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ANTICIPATING LINEAR STOCHASTIC DIFFERENTIAL EQUATIONS WITH ADAPTED COEFFICIENTS

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Abstract. Stochastic differential equations with adapted integrands and initial conditions are well studied within Itô's theory. However, such a general theory is not known for corresponding equations with anticipation. We use examples to illustrate essential ideas of the Ayed–Kuo integral and techniques for dealing with anticipating stochastic differential equations. We prove the general form of the solution for a class of linear stochastic differential equations with adapted coefficients and anticipating initial condition, which in this case is an analytic function of a Wiener integral. We show that for such equations, the conditional expectation of the solution is not the same as the solution of the corresponding stochastic differential equation with the initial condition as the expectation of the original initial condition. In particular, we show that there is an extra term in the stochastic differential equation, and give the exact form of this term.

1. Introduction

Let $B(t)$, where $t \in [a, b]$, be a Brownian motion starting at 0 and let $\{\mathcal{F}_t\}$ be the filtration generated by $B(t)$, that is, $\mathcal{F}_t = \sigma\{B(s); a \leq s \leq t\}$. In the framework of Itô's calculus, a stochastic differential equation

$$\begin{cases} dX(t) = \alpha(t, X(t))\, dB(t) + \beta(t, X(t))\, dt, \quad t \in [a, b], \\ X(a) = \xi, \end{cases}$$

with the initial condition ξ being $\mathcal{F}_a$-measurable, is a symbolical representation of the stochastic integral equation

$$X(t) = \xi + \int_a^t b(s, X(s))\, ds + \int_a^t \sigma(s, X(s))\; dB(s), \quad t \in [a, b],$$

where $\int_a^t \sigma(s, X(s))\; dB(s)$ is defined as an Itô integral. In Itô's framework, we require both the coefficients $b(t, x, \omega)$ and $\sigma(t, x, \omega)$ to be adapted apart from usual integrability constraints, and and the initial condition ξ to be measurable with respect to the initial σ-algebra $\mathcal{F}_a$. The question of how the stochastic integral can be defined when any of these quantities are not adapted (called *anticipating*) has been an open question in the field of stochastic analysis for past decades.

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There have been numerous approaches for solving this problem, a few of which are through the expansion of filtration, white-noise theory, Skorokhod integral, and numerous others. For a brief overview of these, see [7].

In 2008, Ayed and Kuo[1], gave a new definition of integrating anticipating stochastic processes. This naturally led to the question of the corresponding stochastic differential equations. The anticipation in a stochastic differential equation can come from the initial condition or from the coefficients. Let us look at two simple examples of such stochastic differential equations.

Example 1.1. The anticipation in the stochastic differential equation

$$\begin{cases} dZ(t) = Z(t)\,dB(t), \quad t \in [a,b], \\ Z(a) = B(b), \end{cases}$$

arises solely from the initial condition since $B(b)$ is not measurable with respect to $\mathcal{F}_a$.

Example 1.2. The anticipation in the stochastic differential equation

$$\begin{cases} dZ(t) = B(b)Z(t)\,dB(t), \quad t \in [a,b], \\ Z(a) = 1, \end{cases}$$

comes purely from the coefficient.

The paper is organized as follows. In section 2, we introduce the relevant notation and the background theory for defining the general integral. This allows us to assign meaning to anticipating stochastic differential equations. Section 3 contains motivating examples of anticipating stochastic differential equations. In section 4, we find the general solution of linear stochastic differential equations with adapted coefficients and anticipating initial conditions. In section 5, we show a relation between the solution found in section 4 and its conditional expectation.

2. The Ayed–Kuo Stochastic Integral

From here on, we fix an interval $[a,b]$ and assume $t \in [a,b]$ unless otherwise specified. We also fix a Brownian motion $B(t)$ and a filtration $\{\mathcal{F}_t\}$ satisfying the following conditions:

(i) $B(t)$ is adapted to $\{\mathcal{F}_t\}$.
(ii) For any $t \leq s$, $B(s) - B(t)$ and $\mathcal{F}_t$ are independent.

A stochastic process $\phi(t)$ is called *instantly independent* with respect to $\{\mathcal{F}_t\}$ if for each $t \in [a,b]$, the random variable $\phi(t)$ and the σ-algebra $\mathcal{F}_t$ are independent.

The new *stochastic integral* of a stochastic process $\Phi(t)$ introduced in [1] is defined in the following three steps.

(1) Suppose $f(t)$ is an $\mathcal{F}_t$-adapted continuous stochastic process and $\phi(t)$ be an continuous stochastic processes that is instantly independent with respect to $\mathcal{F}_t$. Then the stochastic integral of $\Phi(t) = f(t)\phi(t)$ is defined by

$$\int_a^b f(t)\phi(t)\,dB(t) = \lim_{\|\Delta_n\| \to 0} \sum_{j=1}^n f(t_{j-1})\phi(t_j)(B(t_j) - B(t_{j-1})),$$

provided that the limit exists in probability.

(2) For a process of the form $\Phi(t) = \sum_{i=1}^{n} f_i(t)\phi_i(t)$, the stochastic integral is defined by

$$\int_a^b \Phi(t)\, dB(t) = \sum_{i=1}^{n} \int_a^b f_i(t)\phi_i(t)\, dB(t).$$

(3) Let $\Phi(t)$ be a stochastic process such that there is a sequence $(\Phi_n(t))_{n=1}^{\infty}$ of stochastic processes of the form in step 2 satisfying
   (a) $\int_a^b |\Phi_n(t) - \Phi(t)|^2\, dt \to 0$ as $n \to \infty$, and
   (b) $\int_a^b \Phi_n(t)\, dB(t)$ converges in probability as $n \to \infty$.
   Then the stochastic integral of $\Phi(t)$ is defined by

$$\int_a^b \Phi(t)\, dB(t) = \lim_{n\to\infty} \int_a^b \Phi_n(t)\, dB(t) \quad \text{in probability.}$$

This integral is well defined, as demonstrated in Lemma 2.1 of [3].

Example 2.1 (Equation (1.6) in [1])**.** By writing $B(1) = B(t) + (B(1) - B(t))$ and then following step 1 in the above definition, we obtain

$$\int_0^t B(1)\, dB(s) = B(1)B(t) - t, \quad 0 \le t \le 1. \tag{2.1}$$

Now, we look at an extension of Itô's formula that can also account for instantly independent processes. First, Let X_t and $Y^{(t)}$ be stochastic processes of the form

$$X_t = X_a + \int_a^t g(s)\, dB(s) + \int_a^t h(s)\, ds, \tag{2.2}$$

$$Y^{(t)} = Y^{(b)} + \int_t^b \xi(s)\, dB(s) + \int_t^b \eta(s)\, ds, \tag{2.3}$$

where $g(t), h(t)$ are adapted (so X_t is an Itô process), and $\xi(t), \eta(t)$ are instantly independent such that $Y^{(t)}$ is also instantly independent.

