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SCS 62: Continuous Posets: Injective Hull and MacNeille Completion

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In [H] 3.14 it was shown that the continuous posets endowed with their Scott topology are precisely those sober spaces which have an injective hull in the category $\underline{T_o}$ of T_o-spaces and continuous maps (or, equivalently, in the category <u>Sob</u> of sober spaces and continuous maps). The injective hull is, on the level of the specialization order, a join-dense completion of a continuous poset into a continuous lattice. On the other hand, M.Erné ([E_1] p.54) has provided an example of a continuous poset whose MacNeille completion (object) fails to be a continuous lattice. Recall that the MacNeille completion is the smallest completion of a poset contained in every other completion. Some of the concepts used below have been introduced by M.Erné in [E_1],[E_2]:

For a poset (=partially ordered set) P the following conditions (a),(b),(c),(d),(e) are equivalent: (a) P is up-complete and for every $p \in P$, $\{x \in P | x \ll^* p\}$ is non-empty and up-directed with supremum p (where $a \ll^* b$ iff whenever the normal ideal, or "cut", generated by a Frink ideal F of P contains b, then F contains a - cf. [E_1] p.48). (b) P is a continuous poset and $a \ll^* b$ in P is equivalent to $a \ll b$. (c) P is a continuous poset and the MacNeille completion $e: P \hookrightarrow M$ (consisting of the normal ideals of P) enjoys the following properties: M is a continuous lattice and $e: (P, \sigma_P) \longrightarrow (M, \sigma_M)$ is an embedding with regard to the Scott topologies σ_P and σ_M of P and M, respectively. (d) P is a continuous poset and the injective hull $(P, \sigma_P) \hookrightarrow (L, \sigma_L)$ in the category $\underline{T_o}$ of T_o-spaces and continuous maps induces the MacNeille completion of P.

(e) P is a continuous poset, the MacNeille completion object M of P is a continuous lattice, $\ll^*$ interpolates in P (i.e. whenever $a \ll^* b$ in P, then $a \ll^* c$ and $c \ll^* b$ for some $c \in P$), and the union of any finite number of strongly Scott-closed subsets of P is strongly Scott-closed (a strongly Scott-closed subset K of P is a lower set of P such that whenever F is a Frink ideal of P contained in K, then the normal ideal generated by F in P is contained in K - cf. [E_1] p.49), [or equivalently, every Scott-closed set is strongly Scott-closed]. - By an example, due to M.Erné ([E_1] p.53) it is seen, then, that the MacNeille completion $P \hookrightarrow M$ of a continuous poset P need not be the underlying order-embedding of the injective hull of (P, σ_P) in the category $\underline{T}_o$, even if M is a continuous lattice. - On the other hand, by an example due to G.Markowsky ([M] p.3o4) it is shown that the MacNeille completion object of an up-complete poset P may be a continuous lattice, even if P fails to be a continuous poset. - Furthermore, a new representation of the injective hull of a continuous poset P (or rather of a T_o-space X with an injective hull) is given which is in terms of "minimal" Frink ideals of P (resp., of X in its specialization order), viz. minimal with regard to having the same Scott closure. (This has been instrumental in the study of the MacNeille completion of P reported above.)

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Anyone interested in a copy of the manuscript, before the above mentioned Proceedings are distributed, may write to me.

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