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GENERAL EQUILIBRIUM ASSET PRICING UNDER REGIME SWITCHING

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Abstract. We have investigated the asset pricing problem in a general equilibrium in an economy with two states. Based on the assumption of a CRRA utility function, we have derived a partial differential equation satisfied by the representative agent's cost function. In the case when the representative agent doesn't have intermediate consumption, we have found an explicit solution of the cost function. A closed-form expression for the riskless interest rate has been derived. We have also provided a partial differential equation satisfied by any contingent claim. Based on the stochastic discount factor computed, we have suggested an explanation for the equity premium puzzle.

1. Introduction

Cox, Ingersoll, and Ross (1985a) develop a continuous time general equilibrium model of an economy with a single physical good. They model the economy by a vector of state variables whose dynamics follow a system of stochastic differential equations. Based on their assumptions, they derive a partial differential equation which asset prices must satisfy.

In this paper, we also develop a continuous time general equilibrium model. In our model, there are a finite number of risky assets and a riskless asset. This assumption is more reasonable than the one single good economy in the paper of Cox, Ingersoll and Ross (1985). We assume the economy has two states, a 'good' state and a 'bad' state, (perhaps 'recession' and 'expansion'). There are two types of shocks in the economy: small shocks and large shocks. The small shocks which only affect the individual price movements are modeled by Brownian motions. The large shocks, the states of the economy, are modeled by a continuous time Markov Chain. The states of the economy affect the expected returns and the variances of the assets. We assume in different states, the means and variances of the instantaneous returns are different. This is consistent with the empirical evidence that during an economic expansion a firm's cash flows grow more quickly and remain more stable than during an economic recession, and thus, the means of individual asset returns in general are higher with lower volatilities.

We first inspect the properties of the equilibrium interest rate. In equilibrium we also determine a partial differential equation satisfied by the price of a contingent

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claim written on a risky asset. The stochastic discount factor is investigated and a closed-form expression obtained.

Many studies have investigated general equilibrium. Merton (1973) assumes there are stochastic investment opportunities and derived an intertemporal capital asset pricing model, (ICAPM). Breeden (1979) derives an Intertemporal Consumption Capital Asset Pricing Model, (ICCAPM), with uncertain consumption-goods prices and uncertain investment opportunities. We shall investigate both the ICAPM and the ICCAPM in a Markov regime switching framework in a following paper. Huang (1985) observes that with only one agent, the Cox, Ingersoll and Ross (1985) model is not a real general equilibrium model. The paper utilizes a martingale representation technique to characterize equilibrium portfolio policies. He and Leland (1993) derive necessary and sufficient conditions that must be satisfied by equilibrium asset price processes in a pure exchange economy. The results supplement those of the ICAPM by Merton (1973). Lioui and Poncet (2003) derive the general equilibrium of a dynamic financial market in which the investor's opportunity set includes nonredundant forward contracts. This has been done in the same framework as in Cox, Ingersoll and Ross (1985). More recently, Hugonnier, Morollec and Sundaresan (2005) discuss the irreversible investment problem in general equilibrium. In their framework, the paper of Cox Ingersoll and Ross (1985a) was considered as a benchmark case.

Markov regime switching is appropriate to model changes in the economy. Kazemi (1992) presents an intertemporal model of asset prices in a Markov economy with a limiting stationary distribution. By assuming that state variables follow Markov processes with limiting stationary density functions, Kazemi (1992) shows that the rate of return on a long term default free discount bond will be perfectly negatively correlated with the growth rate of the representative investor's marginal utility of consumption. Whitelaw (2000) shows that a two-regime exchange economy, estimated using consumption growth data, is able to duplicate two interesting features of the empirical relation between expected returns and volatility at the market level. Whitelaw (2000) demonstrates not only that a negative and time-varying relation between expected returns and volatility is consistent with rational expectations but also that such a relationship is consistent with aggregate consumption data in a representative agent framework.

Other authors incorporated Markov switching into general equilibrium models. For instance, Chourdakis and Tzavalis (2000) prices options in a general equilibrium framework. The paper further assumes that the dividend growth rate followed a Markov regime switching model. Then closed-form solutions of the bond price and option prices are derived. Wu and Zhen (2005) develop a general equilibrium model of the term structure of interest rates under regime switching risk framework.

We contribute to the theory in two ways: firstly, in our framework, there are a finite number of risky assets and one riskless asset. This assumption is consistent with modern finance and more reasonable for financial markets. Secondly, we derive a partial differential equation which satisfied by any contingent claims written on the basic assets. Although we are dealing with an incomplete market, we do not have to complete the market by adding assets. The unique risk neutral measure is determined by the representative agent's risk aversion in general equilibrium. This is one of the merits of general equilibrium models. Because of the properties of the Markov chain, our expressions are simpler than those of Cox, Ingersoll, and Ross (1985). This is important for practical applications of our model. Thirdly, the partial differential equation satisfied by contingent claims in our model only include terms related to the state variables of the economy, and the dynamics of the underlying assets, whereas the one in Cox, Ingersoll, and Ross (1985) also has terms related to wealth level of the agent. This is worth highlighting since the value of the derivative does not really depend on the wealth level of the representative agent, but only the economy sate and the price of the underlying asset.