Theorem 2.2 (Theorem 3.2 of [3])**.** *Suppose $\{X_t^{(i)}\}_{i=1}^n$ and $\{Y_j^{(t)}\}_{j=1}^m$ are stochastic processes of the form given by equations* (2.2) *and* (2.3)*, respectively. Suppose $\theta(t, x_1, \ldots, x_n, y_1, \ldots, y_m)$ is a real-valued function that is C^1 in t and C^2 in other variables. Then the stochastic differential of $\theta(t, X_t^{(1)}, \ldots, X_t^{(n)}, Y_1^{(t)}, \ldots, Y_m^{(t)})$ is given by*

$$\begin{aligned} &d\theta(t, X_t^{(1)}, \ldots, X_t^{(n)}, Y_1^{(t)}, \ldots, Y_m^{(t)}) \\ &= \theta_t\, dt + \sum_{i=1}^{n} \theta_{x_i} dX_t^{(i)} + \sum_{j=1}^{m} \theta_{y_j} dY_j^{(t)} \\ &\quad + \frac{1}{2} \sum_{i,k=1}^{n} \theta_{x_i x_k} dX_t^{(i)} dX_t^{(k)} - \frac{1}{2} \sum_{j,l=1}^{m} \theta_{y_j y_l} dY_j^{(t)} dY_l^{(t)}. \end{aligned}$$

Now, we come to an important class of processes that occur ubiquitously in solutions of stochastic differential equations.

Definition 2.3. The *exponential process* associated with adapted stochastic processes $\alpha(t)$ and $\beta(t)$ is defined as

$$\mathcal{E}_{\alpha,\beta}(t) = \exp\left[\int_a^t \alpha(s)\,dB(s) + \int_a^t \left(\beta(s) - \frac{1}{2}\alpha(s)^2\right)\,ds\right]. \tag{2.4}$$

If $\beta \equiv 0$, then we write

$$\mathcal{E}_{\alpha}(t) = \exp\left[\int_a^t \alpha(s)\,dB(s) - \frac{1}{2}\int_a^t \alpha(s)^2\,ds\right].$$

Remark 2.4. The exponential process $\mathcal{E}_{\alpha,\beta}(t)$ is an Itô process satisfying the stochastic differential equation

$$\begin{cases} d\mathcal{E}_{\alpha,\beta}(t) = \alpha(t)\mathcal{E}_{\alpha,\beta}(t)\,dB(t) + \beta(t)\mathcal{E}_{\alpha,\beta}(t)\,dt, & t \in [a,b], \\ \mathcal{E}_{\alpha,\beta}(a) = 1. \end{cases} \tag{2.5}$$

Similarly, the exponential process $\mathcal{E}_{\alpha}(t)$ is an Itô process satisfying the stochastic differential equation

$$\begin{cases} d\mathcal{E}_{\alpha}(t) = \alpha(t)\mathcal{E}_{\alpha}(t)\,dB(t), & t \in [a,b], \\ \mathcal{E}_{\alpha}(a) = 1. \end{cases} \tag{2.6}$$

The proof of the result follows from a direct application of Itô's formula.

3. Motivating Examples of Anticipating SDEs

We use examples to demonstrate the non-trivial nature of the extension regardless of the origin of the anticipation. These serve as motivations for our main results. In this section, we fix $t \in [0,1]$.

3.1. Anticipation due to coefficients. In the following examples, we progressively increase the complexity of the diffusion coefficient of the stochastic differential equation and see its effect on the solution. This will help us to develop our intuition about the non-trivial nature of the results related to anticipating coefficients.

Example 3.1. Let α be a constant. The process

$$\mathcal{E}_{\alpha}(t) = \exp\left[\alpha B(t) - \frac{1}{2}\alpha^2 t\right], \quad t \in [0,1]$$

is a solution of the stochastic differential equation

$$\begin{cases} d\mathcal{E}_{\alpha}(t) = \alpha\mathcal{E}_{\alpha}(t)dB(t), & t \in [0,1], \\ \mathcal{E}_{\alpha}(0) = 1. \end{cases}$$

Example 3.2. Suppose $\alpha(t)$ is a deterministic function. The process

$$\mathcal{E}_{\alpha}(t) = \exp\left[\int_0^t \alpha(s)\,dB(s) - \frac{1}{2}\int_0^t \alpha(s)^2\,ds\right], \quad t \in [0,1]$$

is a solution of the stochastic differential equation

$$\begin{cases} d\mathcal{E}_{\alpha}(t) = \alpha(t)\mathcal{E}_{\alpha}(t)dB(t), & t \in [0,1], \\ \mathcal{E}_{\alpha}(0) = 1. \end{cases}$$

Example 3.3. Consider the adapted coefficient $\alpha(t) = B(t)$. The process

$$X(t) = \exp\left[\frac{1}{2}\left(B(t)^2 - t - \int_0^t B(s)^2\,ds\right)\right], \quad t \in [0,1]$$

is a solution of the stochastic differential equation

$$\begin{cases} dX(t) = B(t)X(t)\,dB(t), & t \in [0,1], \\ X(0) = 1. \end{cases}$$

From the above examples and equation (2.1), one might guess that the process

$$\begin{aligned} Z(t) &= \exp\left[\int_0^t B(1)\,dB(s) - \frac{1}{2}\int_0^t B(1)^2\,ds\right] \\ &= \exp\left[B(1)B(t) - t - \frac{1}{2}B(1)^2 t\right], \qquad t \in [0,1] \end{aligned}$$

is a solution of the stochastic differential equation

$$\begin{cases} dZ(t) = B(1)Z(t)\,dB(t), & t \in [0,1], \\ Z(0) = 1. \end{cases}$$

But this is not true. In fact, we can apply the generalized Itô formula to derive the following result.

Theorem 3.4 (Theorem 3.3 of [2]). *The stochastic process*

$$Z(t) = \exp\left[B(1)B(t) - t - \frac{1}{2}B(1)^2 t\right]$$

is a solution of

$$\begin{cases} dZ(t) = B(1)Z(t)\,dB(t) + B(1)(B(t) - tB(1))Z(t)dt, & t \in [0,1], \\ Z(0) = 1. \end{cases}$$

Then what is the solution of the following stochastic differential equation?

$$\begin{cases} dZ(t) = B(1)Z(t)\,dB(t), & t \in [0,1], \\ Z(0) = 1. \end{cases}$$

The answer is given by the following theorem.

Theorem 3.5 (Theorem 3.1 of [2]). *The process*

$$Z(t) = \exp\left[B(1)\int_0^t e^{-(t-s)}\,dB(s) - \frac{1}{4}B_1^2(1 - e^{-2t}) - t\right], \quad t \in [0,1]$$

is a solution of the stochastic differential equation

$$\begin{cases} dZ(t) = B(1)Z(t)\,dB(t), & t \in [0,1], \\ Z(0) = 1. \end{cases}$$

The above examples demonstrate the non-trivial nature of anticipating coefficients.

