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# AN EXTENSION OF THE ITÔ INTEGRAL*

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ABSTRACT. We introduce the concept of instant independence for certain anticipating stochastic processes and take the class of instantly independent stochastic processes as a counterpart of adapted stochastic processes for the Itô theory of stochastic integration. Then we define the stochastic integral of a stochastic process which is a linear combination of the products of instantly independent and adapted stochastic processes. The crucial idea is to use the right endpoints as the evaluation points for the instantly independent factors, while the left endpoints are used for the adapted factors. We prove a special case of Itô's formula for this new stochastic integral and present some examples of stochastic differential equations.

## 1. Introduction

Let $B(t), t \geq 0$, be a Brownian motion and let $\mathcal{F}_t$ be the $\sigma$-field generated by $\{B(s); 0 \leq s \leq t\}$. The well-known Itô integral is a stochastic integral

$$I(f) = \int_a^b f(t)\, dB(t)$$

defined for a stochastic process $f(t)$ being adapted to the filtration $\{\mathcal{F}_t; t \geq 0\}$ and $\int_a^b |f(t)|^2\, dt < \infty$ almost surely. In particular, when $E\int_a^b |f(t)|^2\, dt < \infty$, we have the following equalities

$$E\big(I(f)\big) = 0, \quad E\big(|I(f)|^2\big) = E\int_a^b |f(t)|^2\, dt. \tag{1.1}$$

See, for instance, chapters 4 and 5 of the book [6] among many standard books on stochastic integration.

In the 1976 International Symposium on Stochastic Differential Equations at Kyoto, K. Itô raised the question on how to define the stochastic integral

$$\int_0^t B(1)\, dB(s), \quad 0 \leq t \leq 1, \tag{1.2}$$

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* A few days after this paper was written, we learned the extremely sad news that Professor Kiyosi Itô passed away at 9:26 am, November 10, 2008. The ideas in this paper are influenced by his 1976 Kyoto lecture. It is with the deepest sorrow and fondest memory that we publish this paper in memory of Professor Kiyosi Itô.

which is not an Itô integral since the integrand $B(1)$ is not adapted to the filtration $\{\mathcal{F}_t; t \geq 0\}$. Itô's idea in [4] is to enlarge the filtration by letting $\mathcal{G}_t$ be the $\sigma$-field generated by $\mathcal{F}_t$ and $B(1)$. Then the integrand $B(1)$ is obviously adapted to the filtration $\{\mathcal{G}_t; t \geq 0\}$. However, $B(t)$ is no more a Brownian motion with respect to this enlarged filtration since $B(t) - B(s)$ and $\mathcal{G}_s$ are not independent for any $s \leq t$. But $B(t)$ is a $\{\mathcal{G}_t\}$-quasimartingale with the decomposition

$$B(t) = M(t) + A(t), \quad 0 \leq t \leq 1,$$

where $A(t) = \int_0^t \frac{B(1)-B(u)}{1-u}\, du$ and $M(t) = B(t) - A(t)$. Hence the stochastic integral in Equation (1.2) can be defined as one with respect to a quasimartingale $B(t)$ and it turns out that

$$\int_0^t B(1)\, dB(s) = B(1)B(t), \quad 0 \leq t \leq 1. \tag{1.3}$$

In 1972 and 1978 Hitsuda [2][3] defined a stochastic integral for non-adapted integrands and obtained a special case of Itô's formula for anticipating stochastic integrals. His idea was used by Skorokhod [10] in 1975 to define such stochastic integral via homogeneous chaos expansion. Their stochastic integral can be defined in terms of the adjoint of white noise differentiation, see chapter 13 of the book [5]. In the recent years there have been many papers dealing with anticipating stochastic integrals and related applications, see, e.g, [1][7][8][9].

