, by hypothesis, preserves $\ll$, 1, $\wedge$ and $\bigvee$ there corresponds a continuous perfect map:</u> Since f preserves $\bigvee$, and L and M are complete lattices,

f has a right adjoint $M \to L$ taking $x \in M$ into

$$\sup f^{-1}[\downarrow x].$$

This right adjoint takes meet-prime elements of M into meet-prime elements of L. Thus it defines a map

$$\mathrm{Spec}^* f : \mathrm{Spec}^* M \to \mathrm{Spec}^* L$$

which is continuous with regard to the standard topologies (cf. e.g. [C] IV-1.26). Indeed,

$$\begin{array}{ccc} \underline{O}\mathrm{Spec}^* M & \xrightarrow{\underline{O}\mathrm{Spec}^* f} & \underline{O}\mathrm{Spec}^* L \\ \downarrow & & \downarrow \\ M & \xrightarrow{\quad f \quad} & L \end{array}$$

commutes, hence $\mathrm{Spec}^* f$ is a perfect map.

Likewise, $Dh: \mathrm{DFilt}_\sigma L \to \mathrm{DFilt}_\sigma M$ induces a continuous perfect map

$$\mathrm{Spec}^* Dh : \mathrm{Spec}^* \mathrm{DFilt}_\sigma M \to \mathrm{Spec}^* \mathrm{DFilt}_\sigma L$$

which assigns to a meet-prime element $\underline{F}$ of $\mathrm{DFilt}_\sigma M$ the meet-prime element

$$\sup (Dh)^{-1}[\downarrow \underline{F}]$$

of $\mathrm{DFilt}_\sigma L$ - where sup (the supremum) is taken in the complete lattice $\mathrm{DFilt}_\sigma L$.

(6) <u>We want to show that</u>

$$\begin{array}{ccc} \mathrm{Spec}^* \mathrm{DFilt}_\sigma M & \xrightarrow{\mathrm{Spec}^* Dh} & \mathrm{Spec}^* \mathrm{DFilt}_\sigma L \\ k_M \uparrow & & \uparrow k_L \\ \mathrm{Spec}^* M & \xrightarrow[\mathrm{Spec}^* f]{} & \mathrm{Spec}^* L \end{array}$$

<u>commutes</u> - where

$$k_M(x) = \{F \in \mathrm{Filt}_\sigma M \mid x \in \operatorname{int} F\}$$

for $x \in \mathrm{Spec}^* M$, and, analogously

$$k_L(y) = \{G \in \mathrm{Filt}_\sigma L \mid y \in \operatorname{int} G\}$$

for $y \in \mathrm{Spec}^* L$. Thus we have to show that, for every $x \in \mathrm{Spec}^* M$,

$$\begin{aligned} (\mathrm{Spec}^* D(h))\, k_M(x) &= \sup (Dh)^{-1}[\downarrow k_M(x)] \\ &= \sup \{\underline{G} \in \mathrm{DFilt}_\sigma L \mid Dh(\underline{G}) \subseteq k_M(x)\} \end{aligned}$$

coincides with

$$k_L(\sup f^{-1}[\downarrow x]) \quad = \quad \{F \in \mathrm{Filt}_\sigma L \mid \sup f^{-1}[\downarrow x] \in \operatorname{int} F\}.$$

Indeed, we shall establish that

$$k_L(\sup f^{-1}[\downarrow x])$$

is the greatest element of

$$\Theta := \{\underline{G} \in DFilt_\sigma L \mid Dh(\underline{G}) \subseteq k_M(x)\}.$$

Suppose first that

$$F \in Dh(k_L(supf^{-1}[\downarrow x]),$$

i.e.

$$supf^{-1}[\downarrow x] \in inth(F) .$$

Since

$$inth(F) = int\varphi intf^{-1}[F] = intf^{-1}[F],$$

this implies

$$x \geq fsupf^{-1}[\downarrow x] \in fintf^{-1}[F],$$

since f preserves suprema, and

$$fintf^{-1}[F] \subseteq intF = \{x \in F \mid x' \ll x \text{ for some } x' \in F\},$$

since f preserves $\ll$. Thus we have $x \in intF$, or, equivalently,

$$F \in k_M(x) .$$

In all, this proves that

$$k_L(supf^{-1}[\downarrow x]) \in \Theta .$$

Now suppose that

$$\underline{G} \in \Theta \text{ and } G \in \underline{G} .$$

Let

$$S := \uparrow f[G].$$

We observe first that $S \in Filt_\sigma M$.