3.2. Anticipation due to initial condition. We start our discussion on stochastic differential equations with anticipating initial conditions with the following example.

Example 3.6 (Examples 4.1-3 of [1])**.**

$$\begin{cases} dX(t) = X(t)dB(t), & t \in [0,1], \\ X(0) = x, & x \in \mathbb{R}. \end{cases} \tag{3.1}$$

It is well known that the solution to (3.1) is

$$X(t) = xe^{B(t)-\frac{1}{2}t}$$

However, if we take this solution and replace x with $B(1)$ and apply the generalized Itô formula to the resultant expression, we obtain a different stochastic differential equation. In particular,

$$Y(t) = B(1)e^{B(t)-\frac{1}{2}t} \tag{3.2}$$

is a solution of

$$\begin{cases} dY(t) = Y(t)dB(t) + \dfrac{1}{B(1)}Y(t)dt, & t \in [0,1], \\ Y(0) = B(1). \end{cases} \tag{3.3}$$

Here, the initial condition is outside the classical theory of Itô calculus since B(1) is not $\mathcal{F}_0$-measurable. We can use the generalized Itô formula along with the Picard iteration method to show that $Y(t)$ is indeed the unique solution.

On the other hand, if we replace all the $B(1)$ terms in (3.3) with $x \in \mathbb{R}$ then we obtain the following stochastic differential equation

$$\begin{cases} dZ(t) = Z(t)dB(t) + \dfrac{1}{x}Z(t)dt, & t \in [0,1], \\ Z(0) = x, & x \in \mathbb{R}. \end{cases}$$

with its solution

$$Z(t) = xe^{B(t)-\frac{1}{2}t+\frac{1}{x}t}$$

The differences in (3.1) and (3.3) demonstrates that replacing the non anticipating term in the solution with an anticipating term yields an extra drift term in the SDE. Furthermore, by replacing all the anticipating terms in (3.3) with a real number, we obtained an extra drift factor in (3.2). These examples highlights some of the differences and interesting patterns between adapted linear stochastic differential equations and non-adapted ones.

Example 3.7 (Section 3 of [5])**.** Consider the following motivational example:

$$\begin{cases} dX(t) = X(t)dB(t), & t \in [0,1], \\ X(0) = B(1). \end{cases}$$

Equation (3.1) would suggest that our solution would be (3.2). However, that is not the case. We have an extra drift term as demonstrated by (3.3). With that in mind, we "guess" that the solution has the form

$$X(t) = (B(1) - \xi(t))\, e^{B(t)-\frac{1}{2}t}$$

with ξ being a deterministic function that needs to be determined. Via a simple application of the generalized Itô formula to the function $\theta(t,x,y) = (y-\xi(t))e^{x-\frac{1}{2}t}$, we get that

$$dX(t) = (B(1)-\xi(t))e^{B(t)-\frac{1}{2}t}\, dB(t) + \left[e^{B(t)-\frac{1}{2}t} - \xi'(t)e^{B(t)-\frac{1}{2}t}\right] dt.$$

The dt term in the above equation must be zero for $X(t)$ to be a solution. Therefore, by solving the following ordinary differential equation

$$\begin{cases} \xi'(t) = 1\,, \quad t \in [0,1], \\ \xi(0) = 0, \end{cases}$$

we get our solution

$$X(t) = (B(1)-t)e^{B(t)-\frac{1}{2}t}.$$

This is the inspiration for the following theorem that provides solutions for a class of stochastic differential equations with anticipating initial conditions.

Theorem 3.8 (Theorem 5.1 of [8])**.** *Let $\alpha(t)$, $h(t) \in L^2[0,1]$, $\beta(t) \in L^1[0,1]$. Assume that $\psi(t)$ is a C^2 function. Then the unique solution of the stochastic differential equation*

$$\begin{cases} dX(t) = \alpha(t)X(t)dB(t) + \beta(t)X(t)\, dt, \quad t \in [0,1], \\ X(0) = \psi\left(\int_0^1 h(s)dB(s)\right), \end{cases}$$

is given by the equation

$$X(t) = \psi\left(\int_0^1 h(s)dB(s) - \int_0^t \alpha(s)h(s)ds\right)\mathcal{E}_{\alpha,\beta}(t),$$

where $\mathcal{E}_{\alpha,\beta}(t)$ is the stochastic process defined in equation (2.4)*.*

Remark 3.9. In Theorem 4.1 of [5], the authors proved a similar result for the particular case where $h(t) \equiv 1$ and ψ is a function on $\mathbb{R}$ having power series expansion at $t=0$ with infinite radius of convergence. In Theorem 3.8 and in [5], $\alpha(t)$ is assumed to be deterministic.

Example 3.10 (Example 5.2 of [8])**.** Consider the stochastic differential equation

$$\begin{cases} dX(t) = X(t)dB(t), \quad t \in [0,1], \\ X(0) = \int_0^1 B(s)ds. \end{cases}$$

We can use stochastic integration by parts and the results of Theorem 3.8 to obtain the solution,

$$X(t) = \left(\int_0^1 B(s)ds - (t - \frac{1}{2}t^2)\right)e^{B(t)-\frac{1}{2}t}.$$

Thus we have solutions for a class of linear stochastic differential equations with deterministic coefficients.

4. Anticipating Stochastic Differential Equations

Motivated by the earlier examples and theorems, we turn to the main result of this paper. In Theorem 3.8 , we had assumed that $\alpha(t) \in L^2[0,1]$ and $\beta(t) \in L^1[0,1]$. In the following theorem, we generalize that condition to allow both $\alpha(t)$ and $\beta(t)$ to be adapted to the filtration generated by the Brownian motion.

Hypothesis 4.1. Assume that $\alpha(t)$, $\beta(t)$ and $h(t)$, where $t \in [a,b]$, have the following properties:

(1) $\alpha(t)$ is an adapted process with $\mathbb{E}\left(\int_a^b |\alpha(t)|^2 \, dt\right) < \infty$,
(2) $\beta(t)$ is an adapted process with $\mathbb{E}\left(\int_a^b |\beta(t)| \, dt\right) < \infty$,
(3) $h(t) \in L^2[a,b]$ is a deterministic function.