The paper is organized in the following way. Section 2 presents the basic framework and assumptions. In section 3 we develop the model and investigate the properties of the presentative agent's cost function. The expression of the equilibrium interest rate is proposed. Section 4 derives the partial differential equation which the prices of any contingent claim must satisfy. In section 5, we also give the expression for the stochastic discount factor in the general equilibrium framework. Section 6 concludes the paper with several remarks and possible extensions of our results.

2. The Basic Model

Assume that there are $n+1$ assets, S_i, $0 \leq j \leq n$, in the market. The riskless asset, S_0, follows the dynamics

$$dS_0 = S_0 r_t dt,$$

where r_t is the local riskless rate of return. We shall show it is endogenously determined in equilibrium in Theorem 3.3 in the following section.

Under the physical probability measure $\mathbb{P}$, the remaining n assets follow the dynamics

$$d\mathbf{S_t} = \mathbf{D}\left[\mathbf{S_t}\right] \boldsymbol{\alpha}_t dt + \mathbf{D}\left[\mathbf{S_t}\right] \boldsymbol{\sigma}_t d\mathbf{Z_t}. \tag{2.1}$$

Here

$$\mathbf{S_t} = \begin{bmatrix} S_{1,t} \\ \vdots \\ S_{n,t} \end{bmatrix},$$

$$\mathbf{D}\left[\mathbf{S_t}\right] = \begin{bmatrix} S_{1,t} & 0 & \cdots & 0 \\ 0 & S_{2,t} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & S_{n,t} \end{bmatrix},$$

$\mathbf{Z} = \{\mathbf{Z_t}, t \geq 0\}$ is an m-dimensional $\mathbb{P}$-Brownian motion, so

$$\mathbf{Z_t} = \begin{bmatrix} Z_{1,t} \\ \vdots \\ Z_{m,t} \end{bmatrix}.$$

We suppose the economy has two states whose dynamics are modeled by a time homogeneous continuous time Markov Chain $X = \{X_t, t \geq 0\}$. The state space of

X is taken to be the two unit vectors $e_1 = (1, 0)'$, $e_2 = (0, 1)'$. We assume that X and $\mathbf{Z}$ are independent. In principle our analysis can be extended to the situation where the economy has N states.

Denote the rate matrix of X by $A = (a_{st})$, $1 \leq s, t \leq 2$. A is a $Q-$matrix. Write $a_{11} = -a_1$, and $a_{22} = -a_2$ so $A = \begin{pmatrix} -a_1 & a_2 \\ a_1 & -a_2 \end{pmatrix}$. The semimartingale representation of the Markov chain is (see Elliott (1993) for a proof)

$$X_t = X_0 + \int_0^t AX_s ds + M_t,$$

where $M = \{M_t, t \geq 0\}$ is an $\Re^2$ valued Martingale process.

We assume that the drift and volatility terms in equation (2.1) are stochastic and driven by the state of the economy, X. Therefore, there are two vectors $\boldsymbol{\alpha}(i) = (\alpha_1(i), \alpha_2(i), \cdots, \alpha_n(i))' \in \Re^2$, $i = 1, 2$, and at time t, $\boldsymbol{\alpha}_t = \sum_{i=1}^{2} \langle X_t, e_i \rangle \boldsymbol{\alpha}(i)$. Here $\langle \cdot, \cdot \rangle$ is the inner product operator. Similarly, suppose there are two volatility matrices $\boldsymbol{\sigma}_i$, $i = 1, 2$. Here, $\boldsymbol{\sigma}_i$ is an $n \times m$ matrix, and $\boldsymbol{\sigma}_t = \sum_{i=1}^{2} \langle X_t, e_i \rangle \boldsymbol{\sigma}_i$. The *j-th* row of the volatility matrix $\boldsymbol{\sigma}_i$ is the $1 \times n$ vector $\boldsymbol{\sigma}_i(j) = e_j' \boldsymbol{\sigma}_i$.

In the economy, there are a fixed number of homogeneous agents maximizing their intertemporal utility. The representative agent maximizes the cost function

$$J(W_t, X_t, t) = \sup_{\pi_t, C_t} \mathbb{E}\left[\int_t^T U(C_s, s) ds + B(W_T, T) \Big| \mathbf{S_t} = \mathbf{S}, X_t = X\right], \quad (2.2)$$

subject to the budget constraint

$$dW_t = [\boldsymbol{\pi}_t'(\boldsymbol{\alpha}_t - r_t \mathbf{1}) W_t + r_t W_t - C_t] dt + \boldsymbol{\pi}_t' \boldsymbol{\sigma}_t W_t d\mathbf{Z_t}. \quad (2.3)$$

The budget constrain means that the representative agent's wealth change equals the sum of the instantaneous changes of the value of the investment in the risky assets and the riskless asset, after subtracting the instantaneous consumption.