The purpose of this short paper is to present a new idea which is more in line with Itô's methods to define a stochastic integral for adapted integrands on the one hand, and for anticipating integrands on the other hand, namely,

(1) we keep the left endpoint as the evaluation point for the adapted part as in the Itô integral. But we use the right endpoint as the evaluation point for the anticipating part of an integrand.
(2) we keep the Brownian motion $B(t)$ as an integrator, but we decompose an integrand into a sum of stochastic processes with special properties (to be specified in Section 2).

Let us use the example $\int_0^t B(1)\, dB(s)$ to illustrate our new idea. Decompose the integrand $B(1)$ into the sum

$$B(1) = \big(B(1) - B(s)\big) + B(s).$$

The integral of the second part, i.e., $\int_0^t B(s)\, dB(s)$ is an Itô integral given by

$$\int_0^t B(s)\, dB(s) = \frac{1}{2}\big(B(t)^2 - t\big). \tag{1.4}$$

On the other hand, the first part $B(1) - B(s)$ is anticipating. We define its integral as the following limit in probability,

$$\int_0^t \big(B(1) - B(s)\big)\, dB(s) = \lim_{\|\Delta\| \to 0} \sum_{i=1}^n \big(B(1) - B(s_i)\big)\big(B(s_i) - B(s_{i-1})\big),$$

where $\Delta = \{s_0, s_1, s_2, \ldots, s_n\}$ is a partition of the interval $[0, t]$ with $s_0 = 0$ and $s_n = t$. Note that we have taken the right endpoint $s_i$ of the subinterval $[s_{i-1}, s_i]$

as the evaluation point for the integrand. Then we can find the limit

$$\begin{aligned}
&\sum_{i=1}^{n} \big(B(1)-B(s_i)\big)\big(B(s_i)-B(s_{i-1})\big) \\
&\quad = B(1)\sum_{i=1}^{n} \big(B(s_i)-B(s_{i-1})\big) - \sum_{i=1}^{n} B(s_i)\big(B(s_i)-B(s_{i-1})\big) \\
&\quad = B(1)B(t) - \Big\{ \sum_{i=1}^{n} \big(B(s_i)-B(s_{i-1})\big)^2 + \sum_{i=1}^{n} B(s_{i-1})\big(B(s_i)-B(s_{i-1})\big) \Big\} \\
&\quad \to B(1)B(t) - \Big\{ t + \frac{1}{2}\big(B(t)^2 - t\big) \Big\}.
\end{aligned}$$

Therefore, we have the stochastic integral

$$\int_0^t \big(B(1)-B(s)\big)\, dB(s) = B(1)B(t) - \frac{1}{2}\big(B(t)^2 + t\big), \quad 0 \le t \le 1. \tag{1.5}$$

It follows from Equations (1.4) and (1.5) that

$$\int_0^t B(1)\, dB(s) = B(1)B(t) - t, \quad 0 \le t \le 1, \tag{1.6}$$

which is clearly different from the right-hand side of Equation (1.3) given by Itô. In particular, our integral has expectation 0, while the one given by Itô has nonzero expectation. It seems to us that zero expectation is more in line with the first equality in Equation (1.1).

We will use the above idea to define a new stochastic integral. Then we will derive a special case of Itô's formula for this integral and solve a simple stochastic differential equation involving anticipating initial condition and integrands. The general case will be explored in later papers.

## 2. Stochastic Integral

From now on we fix a Brownian motion $B(t)$ and a filtration $\{\mathcal{F}_t\}$ satisfying the conditions:

(a) For each $t$, $B(t)$ is $\mathcal{F}_t$-measurable;

(b) For any $s \le t$, $B(t) - B(s)$ and $\mathcal{F}_s$ are independent.

Although both the integrands $B(1)-B(t)$ and $B(1)$ in Equations (1.5) and (1.6) are anticipating, there is an intrinsic difference between them, namely, $B(1)-B(t)$ is independent of the $\sigma$-field $\mathcal{F}_t$ while $B(1)$ is not. This is the motivation for the following definition.

**Definition 2.1.** A stochastic process $\varphi(t), a \le t \le b$, is said to be *instantly independent* with respect to $\{\mathcal{F}_t\}$ if $\varphi(t)$ and $\mathcal{F}_t$ are independent for each $t$.