Theorem 4.2. *Let $\alpha(t)$, $\beta(t)$, and $h(t)$ satisfy Hypothesis 4.1, and $\psi \in C^2(\mathbb{R})$. Then the solution of the stochastic differential equation*

$$\begin{cases} dZ(t) = \alpha(t)Z(t)\, dB(t) + \beta(t)Z(t)\, dt, \quad t \in [a,b], \\ Z(0) = \psi\Big(\displaystyle\int_a^b h(s)\, dB(s)\Big), \end{cases} \tag{4.1}$$

is given by

$$Z(t) = \psi\Big(\int_a^b h(s)\, dB(s) - \int_a^t h(s)\alpha(s)\, ds\Big)\mathcal{E}_{\alpha,\beta}(t). \tag{4.2}$$

Proof. Suppose $Z(t) = \psi\big(\int_a^b h(s)\, dB(s) - Q(t)\big)\mathcal{E}_{\alpha,\beta}(t)$. We need to determine the Itô process $Q(t)$ with $Q(a) = 0$. In order to apply the generalized Itô formula, we write

$$Z(t) = \psi\Big(\int_a^t h(s)\, dB(s) - Q(t) + \int_t^b h(s)\, dB(s)\Big)\mathcal{E}_{\alpha,\beta}(t). \tag{4.3}$$

We define the instantly independent process $Y^{(t)} = \int_t^b h(s)\, dB(s)$ and the following adapted processes

$$X_t^{(1)} = \mathcal{E}_{\alpha,\beta}(t), \quad X_t^{(2)} = \int_a^t h(s)\, dB(s) - Q(t).$$

From the definitions of $X_t^{(1)}$, $X_t^{(2)}$, and $Y^{(t)}$ above, we get the differentials

$$\begin{aligned} dX_t^{(1)} &= \alpha(t)X_t^{(1)}\, dB(t) + \beta(t)X_t^{(1)}\, dt, \\ dX_t^{(2)} &= h(t)\, dB(t) - dQ(t), \\ (dX_t^{(1)})^2 &= \alpha(t)^2(X_t^{(1)})^2\, dt, \\ (dX_t^{(2)})^2 &= h(t)^2\, dt - 2h(t)\, dB(t)dQ(t) + (dQ(t))^2, \\ dX_t^{(1)}dX_t^{(2)} &= h(t)\alpha(t)X_t^{(1)}\, dt - \alpha(t)X_t^{(1)}\, dB(t)dQ(t), \\ dY^{(t)} &= -h(t)\, dB(t), \\ (dY^{(t)})^2 &= h(t)^2\, dt. \end{aligned}$$

Now, define $\theta(x_1, x_2, y) = \psi(x_2 + y)x_1$, so that $Z(t) = \theta\left(X_t^{(1)}, X_t^{(2)}, Y^{(t)}\right)$. From this, we get the partial derivatives

$$\begin{aligned} \theta_{x_1} &= \psi, & \theta_{x_1x_1} &= 0, \\ \theta_{x_2} &= \psi' x_1, & \theta_{x_2x_2} &= \psi'' x_1, \\ \theta_y &= \psi' x_1, & \theta_{x_1x_2} &= \psi', \\ \theta_{yy} &= \psi'' x_1. \end{aligned}$$

Applying Theorem 2.2 and putting everything together, we can easily find the stochastic differential of $Z(t)$:

$$\begin{aligned} dZ(t) &= d\theta(X_t^{(1)}, X_t^{(2)}, Y^{(t)}) \\ &= \theta_{x_1} dX_t^{(1)} + \theta_{x_2} dX_t^{(2)} \\ &\quad + \frac{1}{2}\theta_{x_1x_1}(dX_t^{(1)})^2 + \frac{1}{2}\theta_{x_2x_2}(dX_t^{(2)})^2 \\ &\quad + \theta_{x_1x_2}(dX_t^{(1)})(dX_t^{(2)}) \\ &\quad + \theta_y\, dY^{(t)} - \frac{1}{2}\theta_{yy}(dY^{(t)})^2 \\ &= \psi \cdot \Big(\alpha(t)X_t^{(1)}\, dB(t) + \beta(t)X_t^{(1)}\, dt\Big) + \psi' \cdot X_t^{(1)} \left(\cancel{h(t)\, dB(t)} - dQ(t)\right) \\ &\quad + 0 + \frac{1}{2}\psi'' \cdot X_t^{(1)} \left(\cancel{h(t)^2\, dt} - 2h(t)\, dB(t)dQ(t) + (dQ(t))^2\right) \\ &\quad + \psi' \Big(h(t)\alpha(t)X_t^{(1)}\, dt - \alpha(t)X_t^{(1)}\, dB(t)dQ(t)\Big) \\ &\quad - \cancel{\psi' \cdot X_t^{(1)} h(t)\, dB(t)} - \cancel{\frac{1}{2}\psi'' \cdot X_t^{(1)} h(t)^2\, dt} \\ &= \psi \cdot \Big(\alpha(t)X_t^{(1)}\, dB(t) + \beta(t)X_t^{(1)}\, dt\Big) \\ &\quad + \psi' \cdot X_t^{(1)} \cdot (-dQ(t) + h(t)\alpha(t)\, dt - \alpha(t)\, dB(t)dQ(t)) \\ &\quad + \frac{1}{2}\psi'' \cdot X_t^{(1)} \cdot \left(-2h(t)\, dB(t)dQ(t) + (dQ(t))^2\right). \end{aligned}$$

Therefore, in order for $Z(t)$ to be the solution of equation (4.1), we need to satisfy the following conditions

$$dQ(t) = h(t)\alpha(t)\, dt - \alpha(t)\, dB(t)dQ(t) \tag{4.4}$$

$$(dQ(t))^2 = 2h(t)\, dB(t)dQ(t) \tag{4.5}$$

From equation (4.4), we see that if $dQ(t)$ contains only a dt term (no $dB(t)$ term), then $dQ(t)\, dB(t) = 0$. On the other hand, if $dQ(t)$ contains a $dB(t)$ term, then $dQ(t)\, dB(t) = \gamma(t)\, dt$ for some $\gamma(t)$. Then we have $dQ(t) = (h(t) - \gamma(t))\alpha(t)\, dt$, which again gives $dQ(t)\, dB(t) = 0$. Therefore, in either case, $dQ(t) = h(t)\alpha(t)\, dt$. Note that this also agrees with equation (4.5).

Imposing the initial condition $Q(a) = 0$, we get that $Q(t) = \int_a^t h(t)\alpha(t)\, dt$. Putting this in the assumed form of the solution, we get our result. □

Now we look at a specific case of Theorem 4.2 where $h(t) \equiv 1$.

Corollary 4.3. *Under the same assumptions for $\alpha(t)$, $\beta(t)$ and ψ as in Theorem 4.2, the solution of the stochastic differential equation*

$$\begin{cases} dZ(t) = \alpha(t)Z(t)\,dB(t) + \beta(t)Z(t)\,dt, \quad t \in [a,b], \\ Z(0) = \psi\Big(B(b) - B(a)\Big), \end{cases} \tag{4.6}$$

is given by

$$Z(t) = \psi\Big(B(b) - B(a) - \int_a^t \alpha(s)\,ds\Big)\mathcal{E}_{\alpha,\beta}(t).$$

Remark 4.4. This corollary extends Theorem 4.1 of [5] to include adapted coefficients for the anticipating stochastic differential equation.