Here $B(W_T, T)$ is the bequest or terminal reward function of the representative agent, W_t denotes the representative agent's wealth level at time t and $\mathbf{1}$ is the $n \times 1$ vector $(1, 1, \cdots, 1)'$. $U(\cdot)$ is the utility function of the representative agent, which is strictly increasing and concave. $\boldsymbol{\pi}_t' = (\pi_1, \pi_2, \cdots, \pi_n)_t'$ is an $n \times 1$ vector which represents the proportions of investment in the n risky assets at time t. The representative invests the remaining of his wealth in the riskless asset, that is, his proportion of investment in the riskless assets is $1 - \sum_{i=1}^{n} \pi_i$. C_t is the instantaneous consumption at time t. Write $J_i = J(W_t, e_i, t)$ for $i = 1, 2$, $J(W_t, X_t, t) = \langle \vec{J}, X_t \rangle$. Here $\vec{J} = (J_1, J_2)$ is a 2×1 row vector consisting of elements J_1, J_2. With J_W denoting $\frac{\partial J}{\partial W_t}$ and $J_{WW} = \frac{\partial^2 J}{\partial W_t^2}$, we have $J_W = \langle \vec{J}_W, X_t \rangle$, and $J_{WW} =$

$\langle \vec{J}_{WW}, X_t \rangle$. Therefore,

$$\begin{aligned} dJ &= \langle d\vec{J}, X \rangle + \langle \vec{J}, dX \rangle \\ &= \sup_{\boldsymbol{\pi}_t, C_t} \Big(J_W \left[\boldsymbol{\pi}_t' \left(\boldsymbol{\alpha}_t - r_t \mathbf{1} \right) W_t + r_t W_t - C_t \right] \\ &\quad + \frac{1}{2} J_{WW} W_t^2 \boldsymbol{\pi}_t' \boldsymbol{\sigma}_t \boldsymbol{\sigma}_t' \boldsymbol{\pi}_t + \langle \vec{J}, AX_t \rangle \Big) dt \\ &\quad + J_W W_t \boldsymbol{\pi}' \sigma_t d\mathbf{Z} + \langle \vec{J}, dM \rangle. \end{aligned}$$

Define the partial differential generator associated with the above dynamics to be:

$$\mathcal{A}^{\boldsymbol{\pi}_t, C_t} (J) = J_W \left[\boldsymbol{\pi}_t' \left(\boldsymbol{\alpha}_t - r_t \mathbf{1} \right) W_t + r_t W_t - C_t \right] + \frac{1}{2} J_{WW} W_t^2 \boldsymbol{\pi}_t' \boldsymbol{\sigma}_t \boldsymbol{\sigma}_t' \boldsymbol{\pi}_t + \langle \vec{J}, AX_t \rangle.$$

Based on the model and assumptions, we shall discuss the valuation of assets in the following sections.

3. General Equilibrium Model with Representative Agent

In this section, we investigate the general equilibrium of the economy. We first derive a partial differential equation for the cost function with intermediate consumption. We then discuss the property of the riskless interest rate in equilibrium. An explicit solution of the cost function is given without intermediate consumption as a special case.

3.1. General Equilibrium with Intermediate Consumption. Suppose the agents can invest in both the risk free assets and the riskless asset, (borrowing and saving at the same riskless rate r_t). Consider a representative agent who seeks to optimize the cost function (2.2) subject to the budget constraint (2.3).

The Bellman principle implies that $J(W_t, X_t, t)$ should solve

$$\begin{cases} \dfrac{\partial J}{\partial t} + \sup\limits_{\boldsymbol{\pi}_t, C_t} \left\{ \mathcal{A}^{\boldsymbol{\pi}_t, C_t} (J) + U(C_t, t) \right\} = 0, \\ \qquad\qquad J(W_T, X_T, T) = B(W_T, T), \\ \qquad\qquad J(0, X_t, t) = 0. \end{cases} \tag{3.1}$$

Define

$$\Psi = \frac{\partial J}{\partial t} + \sup_{\boldsymbol{\pi}_t, C_t} \left\{ \mathcal{A}^{\boldsymbol{\pi}_t, C_t} (J) + U(C_t, t) \right\}.$$

Since the supremum can be characterized by a first order condition and the second order condition has been secured by the strict concavity of the utility function, we can rule out the case that the supremum is obtained on the boundary of the budget constraint, a so-called corner solution. Therefore, assuming an interior solution, we can characterize the optimal portfolio and consumption plan according to the first order conditions[1].

$$\begin{aligned} \Psi_C = U_C - J_W &= 0, \\ \Psi_\pi = J_W W_t \left(\boldsymbol{\alpha}_t - r_t \mathbf{1} \right) + J_{WW} W_t^2 \boldsymbol{\pi}_t' \boldsymbol{\sigma}_t \boldsymbol{\sigma}_t' \boldsymbol{\pi}_t &= 0, \end{aligned}$$

[1] We assume strictly postive optimal consumption of the agent in equilibrium.

Suppose the agent has the utility function:

$$U(C_t, t) = e^{-\rho t}\frac{C_t^{1-\gamma}}{1-\gamma}.$$

Here γ is the coefficient of relative risk aversion, and ρ is a constant. Then we obtain the following results.

Lemma 3.1. *In equilibrium, the cost function satisfies the following partial differential equation*

$$\begin{aligned}\frac{\partial J}{\partial t} + J_W r_t W_t - \frac{(J_W)^2}{2J_{WW}}\left[\boldsymbol{\alpha}_t - r_t\mathbf{1}\right]'\left[\boldsymbol{\sigma}_t\boldsymbol{\sigma}_t'\right]^{-1}\left[\boldsymbol{\alpha}_t - r_t\mathbf{1}\right]\\ +J_W\left(e^{\rho t}J_W\right)^{-\frac{1}{\gamma}}\frac{\gamma}{1-\gamma} + \langle\vec{J}, AX_t\rangle = 0.\end{aligned}$$

with the boundary condition $J(0, X_t, t) = 0$.