Evidently, we have

$$G \subseteq \varphi intf^{-1}[S],$$

hence

$$h(S) = \varphi intf^{-1}[S] \in \underline{G},$$

hence

$$S \in Dh(\underline{G}) .$$

Since $\underline{G} \in \Theta$ (by hypothesis), we infer that

$$x \in intS \subseteq S$$

By the very definition of S, we may infer that

$$f(a) \leq x$$

for some $a \in G$. Since G is an upper set and $a \in f^{-1}[\downarrow x]$, it results that

$$supf^{-1}[\downarrow x] \in G.$$

This implies that

$$supf^{-1}[\downarrow x] \in intG,$$

or, equivalently,

$$G \in k_L(supf^{-1}[\downarrow x]),$$

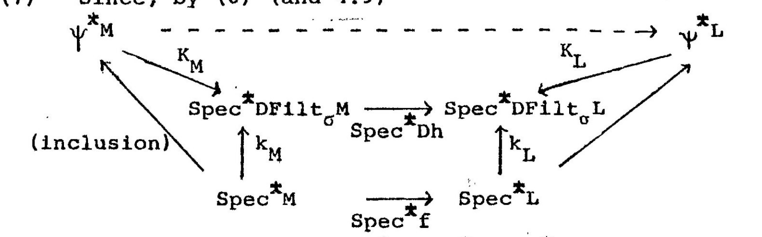
since $supf^{-1}[\downarrow x]$ is meet-prime. (A member G of $Filt_\sigma L$ containing a meet-prime element p of L must contain p in its interior, cf. 3.6).

In all this proves

$$\underline{G} \subseteq k_L(supf^{-1}[\downarrow x]),$$

whenever $\underline{G} \in \Theta$, as claimed.

(7) Since, by (6) (and 4.9)



commutes, the dotted arrow $\psi^*(u): \psi^*M \to \psi^*L$ resulting from the fact that K_M and K_L are homeomorphisms (by 4.9) is - in view of (1) and (2) above - the desired morphism $\psi^*(u): \psi^*X \hookrightarrow \psi^*Y$.

This completes the proof.

6.9 REMARK:

Possibly there is an alternative (shorter) proof for the existence in 6.8 relying on the explicit description of the Fell topology of $\underline{H}(X)$ and $\underline{H}(Y)$. When I tried to work it out, I ran into a difficulty which - possibly - cannot easily be circumvented. - Also, I hope that the techniques employed in the given proof will turn out to be useful for further research.

6.1o REMARK:

It may be noted that the functor ψ is not a reflector. Indeed the full subcategory $\underline{A}$ of locally quasi-compact strongly sober spaces of the category $\underline{B}$ of locally quasi-compact sober spaces and continuous perfect maps is not reflective in $\underline{B}$:

The one-element space 1 is the terminal object of $\underline{A}$, but it is not preserved by the embedding $\underline{A} \hookrightarrow \underline{B}$, since there is no $\underline{B}$-morphism from a non-quasicompact object of $\underline{B}$ to 1.

On the other hand, J.M.G.Fell has observed a certain universal property of $\underline{H}(X)$ ([Fe_2]p.476). However, the hypothesis employed there seems to be closer to the conclusion of 6.8 rather than to its hypothesis.

§ 7 Concluding remarks

7.1 Suppose P is a continuous poset. Let

$$e: (P, \sigma_P) \hookrightarrow (L, \sigma_L)$$

denote any representation of the injective hull of (P, σ_P) - where L is a continuous lattice (by virtue of the analysis given in [H_4]3.14). The closure of $e[P]$ in L with regard to the Lawson topology (=<u>CL</u>-topology) λ_L of L, (<u>without any topology</u>, but) endowed with the partial order inherited from L, is denoted by C. By corestriction, we obtain an order-extension

$$P \hookrightarrow C,$$

called the <u>CL-compactification</u> in [H_9]. It is shown in [H_9]2.1 that C is a continuous poset, that the Lawson topology λ_C of C is the trace of the Lawson topology λ_L of L, and that for the respective Scott topologies

$$(P, \sigma_P) \hookrightarrow (C, \sigma_C)$$

is a (topological) embedding. Furthermore, it is observed in [H_9]7.4 that

$$(C, \lambda_C) = \underline{H}(X),$$

the Fell compactification of the locally quasicompact (sober) space $X := (P, \sigma_P)$ - where, as noted in [H_9]7.5, "=" (instead of "≅") is correct if we choose $(L, \sigma_L) := \gamma(P, \sigma_P)$ (as we shall do here and in the following).