We apply these new results to obtain solutions for some examples of stochastic differential equations with anticipating initial conditions and adapted coefficients. In the first example, the diffusion and drift terms are adapted while the anticipation comes from $X(0) = B(1)$. The second example demonstrates a case where the initial condition is a Riemann integral of a Brownian motion.

Example 4.5. Consider the stochastic differential equation

$$\begin{cases} dX(t) = B(t)X(t)dB(t) + X(t)\,dt, \quad t \in [0,1], \\ X(0) = B(1). \end{cases}$$

Here $\alpha(t) = B(t)$, $\beta(t) \equiv 1, h(t) \equiv 1$, and $\psi(x) = x$. Thus, by Corollary 4.3, we have the solution

$$X(t) = \left(B(1) - \int_0^t B(s)\,ds\right)\exp\left[\frac{1}{2}\left(B^2(t) + t - \int_0^t B^2(s)\,ds\right)\right].$$

Example 4.6. Consider the stochastic differential equation

$$\begin{cases} dX(t) = B(t)X(t)\,dB(t), \quad t \in [0,1], \\ X(0) = \displaystyle\int_0^1 B(s)\,ds. \end{cases}$$

As in Example 3.10, we use stochastic integration by parts to modify the initial condition. Namely,

$$\int_0^1 B(s)\,ds = \int_0^1 (1-s)\,dB(s).$$

Hence with $\alpha(t) = B(t)$, $\beta(t) \equiv 0$, $h(t) = 1 - t$, and $\psi(x) = x$ in Theorem 4.2, we have the solution

$$X(t) = \left(\int_0^1 B(s)\,ds - \int_0^t (1-s)B(s)\,ds\right)\exp\left[\frac{1}{2}\left(B(t)^2 - t - \int_0^t B(s)^2\,ds\right)\right].$$

5. Conditional Expectation of Solutions of SDEs

Given a stochastic process $Z(t)$, its conditional expectation plays a key role in the theory of the generalized stochastic integral [4]. It allows us to project our anticipating stochastic differential equation into the realm of classical Itô theory. As such, analysis of $X(t) = E\left(Z(t)|\mathcal{F}_t\right)$ is of particular interest as it provides a better lens for understanding the anticipating nature of the process itself. It is natural to ask which stochastic differential equation would $X(t)$ satisfy? How different are $dX(t)$ and $dZ(t)$? With that motivation, we show the following result.

Theorem 5.1. *Let $\alpha(t)$, $\beta(t)$, and $h(t)$ satisfy Hypothesis 4.1, and ψ an analytic function on the reals. Suppose that $Z_1(t)$ and $Z_2(t)$ are the solutions of the linear stochastic differential equations*

$$\begin{cases} dZ_1(t) = \alpha(t)Z_1(t)\,dB(t) + \beta(t)Z_1(t)\,dt, \quad t \in [a,b], \\ Z_1(a) = \psi\Big(\displaystyle\int_a^b h(s)\,dB(s)\Big), \end{cases} \tag{5.1}$$

and

$$\begin{cases} dZ_2(t) = \alpha(t)Z_2(t)\,dB(t) + \beta(t)Z_2(t)\,dt, \quad t \in [a,b], \\ Z_2(a) = \psi'\Big(\displaystyle\int_a^b h(s)\,dB(s)\Big), \end{cases} \tag{5.2}$$

respectively. Let $X_1(t) = \mathbb{E}\left(Z_1(t)|\mathcal{F}_t\right)$ and $X_2(t) = \mathbb{E}\left(Z_2(t)|\mathcal{F}_t\right)$. Then $X_1(t)$ satisfies the stochastic differential equation

$$\begin{cases} dX_1(t) = \alpha(t)X_1(t)\,dB(t) + \beta(t)X_1(t)\,dt + h(t)X_2(t)\,dB(t), \quad t \in [a,b], \\ X_1(a) = \mathbb{E}\Big[\psi\Big(\displaystyle\int_a^b h(s)\,dB(s)\Big)\Big]. \end{cases} \tag{5.3}$$

Remark 5.2. In Theorem 4.1 of [8], the authors proved a similar result for the special case where α is deterministic, β is adapted, and $h \equiv 1$.

Proof. By the assumption and Theorem 4.2, the solution processes $Z_1(t)$ can be written as

$$\begin{aligned} Z_1(t) &= \mathcal{E}_{\alpha,\beta}(t) \cdot \psi\left(\Big(\int_a^t h(s)\,dB(s) - \int_a^t h(s)\alpha(s)\,ds\Big) + \int_t^b h(s)\,dB(s)\right) \\ &= \mathcal{E}_{\alpha,\beta}(t) \cdot \sum_{k=0}^{\infty} \frac{1}{k!}\psi^{(k)}\Big(\int_a^t h(s)\,dB(s) - \int_a^t h(s)\alpha(s)\,ds\Big)\Big(\int_t^b h(s)\,dB(s)\Big)^k. \end{aligned}$$

For brevity, we henceforth denote

$$\psi_k(t) = \psi^{(k)}\Big(\int_a^t h(s)\,dB(s) - \int_a^t h(s)\alpha(s)\,ds\Big) \tag{5.4}$$

In this notation, the expression for $Z_1(t)$ becomes

$$Z_1(t) = \mathcal{E}_{\alpha,\beta}(t) \cdot \sum_{k=0}^{\infty} \frac{1}{k!}\psi_k(t)\Big(\int_t^b h(s)\,dB(s)\Big)^k. \tag{5.5}$$

Note that $\mathcal{E}_{\alpha,\beta}(t)$ and $\psi_k(t)$ are adapted for all k. Moreover, since $h(t)$ is deterministic, $\int_t^b h(s)\,dB(s)$ is a Wiener integral, and therefore, $\int_t^b h(s)\,dB(s)$ has the Gaussian distribution with mean 0 and variance

$$V(t) = \int_t^b h(s)^2\,ds. \tag{5.6}$$

Therefore, for any k, we have $\mathbb{E}\left[\left(\int_t^b h(s)\,dB(s)\right)^{2k+1}\right] = 0$ and

$$\mathbb{E}\left[\left(\int_t^b h(s)\,dB(s)\right)^{2k}\right] = V(t)^k(2k-1)!!,$$

where !! denotes the double factorial defined as

$$n!! = \prod_{k=0}^{\lceil \frac{n}{2} \rceil}(n-2k)$$

for any natural number n.