Proof. In equilibrium, the net holding of the riskless asset of the representative agent is zero since the representative agent is also the aggregate one. Write C_t^* is the optimal consumption of the representative agent at time t, and $\boldsymbol{\pi}_t^*$ is the optimal investment proportions, or the agent's best investment strategy, at time t, we must have

$$\left(\boldsymbol{\pi}_t^*\right)'\mathbf{1} = 1. \tag{3.2}$$

Substituting equation (3.2) into the first order conditions

$$\begin{aligned}U_C - J_W &= 0,\\ J_W W_t\left(\boldsymbol{\alpha}_t - r_t\mathbf{1}\right) + J_{WW}W_t^2\boldsymbol{\pi}_t'\boldsymbol{\sigma}_t\boldsymbol{\sigma}_t'\boldsymbol{\pi}_t &= 0,\end{aligned}$$

and rearranging, we obtain

$$\begin{cases} U_C\left(C_t^*, t\right) = J_W,\\ \boldsymbol{\pi}_t^* = -\frac{J_W}{J_{WW}W_t}\left(\boldsymbol{\sigma}_t\boldsymbol{\sigma}_t'\right)^{-1}\left[\boldsymbol{\alpha}_t - r_t\mathbf{1}\right].\end{cases} \tag{3.3}$$

Substituting equation (3.3) into equation (3.1), we obtain

$$\begin{aligned}\frac{\partial J}{\partial t} + J_W\left(r_tW_t - C_t^*\right) - \frac{(J_W)^2}{2J_{WW}}\left[\boldsymbol{\alpha}_t - r_t\mathbf{1}\right]'\left[\boldsymbol{\sigma}_t\boldsymbol{\sigma}_t'\right]^{-1}\left[\boldsymbol{\alpha}_t - r_t\mathbf{1}\right]\\ +U(C_t^*, t) + \langle\vec{J}, AX_t\rangle = 0.\end{aligned} \tag{3.4}$$

From equation (3.3)

$$J_W = U_C\left(C_t^*, t\right) = e^{-\rho t}\left(C_t^*\right)^{-\gamma} \Rightarrow C_t^* = \left(e^{\rho t}J_W\right)^{-\frac{1}{\gamma}}. \tag{3.5}$$

This implies

$$U(C_t^*, t) = \frac{J_W\left(e^{\rho t}J_W\right)^{-\frac{1}{\gamma}}}{1-\gamma}. \tag{3.6}$$

Substituting equation (3.5) and (3.6) into equation (3.4) and rearranging, we have

$$\frac{\partial J}{\partial t}+J_W r_t W_t-\frac{(J_W)^2}{2J_{WW}}\left[\boldsymbol{\alpha}_t-r_t\mathbf{1}\right]'\left[\boldsymbol{\sigma}_t\boldsymbol{\sigma}_t'\right]^{-1}\left[\boldsymbol{\alpha}_t-r_t\mathbf{1}\right]$$
$$+J_W\left(e^{\rho t}J_W\right)^{-\frac{1}{\gamma}}\frac{\gamma}{1-\gamma}+\langle\vec{J},AX_t\rangle=0. \tag{3.7}$$

with boundary condition $J(0,X_t,t)=0$ from equation (3.1). □

The partial differential equation (3.7) is difficult to solve explicitly. We can only use a separability argument to reduce the partial differential equation to a non-linear ordinary differential equation. If we assume the agent only maximizes the terminal wealth then $J_W\left(e^{\rho t}J_W\right)^{-\frac{1}{\gamma}}\frac{\gamma}{1-\gamma}=0$, and we can solve the partial differential equation explicitly. We shall present the result in this case in the following subsection.

Lemma 3.2. *A solution of the partial differential equation* (3.7) *has the following form*

$$J(W_t,X_t,t)=e^{-\rho t}\frac{W_t^{1-\gamma}}{1-\gamma}\left(g(X_t,t)\right)^{\gamma}. \tag{3.8}$$

Here $g(X_t,t)=\langle\vec{g}(t),X_t\rangle$ *and* $\vec{g}(t)=(g_1(t),g_2(t))$. *That is*

$$J_i=J(W_t,e_i,t)=e^{-\rho t}\frac{W_t^{1-\gamma}}{1-\gamma}\left(g_i(t)\right)^{\gamma},\forall i=1,2,$$

where $g_i(t)$ *satisfies the following equation:*

$$\begin{cases}\left(r_t(1-\gamma)-\rho\right)g_1+\gamma\dfrac{\partial g_1}{\partial t}+\dfrac{1-\gamma}{2\gamma}\left[\boldsymbol{\alpha}_t-r_t\mathbf{1}\right]'\left[\boldsymbol{\sigma}_t\boldsymbol{\sigma}_t'\right]^{-1}\left[\boldsymbol{\alpha}_t-r_t\mathbf{1}\right]g_1\\ \qquad\qquad +\gamma-a_1g_1+a_1\left(g_2\right)^{\gamma}\left(g_1\right)^{1-\gamma}=0,\\ \left(r_t(1-\gamma)-\rho\right)g_2+\gamma\dfrac{\partial g_2}{\partial t}+\dfrac{1-\gamma}{2\gamma}\left[\boldsymbol{\alpha}_t-r_t\mathbf{1}\right]'\left[\boldsymbol{\sigma}_t\boldsymbol{\sigma}_t'\right]^{-1}\left[\boldsymbol{\alpha}_t-r_t\mathbf{1}\right]g_2\\ \qquad\qquad +\gamma-a_2g_2+a_2\left(g_1\right)^{\gamma}\left(g_2\right)^{1-\gamma}=0.\end{cases} \tag{3.9}$$