Thus the stochastic process $B(1) - B(t), 0 \le t \le 1$, is instantly independent, while $B(1), 0 \le t \le 1$, is not. Moreover, note that if $\varphi(t)$ is adapted and instantly independent, then $\varphi(t)$ must be a deterministic function. In this sense, we can view instantly independent stochastic processes as a counterpart of the adapted stochastic processes for the Itô integral.

In order to explain our new idea in a simple way and also being guided by Theorem 4.7.1 of the book [6], we make the following definition.

**Definition 2.2.** Let $f(t), a \leq t \leq b$, be adapted and let $\varphi(t), a \leq t \leq b$, be instantly independent. Define the stochastic integral of $f(t)\varphi(t)$ by

$$I(f\varphi) = \int_a^b f(t)\varphi(t)\, dB(t) = \lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} f(t_{i-1})\varphi(t_i)\big(B(t_i) - B(t_{i-1})\big) \tag{2.1}$$

provided that the limit in probability exists.

Observe that in Equation (2.1), the evaluation points for $f(t)$ and $\varphi(t)$ on $[t_{i-1}, t_i]$ are the left and right endpoints, respectively. When $\varphi(t) = 1$, this new stochastic integral reduces to the Itô integral. Moreover, for each $i$, we have

$$\begin{aligned} &E\big\{f(t_{i-1})\varphi(t_i)\big(B(t_i) - B(t_{i-1})\big)\big\} \\ &\quad = E\big\{\mathcal{E}\big[f(t_{i-1})\varphi(t_i)\big(B(t_i) - B(t_{i-1})\big) \,|\, \mathcal{F}_{t_i}\big]\big\} \\ &\quad = E\big\{f(t_{i-1})\big(B(t_i) - B(t_{i-1})\big)\mathcal{E}\big[\varphi(t_i) \,|\, \mathcal{F}_{t_i}\big]\big\}. \end{aligned}$$

Note that $\mathcal{E}\big[\varphi(t_i) \,|\, \mathcal{F}_{t_i}\big] = E\big(\varphi(t_i)\big)$ since $\varphi(t)$ is instantly independent. Hence

$$\begin{aligned} &E\big\{f(t_{i-1})\varphi(t_i)\big(B(t_i) - B(t_{i-1})\big)\big\} \\ &\quad = E\big(\varphi(t_i)\big)\, E\big\{f(t_{i-1})\big(B(t_i) - B(t_{i-1})\big)\big\} \\ &\quad = E\big(\varphi(t_i)\big)\, E\big\{\mathcal{E}\big[f(t_{i-1})\big(B(t_i) - B(t_{i-1})\big) \,|\, \mathcal{F}_{t_{i-1}}\big]\big\} \\ &\quad = E\big(\varphi(t_i)\big)\, E\big\{f(t_{i-1})\,\mathcal{E}\big[B(t_i) - B(t_{i-1}) \,|\, \mathcal{F}_{t_{i-1}}\big]\big\} \\ &\quad = 0, \end{aligned}$$

since $\mathcal{E}\big[B(t_i) - B(t_{i-1}) \,|\, \mathcal{F}_{t_{i-1}}\big] = 0$. This shows that $E\big(I(f\varphi)\big) = 0$, a property that we want to keep for the new stochastic integral.

In general, suppose $F(t)$ is a stochastic process given by

$$F(t) = \sum_{k=1}^{N} f_k(t)\varphi_k(t), \tag{2.2}$$

where $f_k(t)$'s are adapted and $\varphi_k(t)$'s are instantly independent. Then we define the stochastic integral of $F(t)$ by

$$I(F) = \int_a^b F(t)\, dB(t) = \sum_{k=1}^{N} \int_a^b f_k(t)\varphi_k(t)\, dB(t).$$

It can be easily checked that $I(F)$ is well-defined, i.e., it is independent of the representation of $F$ in Equation (2.2).