Moreover, $\int_t^b h(s)\,dB(s)$ is independent of $\mathcal{F}_t$ for every t. Using all of these information, we get

$$\begin{aligned} X_1(t) &= \mathcal{E}_{\alpha,\beta}(t)\cdot\sum_{k=0}^{\infty}\frac{1}{(2k)!}\psi_{2k}(t)\ \mathbb{E}\left[\left(\int_t^b h(s)\,dB(s)\right)^{2k}\right] \\ &= \mathcal{E}_{\alpha,\beta}(t)\cdot\sum_{k=0}^{\infty}\frac{1}{(2k)!}\psi_{2k}(t)V(t)^k(2k-1)!! \\ &= \mathcal{E}_{\alpha,\beta}(t)\cdot\sum_{k=0}^{\infty}\frac{1}{(2k)!!}\psi_{2k}(t)V(t)^k, \end{aligned} \tag{5.7}$$

and similarly,

$$X_2(t) = \mathcal{E}_{\alpha,\beta}(t)\cdot\sum_{k=0}^{\infty}\frac{1}{(2k)!!}\psi_{2k+1}(t)V(t)^k, \tag{5.8}$$

Now we look at the differentials. From equations (5.4) and (5.6), we get

$$d\left(V(t)^k\right) = kV(t)^{k-1}\,(-h(t)^2\,dt),$$

and

$$\begin{aligned} d\psi_{2k}(t) &= \psi_{2k+1}(t)(h(t)\,dB(t) - h(t)\alpha(t)\,dt) + \frac{1}{2}\psi_{2k+2}(t)(h(t)^2\,dt) \\ &= \psi_{2k+1}(t)h(t)\,dB(t) + \left(\frac{1}{2}\psi_{2k+2}(t)h(t)^2 - \psi_{2k+1}(t)h(t)\alpha(t)\right)dt. \end{aligned}$$

Using the expressions for $d\left(V(t)^k\right)$ and $d\psi_{2k}(t)$, and Remark 2.4, we get

$$\begin{aligned}
&d\left(\mathcal{E}_{\alpha,\beta}(t)\psi_{2k}(t)V(t)^k\right)\\
&= \psi_{2k}(t)V(t)^k\, d\mathcal{E}_{\alpha,\beta}(t) + \mathcal{E}_{\alpha,\beta}(t)V(t)^k\, d\psi_{2k}(t) + \mathcal{E}_{\alpha,\beta}(t)\psi_{2k}(t)\, dV(t)^k\\
&\quad + \mathcal{E}_{\alpha,\beta}(t)\, d\psi_{2k}(t)\cdot dV(t)^k + \psi_{2k}(t)\, d\mathcal{E}_{\alpha,\beta}(t)\cdot dV(t)^k + V(t)^k\, d\mathcal{E}_{\alpha,\beta}(t)\cdot d\psi_{2k}(t)\\
&= \psi_{2k}(t)V(t)^k\Big(\alpha(t)\mathcal{E}_{\alpha,\beta}(t)\, dB(t) + \beta(t)\mathcal{E}_{\alpha,\beta}(t)\, dt\Big)\\
&\quad + \mathcal{E}_{\alpha,\beta}(t)V(t)^k\Big[\psi_{2k+1}(t)h(t)\, dB(t) + \Big(\frac{1}{2}\psi_{2k+2}(t)h(t)^2 - \cancel{\psi_{2k+1}(t)h(t)\alpha(t)}\Big)dt\Big]\\
&\quad + \mathcal{E}_{\alpha,\beta}(t)\psi_{2k}(t)\left(-kV(t)^{k-1}h(t)^2\, dt\right)\\
&\quad + 0 + 0 + V(t)^k\,\cancel{\Big(\mathcal{E}_{\alpha,\beta}(t)\psi_{2k+1}(t)\alpha(t)h(t)\Big)}\, dt\\
&= \mathcal{E}_{\alpha,\beta}(t)V(t)^k\left(\psi_{2k}(t)\alpha(t) + \psi_{2k+1}(t)h(t)\right)\, dB(t)\\
&\quad + \mathcal{E}_{\alpha,\beta}(t)V(t)^{k-1}\Big(\psi_{2k}(t)V(t)\beta(t) + \frac{1}{2}\psi_{2k+2}(t)V(t)h(t)^2 - k\psi_{2k}(t)h(t)^2\Big)\, dt.
\end{aligned}$$

At this point, we note that

$$\begin{aligned}
\sum_{k=0}^{\infty}\frac{1}{(2k)!!}k\psi_{2k}(t) &= \sum_{k=1}^{\infty}\frac{1}{(2k)(2k-2)!!}k\psi_{2k}(t)\\
&= \frac{1}{2}\sum_{k-1=0}^{\infty}\frac{1}{(2(k-1))!!}\psi_{2(k-1)+2}(t)\\
&= \frac{1}{2}\sum_{k=0}^{\infty}\frac{1}{(2k)!!}\psi_{2k+2}(t). \qquad (5.9)
\end{aligned}$$

Now, since $X_1(t) = \sum_{k=0}^{\infty}\frac{1}{(2k)!!}\mathcal{E}_{\alpha,\beta}(t)\psi_{2k}(t)V(t)^k$ (see equation (5.7)), we get

$$\begin{aligned}
dX_1(t) &= \sum_{k=0}^{\infty}\frac{1}{(2k)!!}d\left(\mathcal{E}_{\alpha,\beta}(t)\psi_{2k}(t)V(t)^k\right)\\
&= \sum_{k=0}^{\infty}\frac{1}{(2k)!!}\mathcal{E}_{\alpha,\beta}(t)V(t)^k\psi_{2k}(t)\alpha(t)\, dB(t)\\
&\quad + \sum_{k=0}^{\infty}\frac{1}{(2k)!!}\mathcal{E}_{\alpha,\beta}(t)V(t)^k\psi_{2k+1}(t)h(t)\, dB(t)\\
&\quad + \sum_{k=0}^{\infty}\frac{1}{(2k)!!}\mathcal{E}_{\alpha,\beta}(t)V(t)^k\psi_{2k}(t)\beta(t)\, dt\\
&\quad + \cancel{\sum_{k=0}^{\infty}\frac{1}{(2k)!!}\frac{1}{2}\mathcal{E}_{\alpha,\beta}(t)V(t)^k\psi_{2k+2}(t)h(t)^2\, dt}\\
&\quad - \cancel{\sum_{k=0}^{\infty}\frac{1}{(2k)!!}k\mathcal{E}_{\alpha,\beta}(t)V(t)^{k-1}\psi_{2k}(t)h(t)^2\, dt}\\
&= \alpha(t)X_1(t)\, dB(t) + h(t)X_2(t)\, dB(t) + \beta(t)X_1(t)\, dB(t),
\end{aligned}$$

where, in the second step, we used the result of equation (5.9). This completes the proof of the theorem. □

The presence of the extra term in the conditional stochastic differential equation in (5.3) poses an interesting question. Note that the stochastic differential equation for $X_1(t)$ is defined via $X_2(t)$. However, $X_2(t)$ is defined in equation (5.8) as an infinite series and a closed form is not guaranteed. It is important to note that $X_2(t)$ arose from taking the first derivative of ψ as the initial condition. Similarly, we can use the second derivative of ψ as the initial condition to arrive at $X_3(t)$. Then, the results of Theorem 5.1 would provide the link between $dX_2(t)$ to $dX_3(t)$. This way, we can form an infinite chain by using the infinite derivatives of the analytic function ψ as initial conditions. However, there is again no guarantee of a closed form. Nevertheless, since we know that the derivative of the exponential function is itself, we have the following example.