Since J_i is not linear in g_i, the above system of ordinary differential equation is nonlinear. If we let $\gamma=1$, that is, we assume that the utility function is a logarithm function, $U(C,t)=e^{-\rho t}\ln C$. Then we could eliminate the nonlinearity in the system. However, this might not be a reasonable assumption because the γ implicit in the option prices is usually different from 1.

Now consider the equilibrium interest rate, r_t. We shall show it is endogenously determined in equilibrium.

Theorem 3.3. *In equilibrium we have*

$$r_t=\frac{\mathbf{1}'\left(\sum\limits_{i=1}^{2}\langle X_t,e_i\rangle\boldsymbol{\sigma}_i\left(\sum\limits_{i=1}^{2}\langle X_t,e_i\rangle\boldsymbol{\sigma}_i\right)'\right)^{-1}\sum\limits_{i=1}^{2}\langle X_t,e_i\rangle\alpha(i)-\gamma}{\mathbf{1}'\left(\sum\limits_{i=1}^{2}\langle X_t,e_i\rangle\boldsymbol{\sigma}_i\left(\sum\limits_{i=1}^{2}\langle X_t,e_i\rangle\boldsymbol{\sigma}_i\right)'\right)^{-1}\mathbf{1}}.$$

Remark 3.4. We can rewrite r_t as $r_t = \langle \vec{r}, X_t \rangle$ with $\vec{r} = (r_1, r_2)$ and

$$r_i = r\left(X_t = e_i\right) = \frac{\mathbf{1}'\left(\boldsymbol{\sigma_i}\boldsymbol{\sigma_i'}\right)^{-1}\boldsymbol{\alpha_i} - \gamma}{\mathbf{1}'\left(\boldsymbol{\sigma_i}\boldsymbol{\sigma_i'}\right)^{-1}\mathbf{1}}, \quad \forall i = 1, 2.$$

Proof. From the proof of Lemma 3.1, we have in equilibrium,

$$\boldsymbol{\pi}_t^* = -\frac{J_W}{J_{WW}W_t}\left(\boldsymbol{\sigma}_t\boldsymbol{\sigma_t'}\right)^{-1}\left[\boldsymbol{\alpha}_t - r_t\mathbf{1}\right],$$

and $\left(\boldsymbol{\pi}_{\mathbf{t}}^*\right)'\mathbf{1} = 1$. These imply that

$$r_t = \left(\boldsymbol{\pi}_{\mathbf{t}}^*\right)'\boldsymbol{\alpha}_t + \frac{J_{WW}W_t}{J_W}\left(\boldsymbol{\pi}_{\mathbf{t}}^*\right)'\boldsymbol{\sigma}_t\boldsymbol{\sigma_t'}\boldsymbol{\pi}_{\mathbf{t}}^*. \tag{3.10}$$

Equation (3.10) connects the equilibrium local riskless rate with the drift vector and the volatility matrix of the risky assets and the risk aversion of the representative agent. In other words, the equilibrium local riskless rate is determined by the drifts, the variance-covariance terms of the risky assets, and the risk aversion of the representative agent.

From Lemma 3.2, we have

$$J(W_t, X, t) = e^{-\rho t}\frac{W_t^{1-\gamma}}{1-\gamma}\left(g(X_t, t)\right)^{\gamma},$$

Therefore,

$$-\frac{J_{WW}W_t}{J_W} = \gamma. \tag{3.11}$$

Substituting equation (3.11) into equation (3.3) and rearranging, we obtain

$$\boldsymbol{\alpha}_t - r_t\mathbf{1} = \gamma\boldsymbol{\sigma}_t\boldsymbol{\sigma_t'}\boldsymbol{\pi}_t^*, \text{ or } \ \boldsymbol{\pi}_t^* = \frac{1}{\gamma}\left(\boldsymbol{\sigma}_t\boldsymbol{\sigma_t'}\right)^{-1}\left[\boldsymbol{\alpha}_t - r_t\mathbf{1}\right]. \tag{3.12}$$

For the risky asset S_i, we have

$$\alpha_{i,t} - r_t = \gamma\boldsymbol{\sigma_{i,t}}\boldsymbol{\sigma_t'}\boldsymbol{\pi}_t^*. \tag{3.13}$$

Combining equation (3.12) and (3.10), we have

$$r_t = \frac{1}{\gamma}\left[\boldsymbol{\alpha}_t - r_t\mathbf{1}\right]'\left(\boldsymbol{\sigma}_t\boldsymbol{\sigma_t'}\right)^{-1}\left(\boldsymbol{\alpha}_t - \gamma\boldsymbol{\sigma}_t\boldsymbol{\sigma_t'}\frac{1}{\gamma}\left(\boldsymbol{\sigma}_t\boldsymbol{\sigma}_t'\right)^{-1}\left[\boldsymbol{\alpha}_t - r_t\mathbf{1}\right]\right).$$