**Example 2.3.** Let us evaluate the stochastic integral $\int_0^t B(1)^2\, dB(s)$ for $t \in \mathbb{R}$. First consider the case $0 \leq t \leq 1$. Express the integrand $B(1)^2$ in the form of Equation (2.2) as

$$B(1)^2 = \big(B(1) - B(s)\big)^2 + 2B(s)\big(B(1) - B(s)\big) + B(s)^2.$$

Then the integral of $B(1)^2$ is the limit in probability of the following sum

$$\sum_{i=1}^{n}\big\{\big(B(1)-B(s_i)\big)^2+2B(s_{i-1})\big(B(1)-B(s_i)\big)+B(s_{i-1})^2\big\}\big(B(s_i)-B(i_{i-1})\big),$$

which can be simplified to equal

$$B(1)^2B(t)-2B(1)\sum_{i=1}^{n}\big(B(s_i)-B(s_{i-1})\big)^2+\sum_{i=1}^{n}\big(B(s_i)-B(s_{i-1})\big)^3.$$

Note that the first and second summations converge to $t$ and 0, respectively, in $L^2(\Omega)$, hence in probability. Therefore,

$$\int_0^t B(1)^2\,dB(s)=B(1)^2B(t)-2B(1)t,\quad 0\le t\le 1.$$

When $t>1$, we have

$$\begin{aligned}\int_0^t B(1)^2\,dB(s)&=\int_0^1 B(1)^2\,dB(s)+\int_1^t B(1)^2\,dB(s)\\&=B(1)^3-2B(1)+B(1)^2\big(B(t)-B(1)\big)\\&=B(1)^2B(t)-2B(1).\end{aligned}$$

In general, for a positive integer $n$, the following formula holds:

$$\int_0^t B(1)^n\,dB(s)=\begin{cases}B(1)^nB(t)-nB(1)^{n-1}t, & \text{if } 0\le t\le 1,\\ B(1)^nB(t)-nB(1)^{n-1}, & \text{if } t>1.\end{cases}$$

**Example 2.4.** We can use the same arguments as those in the previous example to derive the following integral

$$\int_0^t B(\tfrac12)B(1)\,dB(s)=\begin{cases}B(\frac12)B(1)B(t)-tB(\frac12)-tB(1), & \text{if } 0\le t\le \frac12,\\ B(\frac12)B(1)B(t)-tB(\frac12)-\frac12B(1), & \text{if } \frac12<t\le 1,\\ B(\frac12)B(1)B(t)-B(\frac12)-\frac12B(1), & \text{if } t>1.\end{cases}$$

**Example 2.5.** Consider the integral $\int_0^t B(1)B(s)\,dB(s)$. When $0\le t\le 1$, write the integrand in the form of Equation (2.2) as

$$B(1)B(s)=B(s)\big(B(1)-B(s)\big)+B(s)^2.$$

Thus the integral is the limit in probability of the following sum

$$\sum_{i=1}^{n}\big\{B(s_{i-1})\big(B(1)-B(s_i)\big)+B(s_{i-1})^2\big\}\big(B(s_i)-B(s_{i-1})\big).$$

This sum can be easily seen to equal

$$B(1)\sum_{i=1}^{n}B(s_{i-1})\big(B(s_i)-B(s_{i-1})\big)-\sum_{i=1}^{n}B(s_{i-1})\big(B(s_i)-B(s_{i-1})\big)^2,$$

which converges in probability to $B(1)\int_0^t B(s)\,dB(s)-\int_0^t B(s)\,ds$. Therefore,

$$\int_0^t B(1)B(s)\,dB(s)=\frac12B(1)\big(B(t)^2-t\big)-\int_0^t B(s)\,ds,\quad 0\le t\le 1. \tag{2.3}$$

When $t > 1$, write the integral $\int_0^t$ as $\int_0^1 + \int_1^t$ to obtain the equality

$$\int_0^t B(1)B(s)\,dB(s) = \frac{1}{2}B(1)\big(B(t)^2 - t\big) - \int_0^1 B(s)\,ds, \quad t > 1.$$