Example 5.3. Let $\alpha(t)$, $\beta(t)$, and $h(t)$ satisfy Hypothesis 4.1, and let $\psi(x) = e^x$. In this case, $\psi \equiv \psi'$, so $Z_1(t) \equiv Z_2(t)$. Consequently, $X_1(t) = X_2(t)$, which we call $X(t)$ for convenience. Then by Theorem 5.1,

$$X(t) = \mathcal{E}_{\alpha,\beta}(t) \exp\Big(\int_a^t h(s)\, dB(s) - \int_a^t h(s)\alpha(s)\, ds \Big), \quad t \in [a, b], \tag{5.10}$$

and $X(t)$ satisfies the stochastic differential equation

$$\begin{cases} dX(t) = (\alpha(t) + h(t))X(t)\, dB(t) + \beta(t)X(t)\, dt, \\ X(a) = 1. \end{cases} \tag{5.11}$$

In general, the absence of a closed form does not pose significant problems. Recall that the scaled Hermite polynomials $\left\{ \frac{1}{\sqrt{n!\rho^n}} H_n(x; \rho) \right\}$ form an orthonormal basis for the space $L^2(\mathbb{R}, \gamma)$, where γ is the Gaussian measure with mean 0 and variance ρ. Therefore, if we are able to arrive at a closed form reformulation of Theorem 5.1 for Hermite polynomials, we can use this to state the result for conditional expectation of the solution when the initial condition is any $L^2(\mathbb{R}, \gamma)$-function of a Wiener integral. In what follows, we derive such a result.

Recall that the Hermite polynomial of degree n with parameter ρ defined by

$$H_n(x; \rho) = (-\rho)^n e^{\frac{x^2}{2\rho}} D_x^n e^{-\frac{x^2}{2\rho}},$$

where D_x is the differentiation operator with respect to the variable x. From page 334 of [6], we state the following identities:

$$D_x H_n(x; \rho) = n H_{n-1}(x; \rho) \tag{5.12}$$

$$D_\rho H_n(x; \rho) = -\frac{1}{2} D_x^2 H_n(x; \rho) \tag{5.13}$$

$$H_n(x + y; \rho) = \sum_{k=0}^{n} \binom{n}{k} H_{n-k}(x; \rho) y^k \tag{5.14}$$

We use these facts to prove the following lemma.

Lemma 5.4. *The stochastic process* $X(t) = H_n\Big(\int_a^t h(s)\,dB(s); \int_a^t h(s)^2\,ds\Big)$ *with* $h(t) \in L^2[a,b]$ *is a martingale with respect to the filtration generated by the Brownian motion* $B(t)$ *and*

$$dX(t) = nH_{n-1}\Big(\int_a^t h(s)\,dB(s); \int_a^t h(s)^2\,ds\Big)h(t)\,dB(t) \tag{5.15}$$

Proof. Here $x = \int_a^t h(s)\,dB(s)$ and $\rho = \int_a^t h(s)^2\,ds$. So we have $dx = h(t)\,dB(t)$ and $d\rho = h(t)^2\,dt$, and $(dx)^2 = d\rho$. Using Itô's formula, we get

$$\begin{aligned} dX(t) &= D_x H_n(x;\rho)dx + \cancel{\frac{1}{2}D_x^2 H_n(x;\rho)(dx)^2} + \cancel{D_\rho H_n(x;\rho)d\rho} \\ &= nH_{n-1}\Big(\int_a^t h(s)\,dB(s); \int_a^t h(s)^2\,ds\Big)h(t)\,dB(t), \end{aligned}$$

where we used equation (5.13) for the cancellation and (5.12) to get the final term. □

This leads to the following result.

Theorem 5.5. *Let* $\alpha(t)$*,* $\beta(t)$*, and* $h(t)$ *satisfy Hypothesis 4.1. For a fixed* $n \geq 1$*, suppose* $Z(t)$ *is the solution of the linear stochastic differential equation*

$$\begin{cases} dZ(t) = \alpha(t)Z(t)\,dB(t) + \beta(t)Z(t)\,dt, \qquad t \in [a,b], \\ Z(a) = H_n\Big(\int_a^b h(s)\,dB(s); \int_a^b h(s)^2\,ds\Big). \end{cases} \tag{5.16}$$

Then $X(t) = \mathbb{E}\left(Z(t)|\mathcal{F}_t\right)$ *is given by*

$$X(t) = H_n\Big(\int_a^t h(s)\,dB(s) - \int_a^t h(s)\alpha(s)\,ds; \int_a^t h(s)^2\,ds\Big)\mathcal{E}_{\alpha,\beta}(t), \quad t \in [a,b]. \tag{5.17}$$

Moreover, $X(t)$ *satisfies the following stochastic differential equation*

$$\begin{cases} dX(t) = \alpha(t)X(t)\,dB(t) + \beta(t)X(t)\,dt \\ \qquad + nH_{n-1}\Big(\int_a^t h(s)\,dB(s) - \int_a^t h(s)\alpha(s)\,ds; \int_a^t h(s)^2\,ds\Big)\mathcal{E}_{\alpha,\beta}(t)h(t)\,dB(t) \\ X(a) = 0. \end{cases} \tag{5.18}$$

Remark 5.6. For any x and ρ, we have $H_0(x;\rho) = 1$. Hence the stochastic differential equation (5.16) is identically equation (2.5).