Solving for r_t, we finally obtain

$$\begin{aligned} r_t &= \frac{\mathbf{1}'\left(\boldsymbol{\sigma}_t\boldsymbol{\sigma_t'}\right)^{-1}\boldsymbol{\alpha}_t - \gamma}{\mathbf{1}'\left(\boldsymbol{\sigma}_t\boldsymbol{\sigma_t'}\right)^{-1}\mathbf{1}} \qquad (3.14) \\ &= \frac{\mathbf{1}'\left(\sum_{i=1}^{2}\langle X_t, e_i\rangle\boldsymbol{\sigma}_i\left(\sum_{i=1}^{2}\langle X_t, e_i\rangle\boldsymbol{\sigma}_i\right)'\right)^{-1}\sum_{i=1}^{2}\langle X_t, e_i\rangle\alpha(i) - \gamma}{\mathbf{1}'\left(\sum_{i=1}^{2}\langle X_t, e_i\rangle\boldsymbol{\sigma}_i\left(\sum_{i=1}^{2}\langle X_t, e_i\rangle\boldsymbol{\sigma}_i\right)'\right)^{-1}\mathbf{1}}. \end{aligned}$$

□

Note that equation (3.14) is similar to equation (14) in Cox, Ingersoll and Ross (1985b), page 391, except:

- In their paper, they assume $\gamma \equiv 1$,
- Our state variable X_t, which corresponds to their Y_t, is implicit in $\alpha_t = \sum_{i=1}^{2} \langle X_t, e_i \rangle \alpha(i)$, and $\sigma_t = \sum_{i=1}^{2} \langle X_t, e_i \rangle \sigma_i$.

Corollary 3.5. *The local riskless rate of return follows a continuous time Markov chain*

$$dr_t = \langle \vec{r}, AX_t \rangle dt + dM_t^r, \tag{3.15}$$

where $dM_t^r = \langle \vec{r}, dM_t \rangle$ is a scalar Martingale process.

Proof. Following **Theorem 3.3**, we have $r_t = \langle \vec{r}, X_t \rangle$. Applying Ito's formula on r_t, we obtain

$$\begin{aligned} dr_t &= \langle \vec{r}, dX_t \rangle \\ &= \langle \vec{r}, AX_t \rangle dt + \langle \vec{r}, dM_t \rangle. \end{aligned}$$

Finally, we define $dM_t^r = \langle \vec{r}, dM_t \rangle$ to complete the proof. □

From Theorem 3.3, we know that in our framework, the riskless rate can take two values depending on the states of the economy.

3.2. General Equilibrium without Intermediate Consumption. In this subsection we assume that the representative agent has no intermediate consumption. Therefore, the representative agent's objective is to maximize the terminal wealth level. Then the term $J_W \left(e^{\rho t} J_W\right)^{-\frac{1}{\gamma}} \frac{\gamma}{1-\gamma}$ in equation (3.7) is zero, and we can explicitly solve equation (3.7). We also assume as above that the agent is using power utility function

$$U(W_t) = \frac{W_t^{1-\gamma}}{1-\gamma}.$$

Theorem 3.6. *In general equilibrium without intermediate consumption, the representative's cost function is*

$$J(W_t, X_t, t) = \exp\left(-(1-\gamma) \int_0^t r_u du\right) \frac{W_t^{1-\gamma}}{1-\gamma} \langle G, X_t \rangle,$$

where $G = (g_1(t), g_2(t))'$ can be explicitly computed.

4. A Partial Differential Equation for Contingent Claims

In the economy there are markets for contingent claims written on the n risky assets. All agents in the market can invest in the contingent claims. In this section we derive the no-arbitrage partial differential equation for the price of the contingent claims.

Consider a derivative written on the *j-th* risky asset, S_j, $1 \leq j \leq n$. Obviously we could add more than one such derivative written on different risky assets. However, to serve our purpose of obtaining a no arbitrage partial differential equation, considering a single derivative asset is is sufficient. Since the tradable derivative is written on the underling risky asset S_j whose price follows a geometrical Brownian motion, the value of the derivative will depend on all variables of the economy. Write $f_i = f(S_{j,t}, e_i, t)$ for $i = 1, 2$, $f(S_{j,t}, X_t, t) = \langle \vec{f}, X_t \rangle$. Here $\vec{f} = (f_1, f_2)$ is a 2×1 row vector consisting of elements f_1, f_2.

Similar to the paper of Cox, Ingersoll, and Ross (1985a), we assume the prices of the derivative, f, follow the dynamics

$$df = f\alpha_{f,t}dt + f\boldsymbol{\sigma}_{f,t}d\mathbf{Z}_t. \tag{4.1}$$

Here $\alpha_{f,t} = \sum_{i=1}^{2} \langle X_t, e_i \rangle \alpha_f(i)$, and $\boldsymbol{\sigma}_{\boldsymbol{f,t}} = \sum_{i=1}^{2} \langle X_t, e_i \rangle \boldsymbol{\sigma}_{f,i}$, where $\alpha_f(i), i = 1, 2$, the drift of the contingent claim at time t when $X_t = e_i$, are scalars, and

$$\boldsymbol{\sigma}_{f,i} = (\sigma_{f,i}(1), \sigma_{f,i}(2), \cdots, \sigma_{f,i}(m)), \quad i = 1, 2,$$

the variance of the returns of the contingent at time t when $X_t = e_i$, are $1 \times m$ row vectors.