**Example 2.6.** To evaluate the stochastic integral $\int_0^t e^{B(1)}\,dB(t)$ for $0 \le t \le 1$, write the integrand in the form of Equation (2.2) as

$$e^{B(1)} = e^{B(s)}\,e^{B(1)-B(s)}$$

so that the integral is the limit in probability of the following sum

$$\sum_{i=1}^n e^{B(s_{i-1})}\,e^{B(1)-B(s_i)}\big(B(s_i) - B(s_{i-1})\big),$$

which is equal to

$$\begin{aligned}
&e^{B(1)}\sum_{i=1}^n e^{-\big(B(s_i)-B(s_{i-1})\big)}\big(B(s_i) - B(s_{i-1})\big)\\
&= e^{B(1)}\sum_{i=1}^n \Big\{1 - \big(B(s_i) - B(s_{i-1})\big) + \frac{1}{2}\big(B(s_i) - B(s_{i-1})\big)^2\\
&\qquad + \mathrm{o}\big((B(s_i) - B(s_{i-1}))^2\big)\Big\}\big(B(s_i) - B(s_{i-1})\big)\\
&\to e^{B(1)}\big(B(t) - t\big) \quad \text{ in probability.}
\end{aligned}$$

Thus we have the stochastic integral

$$\int_0^t e^{B(1)}\,dB(t) = e^{B(1)}\big(B(t) - t\big), \quad 0 \le t \le 1.$$

Moreover, it is easy to see that

$$\int_0^t e^{B(1)}\,dB(t) = e^{B(1)}\big(B(t) - 1\big), \quad t > 1.$$

**Example 2.7.** Suppose $\psi(x)$ is a continuous function. In order to evaluate the stochastic integral $\int_0^t B(1)\psi\big(B(s)\big)\,dB(s)$ for $0 \le t \le 1$, we write the integrand in the form of Equation (2.2) as

$$B(1)\psi\big(B(s)\big) = \big(B(1) - B(s)\big)\psi\big(B(s)\big) + B(s)\psi\big(B(s)\big).$$

For convenience, let $\Delta B_i = B(s_i) - B(s_{i-1})$. Then

$$\begin{aligned}
&\sum_{i=1}^n \Big\{\big(B(1) - B(s_i)\big)\psi\big(B(s_{i-1})\big) + B(s_{i-1})\psi\big(B(s_{i-1})\big)\Big\}\Delta B_i\\
&= B(1)\sum_{i=1}^n \psi\big(B(s_{i-1})\big)\Delta B_i - \sum_{i=1}^n \psi\big(B(s_{i-1})\big)(\Delta B_i)^2\\
&\to B(1)\int_0^t \psi\big(B(s)\big)\,dB(s) - \int_0^t \psi\big(B(s)\big)\,ds,
\end{aligned}$$

which yields the following formula for $0 \le t \le 1$,

$$\int_0^t B(1)\psi\big(B(s)\big)\,dB(s) = B(1)\int_0^t \psi\big(B(s)\big)\,dB(s) - \int_0^t \psi\big(B(s)\big)\,ds. \tag{2.4}$$

Moreover, for $t > 1$, we can easily show that

$$\int_0^t B(1)\psi\big(B(s)\big)\,dB(s) = B(1)\int_0^t \psi\big(B(s)\big)\,dB(s) - \int_0^1 \psi\big(B(s)\big)\,ds.$$

## 3. Itô's Formula

We now derive a special case of Itô's formula for the new stochastic integral defined in Section 2.