Proof. We first prove equation (5.17). Using Theorem 4.2 and equation (5.14), we can write

$$\begin{aligned}
Z(t) &= \mathcal{E}_{\alpha,\beta}(t) H_n\Big(\int_a^b h(s)\,dB(s) - \int_a^t h(s)\alpha(s)\,ds; \int_a^b h(s)^2\,ds\Big) \\
&= \mathcal{E}_{\alpha,\beta}(t) \sum_{k=0}^{n} \binom{n}{k} H_{n-k}\Big(\int_a^b h(s)\,dB(s); \int_a^b h(s)^2\,ds\Big)\Big(-\int_a^t h(s)\alpha(s)\,ds\Big)^k \\
&= \mathcal{E}_{\alpha,\beta}(t) \sum_{k=0}^{n} \binom{n}{k} J_{n-k}(b)\Big(-\int_a^t h(s)\alpha(s)\,ds\Big)^k,
\end{aligned}$$

where we used the notation

$$J_n(t) = H_n\Big(\int_a^t h(s)\,dB(s); \int_a^t h(s)^2\,ds\Big). \tag{5.19}$$

Using Lemma 5.4, we get $\mathbb{E}\left(J_{n-k}(b)|\mathcal{F}_t\right) = J_{n-k}(t)$. Taking the conditional expectation with the knowledge that $\mathcal{E}_{\alpha,\beta}(t)$ is adapted and that stochastic integrals of adapted processes are adapted,

$$\begin{aligned}
X(t) &= \mathcal{E}_{\alpha,\beta}(t) \sum_{k=0}^{n} \binom{n}{k} \mathbb{E}\left(J_{n-k}(b)|\mathcal{F}_t\right)\Big(-\int_a^t h(s)\alpha(s)\,ds\Big)^k \\
&= \mathcal{E}_{\alpha,\beta}(t) \sum_{k=0}^{n} \binom{n}{k} J_{n-k}(t)\Big(-\int_a^t h(s)\alpha(s)\,ds\Big)^k \\
&= \mathcal{E}_{\alpha,\beta}(t) H_n\Big(\int_a^t h(s)\,dB(s) - \int_a^t h(s)\alpha(s)\,ds; \int_a^t h(s)^2\,ds\Big),
\end{aligned}$$

which proves equation (5.17).

Since $H_n(0;0) = 0$, we see that $X(a) = 0$. Using Itô's formula and equation (5.15),

$$\begin{aligned}
dH_n &= dH_n\Big(\int_a^t h(s)\,dB(s) - \int_a^t h(s)\alpha(s)\,ds; \int_a^t h(s)^2\,ds\Big) \\
&= D_x H_n \cdot (h(t)\,dB(t) - h(t)\alpha(t)\,dt) \\
&\qquad + \cancel{\frac{1}{2} D_x^2 H_n \cdot (h(t)^2\,dt)} + \cancel{D_\rho H_n \cdot (h(t)^2\,dt)} \\
&= nH_{n-1} \cdot h(t)(dB(t) - \alpha(t)\,dt).
\end{aligned}$$

Finally, using equation (5.17), we get

$$\begin{aligned}
dX(t) &= H_n\,\mathcal{E}_{\alpha,\beta}(t) + \mathcal{E}_{\alpha,\beta}(t)\,dH_n + d\mathcal{E}_{\alpha,\beta}(t) \cdot dH_n \\
&= H_n \mathcal{E}_{\alpha,\beta}(t)(\alpha(t)\,dB(t) + \beta(t)\,dt) \\
&\quad + \mathcal{E}_{\alpha,\beta}(t) n H_{n-1} \cdot h(t)(dB(t) - \cancel{\alpha(t)}\,dt) + \cancel{\mathcal{E}_{\alpha,\beta}(t)\alpha(t) n H_{n-1} h(t)}\,dt. \\
&= \alpha(t)X(t)\,dB(t) + \beta(t)X(t)\,dt + nH_{n-1}h(t)\mathcal{E}_{\alpha,\beta}(t)\,dB(t),
\end{aligned}$$

which gives us equation (5.18). □

In Equation (5.18), we specify an explicit form of the extra term in the stochastic differential equation for the conditioned process $X(t)$. We use this result in the following examples.

Example 5.7. Consider the stochastic differential equation

$$\begin{cases} dZ(t) = B(t)Z(t)dB(t), \quad t \in [0,1], \\ Z(0) = B(1). \end{cases}$$

Here $\alpha(t) = B(t)$, $\beta(t) \equiv 0$, $h \equiv 1$, and $B(1) = H_1(B(1);1)$. From Theorem 5.5,

$$X(t) = E\left(Z(t)|\mathcal{F}_t\right) = \left(B(t) - \int_0^t B(s)ds\right) \exp\left[\frac{1}{2}\left(B(t)^2 - t - \int_0^t B(s)^2\,ds\right)\right]$$

and $X(t)$ satisfies the following stochastic differential equation

$$\begin{cases} dX(t) = \left\{B(t)X(t) + \exp\left[\frac{1}{2}\left(B(t)^2 - t - \int_0^t B(s)^2\,ds\right)\right]\right\}dB(t), \quad t \in [0,1], \\ X(0) = 0. \end{cases}$$

Example 5.8. Consider the stochastic differential equation

$$\begin{cases} dZ(t) = B(t)Z(t)dB(t), \quad t \in [0,1], \\ Z(0) = B(1)^2 - 1. \end{cases}$$

From Theorem 5.5,

$$X(t) = \left[\left(B(t) - \int_0^t B(s)ds\right)^2 - t\right] \exp\left[\frac{1}{2}\left(B(t)^2 - t - \int_0^t B(s)^2\,ds\right)\right],$$

and for $t \in [0,1]$, $X(t)$ satisfies the following stochastic differential equation.

$$\begin{cases} dX(t) = \left\{B(t)X(t) + 2\exp\left[\frac{1}{2}\left(B(t)^2 - t - \int_0^t B(s)^2\,ds\right)\right]\right\}dB(t), \\ X(0) = 0. \end{cases}$$

References

1. Ayed, W. and Kuo, H.-H.: An extension of the Itô integral, *Communications on Stochastic Analysis* **2**, no. 3 (2008), 323–333.
2. Hwang, C.-R., Kuo, H.-H., and Saitô, K.: Anticipating Exponential Processes and Stochastic Differential Equations, *Communications on Stochastic Analysis* **13**, no. 3 (2019), Article 9 (12 pages).
3. Hwang, C.-R., Kuo, H.-H., Saitô, K., and Zhai, J.: A general Itô formula for adapted and instantly independent stochastic processes, *Communications on Stochastic Analysis* **10**, no. 3 (2016), 341–362.
4. Hwang, C.-R., Kuo, H.-H., Saitô, K, and Zhai, J: Near-martingale property of anticipating stochastic integration, *Communications on Stochastic Analysis* **11**, no. 4 (2017), 491–504.
5. Khalifa, N., Kuo, H.-H., Ouerdiane, H., and Szozda, B.: Linear stochastic differential equations with anticipating initial conditions, *Communications on Stochastic Analysis* **7**, no. 2 (2013), 245–253.
6. Kuo, H.-H.: *White Noise Distribution Theory*, CRC Press, 1996.
7. Kuo, H.-H.: The Itô calculus and white noise theory: a brief survey toward general stochastic integration, *Communications on Stochastic Analysis* **8**, no. 1 (2014), 111–139.
8. Kuo, H.-H., Sinha, S, and Zhai, J: Stochastic Differential Equations with Anticipating Initial Conditions, *Communications on Stochastic Analysis* **12**, no. 4 (2018), 473–485.

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