Suppose the underlying asset, S_j, pays continuously dividend $\delta_{j,t}$ at time t, then the prices of S_i follows the dynamics

$$dS_{j,t} = S_{j,t}(\alpha_{j,t} - \delta_{j,t})dt + S_{j,t}\boldsymbol{\sigma}_{j,t}d\mathbf{Z_t}.$$

Here, $\alpha_{j,t} = \left\langle (\alpha_j(1), \alpha_j(2))', X_t \right\rangle$, where $\alpha_j(1)$, and $\alpha_j(2)$ are the *j-th* elements of the vector $\boldsymbol{\alpha}(1)$ and $\boldsymbol{\alpha}(2)$ respectively, and

$$\boldsymbol{\sigma}_{j,t} = \sum_{i=1}^{2} \langle X_t, e_i \rangle \boldsymbol{\sigma}_i(j),$$

where $\boldsymbol{\sigma}_i(j) = e_j'\boldsymbol{\sigma}_i$ is the *j-th* row of the volatility matrix $\boldsymbol{\sigma}_i$.

Theorem 4.1. *In equilibrium at time t the price, f, of a contingent claim written on a risky asset, S_j, satisfies the partial differential equation*

$$r_t f = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial S_j} S_{j,t}(r_t - \delta_{j,t}) + \frac{1}{2}\frac{\partial^2 f}{\partial S_j^2}(S_{j,t})^2 \boldsymbol{\sigma}_{j,t}\boldsymbol{\sigma}'_{j,t} + \langle \vec{f}, AX \rangle.$$

Proof. Here we explicitly model a dividend paying risky asset. Our analysis in the preceding sections remains the same if we have dividend paying assets and assume all dividend proceeds are used to repurchase the asset itself. We apply Ito's formula to $f = f(S_j, X_t, t)$ and obtain

$$\begin{aligned} df = &\left(\frac{\partial f}{\partial t} + \frac{\partial f}{\partial S_j} S_{j,t}(\alpha_{j,t} - \delta_i) + \frac{1}{2}\frac{\partial^2 f}{\partial S_i^2}(S_{j,t})^2 \boldsymbol{\sigma}_{j,t}\boldsymbol{\sigma}'_{j,t} + \langle \vec{f}, AX \rangle \right) dt \\ &+ \frac{\partial f}{\partial S_i} S_{i,t}\boldsymbol{\sigma}_{j,t}d\mathbf{Z}_t. \end{aligned} \tag{4.2}$$

Comparing the coefficients of dt and $d\mathbf{Z}_t$ in equations (4.1) and (4.2), we have

$$f\alpha_{f,t} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial S_j} S_{j,t}(\alpha_{j,t} - \delta_i) + \frac{1}{2}\frac{\partial^2 f}{\partial S_i^2}(S_{j,t})^2 \boldsymbol{\sigma}_{j,t}\boldsymbol{\sigma}'_{j,t} + \langle \vec{f}, AX \rangle. \tag{4.3}$$

Adding $r_t f - r_t f$ on the left hand and $r_t \frac{\partial f}{\partial S_j} S_{j,t} - r_t \frac{\partial f}{\partial S_j} S_{j,t}$ on the right hand of equation (4.3) and rearranging the equation, we obtain

$$\begin{aligned} r_t f + f\left(\alpha_{f,t} - r_t\right) = \frac{\partial f}{\partial t} &+ \frac{\partial f}{\partial S_j} S_{j,t}\left(r_t - \delta_i\right) + \frac{\partial f}{\partial S_j} S_{j,t}\left(\alpha_{j,t} - r_t\right) \\ &+ \frac{1}{2}\frac{\partial^2 f}{\partial S_j^2}\left(S_{j,t}\right)^2 \boldsymbol{\sigma}_{j,t}\boldsymbol{\sigma}'_{j,t} + \langle \vec{f}, AX_t \rangle, \end{aligned} \tag{4.4}$$

and

$$f\boldsymbol{\sigma}_{f,t} = \frac{\partial f}{\partial S_j} S_{j,t}\boldsymbol{\sigma_{j,t}}. \tag{4.5}$$

With the derivative, the representative agent can invest in the risky assets, the riskless asset, and the derivative. Assume the representative agent's proportion of investment in the derivative at time t is $\pi_{f,t}$. Then, the objective is to optimize a cost function similar to equation (2.2), subject to budget constraints with extra terms related to $\pi_{f,t}$. In equilibrium, the net holding of the contingent claim of the representative agent is zero since the long positions and short positions of the claim are cancelled. Therefore we still have the representative agent's optimal investment

$$\pi_t^* = \frac{1}{\gamma}\left(\boldsymbol{\sigma}_t\boldsymbol{\sigma'_t}\right)^{-1}\left[\boldsymbol{\alpha}_t - r_t\mathbf{1}\right].$$

For the risky asset S_j, we have

$$\alpha_{j,t} - r_t = \gamma\boldsymbol{\sigma}_{j,t}\boldsymbol{\sigma'_t}\pi_t^*.$$