**Lemma 3.1.** *Suppose $f(x)$ is a continuous function and $\varphi(x)$ is a $C^1$-function. Let $\theta(x,y) = f(x)\varphi(y-x)$. Then for each $t \in [a,T]$,*

$$\begin{aligned}&\sum_{i=1}^n \theta\big(B(s_{i-1}), B(T)\big)\big(B(s_i) - B(s_{i-1})\big)\\ &\longrightarrow \int_a^t \theta\big(B(s), B(T)\big)\,dB(s) + \int_a^t \frac{\partial\theta}{\partial y}\big(B(s), B(T)\big)\,ds\end{aligned} \tag{3.1}$$

*in probability as $\|\Delta\| \to 0$. Here $\Delta = \{s_0, s_1, \ldots, s_n\}$ is a partition of $[a,t]$ with $s_0 = a$ and $s_n = t$.*

*Proof.* For simplicity, let $\Delta B_i = B(s_i) - B(s_{i-1})$. This lemma can be informally proved as follows:

$$\begin{aligned}&\sum_{i=1}^n \theta\big(B(s_{i-1}), B(T)\big)\Delta B_i\\ &= \sum_{i=1}^n f\big(B(s_{i-1})\,\varphi\big(B(T) - B(s_{i-1})\big)\Delta B_i\\ &\approx \sum_{i=1}^n f\big(B(s_{i-1})\Big\{\varphi\big(B(T) - B(s_i)\big) + \varphi'\big(B(T) - B(s_i)\big)\Delta B_i\Big\}\Delta B_i\\ &\to \int_a^t f\big(B(s)\big)\,\varphi\big(B(T) - B(s)\big)\,dB(s) + \int_a^t f\big(B(s)\big)\,\varphi'\big(B(T) - B(s)\big)\,ds,\end{aligned}$$

which yields Equation (3.1) since $\theta(x,y) = f(x)\varphi(y-x)$. □

**Theorem 3.2.** *Let $f(x)$ and $\varphi(x)$ be $C^2$-functions and $\theta(x,y) = f(x)\varphi(y-x)$. Then the following equality holds for $a \le t \le T$,*

$$\begin{aligned}\theta\big(B(t), B(T)\big) &= \theta\big(B(a), B(T)\big) + \int_a^t \frac{\partial\theta}{\partial x}\big(B(s), B(T)\big)\,dB(s)\\ &\quad + \int_a^t \Big\{\frac{1}{2}\frac{\partial^2\theta}{\partial x^2}\big(B(s), B(T)\big) + \frac{\partial^2\theta}{\partial x\partial y}\big(B(s), B(T)\big)\Big\}\,ds.\end{aligned}$$

*Proof.* Let $\Delta = \{s_0, s_1, \ldots, s_n\}$ be a partition of $[a, t]$ with $s_0 = a$ and $s_n = t$ and let $\Delta B_i = B(s_i) - B(s_{i-1})$. Then

$$\theta\big(B(t), B(T)\big) = \sum_{i=1}^{n} \big\{\theta\big(B(s_i), B(T)\big) - \theta\big(B(s_{i-1}), B(T)\big)\big\}$$
$$\approx \sum_{i=1}^{n} \frac{\partial\theta}{\partial x}\big(B(s_{i-1}), B(T)\big)\Delta B_i + \sum_{i=1}^{n} \frac{1}{2}\frac{\partial^2\theta}{\partial x^2}\big(B(s_{i-1}), B(T)\big)(\Delta B_i)^2.$$

Apply Lemma 3.1 to the function $\frac{\partial\theta}{\partial x}$ to show that the first summation converges in probability to

$$\int_a^t \frac{\partial\theta}{\partial x}\big(B(s), B(T)\big)\, dB(s) + \int_a^t \frac{\partial^2\theta}{\partial x\partial y}\big(B(s), B(T)\big)\, ds.$$

On the other hand, the second summation converges in probability to

$$\int_a^t \frac{1}{2}\frac{\partial^2\theta}{\partial x^2}\big(B(s), B(T)\big)\, ds.$$