Since the Scott-open sets of a continuous poset are precisely the Lawson-open upper sets (cf.[H_9]0.5), we can infer from 6.4 (and [H_9]7.4):

7.1.1 <u>PROPOSITION:</u>

For a continuous poset P, the <u>CL</u>-compactification of P endowed with the respective Scott topologies

$$(P, \sigma_P) \hookrightarrow (C, \sigma_C)$$

is (equivalent to) the ψ-extension of (P, σ_P).

The <u>CL</u>-compactification $P \hookrightarrow C$ of a continuous poset P is a bijection iff the Lawson topology λ_P of P is compact

Hausdorff. In view of 1.13 "(3) iff (5)" we can infer

7.1.2 <u>COROLLARY</u>:
The Scott topology σ_P of a continuous poset P is strongly sober iff the Lawson topology λ_P of P is compact (Hausdorff).

7.2 For a distributive continuous lattice L, we have seen in 2.12 that

$$\mathrm{DFilt}_\sigma L \cong \mathrm{DID}(L)$$

is a complete lattice, i.e.

$$\mathrm{DID}(L) \cong \mathrm{IDID}(L) .$$

It is not unlikely that this is true for arbitrary continuous lattices L or, equivalently, that

$$\mathrm{DIDI}(P) = \mathrm{IDIDI}(P)$$

for arbitrary continuous posets P.

If this were true, then the number of non-isomorphic continuous posets which can be built up form a given continuous poset P by applying D(?) and I(?) would be <u>finite</u>. Indeed, there seems to be some evidence from examples that for every continuous poset P the following sharper formula is valid:

$$\mathrm{DIDI}(P) \cong \mathrm{IDID}(P) .$$

Let e.g.

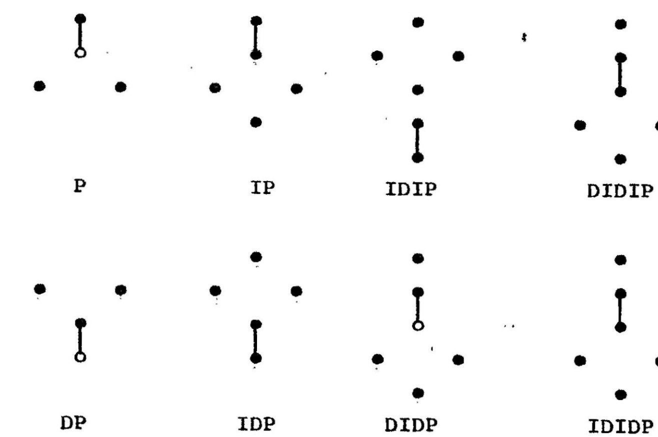
$$P = \{a,b\} \cup (o,1]$$

where (o,1], the real numbers x with $o < x \leq 1$, receives the natural order from the order $\leq$ of P and

$$a < x \quad \text{and} \quad b < x$$

for $x \in (o,1]$ are the only occurences of $\prec$ involving a or b.

We then have the following figures (where $\prec$ is realized as "strictly below", o indicates a missing point, whereas ● designates an existing point):



The construction of I(Q) relies upon the observation in $[H_{1o}]$ §1 that the convergence sets of a continuous poset Q are the Scott closures of the Frink ideals of Q: The Frink ideals of Q are easily computed, and so are their Scott closures.

7.3 Is, for arbitrary (not necessarily locally quasi-compact) spaces X,

$$\psi X$$

an idempotent construction?

7.4 Certainly desired is an external characterization of the ψ-extension $X \hookrightarrow \psi X$ (as well as an intrinsic characterization).

7.5 In $[Fe_1]$ 2.2 J.M.G.Fell has given an interpretation of <u>H</u>(X) in functional-analytic terms in the special case that X is the "dual space" of a C^*-algebra A. It seems to be a natural question whether for every C^*-algebra A there exists an "associated" C^*-algebra A' such that ψX is the "dual space" of A' (and whether there exists a natural morphism between A and A' inducing, in a sense to be made precise, the extension $X \hookrightarrow \psi X$).

7.6 It has been pointed out in $[H_7]$ §8 that the "dual" (of a T_o-space with an injective hull which is in general non-sober) also plays a role in the study of ordinary Hausdorff compactifications, i.e. dense embeddings of a completely regular Hausdorff space into a compact Hausdorff space. Indeed, the analogy with the Fell compactification can be pursued further, bringing into light the role of the injective hull. (This will be explained in detail in a forthcoming memo or paper.) Also, there is an interesting question arising from this analogy (in order to pursue it still further). If the answer is non-vacuous, it will possibly induce a new viewpoint in the study of Hausdorff compactifications.

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