Put these two limits together to get the equality in the theorem. □

**Corollary 3.3.** *Let $f(x)$ and $\varphi(x)$ be $C^2$-functions and $\theta(x, y) = f(x)\varphi(y - x)$. Then the following equality holds for $t > T$,*

$$\theta\big(B(t), B(T)\big) = \theta\big(B(a), B(T)\big) + \int_a^t \frac{\partial\theta}{\partial x}\big(B(s), B(T)\big)\, dB(s)$$
$$+ \frac{1}{2}\int_a^t \frac{\partial^2\theta}{\partial x^2}\big(B(s), B(T)\big)\, ds + \int_a^T \frac{\partial^2\theta}{\partial x\partial y}\big(B(s), B(T)\big)\, ds.$$

*Proof.* Write $\theta\big(B(t), B(T)\big)$ as

$$\theta\big(B(t), B(T)\big) = \theta\big(B(T), B(T)\big) + \big\{\theta\big(B(t), B(T)\big) - \theta\big(B(T), B(T)\big)\big\}. \tag{3.2}$$

On the interval $[T, t]$, since $B(T)$ is $\mathcal{F}_T$-measurable, we can apply Itô's formula for an adapted integrand to get

$$\theta\big(B(t), B(T)\big) - \theta\big(B(T), B(T)\big)$$
$$= \int_T^t \frac{\partial\theta}{\partial x}\big(B(s), B(T)\big)\, dB(s) + \frac{1}{2}\int_T^t \frac{\partial^2\theta}{\partial x^2}\big(B(s), B(T)\big)\, ds. \tag{3.3}$$

Then put Equation (3.3) and the value of $\theta\big(B(T), B(T)\big)$ from Theorem 3.2 into Equation (3.2) to get the formula in this corollary. □

Upon comparing Theorem 3.2 and Corollary 3.3 with Theorem 13.19 in [5] we see that the stochastic integral $\int_a^b F(t)\, dB(t)$ defined in Section 2 equals the Hitsuda-Skorokhod integral $\int_a^b \partial_t^* F(t)\, dt$ (see page 259 in [5]). However, the white noise integral $\int_a^b \partial_t^* F(t)\, dt$ is in general a generalized function and obviously lacks probabilistic interpretation. Thus our stochastic integral can be viewed as a bridge connecting the Itô theory of stochastic integration and white noise theory.

The above Itô's formula can be generalized to the case involving the $t$-variable and more specified values of $B(t)$, e.g., $\theta\big(t, B(t), B(T_1), B(T_2)\big)$ with $T_1 < T_2$. In this case, the stochastic differential $d\theta\big(t, B(t), B(T_1), B(T_2)\big)$ is given by

$$d\theta\big(t, B(t), B(T_1), B(T_2)\big)$$
$$= \begin{cases} \dfrac{\partial\theta}{\partial x}\, dB(t) + \Big(\dfrac{\partial\theta}{\partial t} + \dfrac{1}{2}\dfrac{\partial^2\theta}{\partial x^2} + \dfrac{\partial^2\theta}{\partial x\partial y} + \dfrac{\partial^2\theta}{\partial x\partial z}\Big)\, dt, & \text{if } a \le t \le T_1, \\ \dfrac{\partial\theta}{\partial x}\, dB(t) + \Big(\dfrac{\partial\theta}{\partial t} + \dfrac{1}{2}\dfrac{\partial^2\theta}{\partial x^2} + \dfrac{\partial^2\theta}{\partial x\partial z}\Big)\, dt, & \text{if } T_1 < t \le T_2, \\ \dfrac{\partial\theta}{\partial x}\, dB(t) + \Big(\dfrac{\partial\theta}{\partial t} + + \dfrac{1}{2}\dfrac{\partial^2\theta}{\partial x^2}\Big)\, dt, & \text{if } t > T_2, \end{cases}$$

which can be used to derive the stochastic integral in Example 2.4.

## 4. Stochastic Differential Equations

In this section we present some linear stochastic differential equations to show the difference between the adapted and non-adapted cases.

**Example 4.1.** Consider the stochastic differential equation

$$dX_t = X_t\, dB(t) + \frac{1}{B(1)} X_t\, dt, \quad X_0 = B(1), \quad 0 \le t \le 1,$$

which means the following stochastic integral equation

$$X_t = B(1) + \int_0^t X_s\, dB(s) + \int_0^t \frac{1}{B(1)} X_s\, ds. \tag{4.1}$$

Since $B(1)$ appears as the initial condition, this equation is not of an adapted type. We shall try the iteration method to solve this equation. Let

$$X_t^{(1)} = B(1).$$

Then use Equation (1.6) to find that

$$\begin{aligned} X_t^{(2)} &= B(1) + \int_0^t X_s^{(1)}\, dB(s) + \int_0^t \frac{1}{B(1)} X_s^{(1)}\, ds \\ &= B(1)\big(1 + B(t)\big). \end{aligned}$$

Similarly, use Equation (2.3) to derive

$$\begin{aligned} X_t^{(3)} &= B(1) + \int_0^t X_s^{(2)}\, dB(s) + \int_0^t \frac{1}{B(1)} X_s^{(2)}\, ds \\ &= B(1)\Big(1 + B(t) + \frac{1}{2}B(t)^2 - \frac{1}{2}t\Big). \end{aligned}$$

We can use Equation (2.4) with $\psi(x) = x^2$ to go one more step to show that

$$X_t^{(4)} = B(1)\Big(1 + B(t) + \frac{1}{2}B(t)^2 - \frac{1}{2}t + \frac{1}{3!}B(t)^3 - \frac{1}{2}tB(t)\Big).$$

In general, we can easily see that

$$X_t^{(m)} = B(1) \sum_{n=0}^{m-1} \frac{1}{n!} H_n\big(B(t); t\big), \tag{4.2}$$

where $H_n(x;\rho)$ is the Hermite polynomial of degree $n$ with parameter $\rho$, i.e.,

$$H_n(x;\rho) = \sum_{k=0}^{[\![n/2]\!]} \binom{n}{2k} (2k-1)!!\,(-\rho)^k x^{n-2k},$$

(see, e.g., page 156 in [6].) By letting $m \to \infty$ in Equation (4.2), we get the solution $X_t$ of Equation (4.1)

$$X_t = B(1) \sum_{n=0}^{\infty} \frac{1}{n!} H_n\big(B(t); t\big). \tag{4.3}$$

But we have the well-known generating function

$$e^{tx - \frac{1}{2}\rho t^2} = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x;\rho),$$

(see, e.g., page 159 in [6].) Put $t = 1$, $x = B(t)$, and $\rho = t$ to get

$$e^{B(t) - \frac{1}{2}t} = \sum_{n=0}^{\infty} \frac{1}{n!} H_n\big(B(t); t\big). \tag{4.4}$$

From Equations (4.3) and (4.4), we conclude that $X_t$ is given by

$$X_t = B(1)\, e^{B(t) - \frac{1}{2}t}. \tag{4.5}$$

In fact, we can use Itô's formula in Theorem 3.2 to check that this stochastic process $X_t$ is indeed a solution of Equation (4.1).

**Example 4.2.** Replace $B(1)$ in Equation (4.1) by $x \in \mathbb{R}$. Then the resulting stochastic integral equation

$$Y_t = x + \int_0^t Y_s\, dB(s) + \int_0^t \frac{1}{x} Y_s\, ds$$

is an adapted equation and has the solution given by

$$Y_t = x e^{B(t) - \frac{1}{2}t + \frac{1}{x}t},$$

(see, e.g., page 233 in [6].) This stochastic process $Y_t$ and the stochastic process $X_t$ in Example 4.1 show the role played by $B(1)$.

**Example 4.3.** If we replace $B(1)$ in Equation (4.5) by $x \in \mathbb{R}$, then the resulting stochastic process

$$Z_t = x\, e^{B(t) - \frac{1}{2}t}$$

is the solution of the following adapted stochastic integral equation

$$Z_t = x + \int_0^t Z_s\, dB(s),$$

which is very different from Equation (4.1).

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