For the derivative f, we have

$$\alpha_{f,t} - r_t = \gamma\boldsymbol{\sigma_{f,t}}\boldsymbol{\sigma'_t}\boldsymbol{\pi}_t^*.$$

Therefore,

$$\begin{aligned} \frac{\partial f}{\partial S_j} S_{j,t}\left(\alpha_{j,t} - r_t\right) &= \gamma\frac{\partial f}{\partial S_j} S_{j,t}\boldsymbol{\sigma_{j,t}}\boldsymbol{\sigma'_t}\boldsymbol{\pi}_t^*, \\ f\left(\alpha_{f,t} - r_t\right) &= \gamma f\boldsymbol{\sigma_{f,t}}\boldsymbol{\sigma'_t}\boldsymbol{\pi}_t^*. \end{aligned}$$

From equation (4.5) , the right hand sides of these two equations are equivalent, thus we obtain

$$f\left(\alpha_{f,t} - r_t\right) = \frac{\partial f}{\partial S_j} S_{j,t}\left(\alpha_{j,t} - r_t\right).$$

Therefore, equation (4.4) becomes

$$r_t f = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial S_j} S_{j,t}\left(r_t - \delta_{i,t}\right) + \frac{1}{2}\frac{\partial^2 f}{\partial S_i^2}\left(S_{j,t}\right)^2 \boldsymbol{\sigma}_{j,t}\boldsymbol{\sigma}'_{j,t} + \langle \vec{f}, AX_t \rangle.$$

This is a partial differential equation satisfied by any contingent claim written on any risky asset S_j, $1 \le j \le n$. $\square$

5. Stochastic Discount Factor and Market Price of Risk

In an economy there is no arbitrage if and only of there exists a stochastic discount factor, or pricing kernel. In an equilibrium setting, the pricing kernel is given by the marginal utility of optimal consumption. In a discrete time framework Cochrane (2001) defined a stochastic discount factor, state price density, or the pricing kernel Λ as satisfying the following condition:

Under the law of one price, there exists an adapted stochastic process Λ_t, such that

$$E\left[R_k \Lambda_k \middle| \mathcal{F}_{k-1}\right] = \mathbf{1},$$

where $\mathbf{1}$ is an n-vector of ones, R_k is the gross return on n assets at time k, and $\mathcal{F}_{k-1}$ is the information available at time k.

Definition 5.1. In continuous time, a *stochastic discount factor* is an adapted stochastic process Λ_t, such that, for any risky asset S_j

$$\Lambda_t S_{j,t} = \mathbb{E}\left[\int_t^\infty \Lambda_s D_{j,s} ds \middle| F_t\right]$$

where $D_{j,s} = S_{j,s}\delta_{j,s}$ which is the instantaneous amount of dividend paid by the asset S_j at time t.

In our model, the stochastic discount factor can be computed in the following way.

Theorem 5.2. *In general equilibrium the stochastic discount factor is*

$$d\Lambda_t = -\Lambda_t r_t dt - \Lambda_t \boldsymbol{\lambda_t'} d\mathbf{Z_t}. \tag{5.1}$$

where $\boldsymbol{\lambda_t} = \gamma \boldsymbol{\sigma_t'} \boldsymbol{\pi}_t^*$ *is a* $m \times 1$ *vector called the market price of risk.*

It is known that in a recession, or in a 'bad' state of the economy, the volatility of the markets is higher than in a 'good' state. Therefore, the market price of risk, $\boldsymbol{\lambda} = \gamma \boldsymbol{\sigma_t'} \boldsymbol{\pi}_t^*$, is higher when economy is in a 'recession' state than in a 'good' state.

Write $R_{j,t}$ as the return of the j-*th* risky asset at time t. Then, $R_{j,t}$ follows the dynamics

$$dR_{j,t} = \frac{dS_{j,t}}{S_{j,t}} = (\alpha_{j,t} - \delta_{j,t})\, dt + \boldsymbol{\sigma}_{j,t} d\mathbf{Z_t}.$$

On the other hand, the dynamics of the stochastic discount factor are

$$d\Lambda_t = -\Lambda_t r_t dt - \Lambda_t \boldsymbol{\lambda_t'} d\mathbf{Z_t} = -\Lambda_t r_t dt - \Lambda_t \gamma \left(\boldsymbol{\pi_t^*}\right)' \boldsymbol{\sigma}_t d\mathbf{Z_t}.$$

We note that there is negative quadratic covariation between the return and stochastic discount factor dynamics. This implies a positive risk premium is demanded by the agent who holds the security.

Intuitively, at time t, if $d\mathbf{Z_t}$ is large or at least greater than zero, the owner of the security has a return higher than the expected return, $\alpha_{i,t} - \delta_{i,t}$. At the same time, the stochastic discount factor is lower. The lower stochastic discount factor, the higher is the discount rate. Thus, a higher return is discounted more, and vice versa. Since the expected return of any risky asset is the realized returns weighted by the stochastic discount factor, on average, the return of this particular security is lower. Thus, in equilibrium, the agent demands a positive risk premium for holding this security.

In an equilibrium setting, the pricing kernel is given by the representative agent's marginal utility of optimal consumption. That is:

$$\frac{\partial U}{\partial C}\left(C_t^*, t\right) = l\Lambda_t,$$

where $l > 0$ is constant.

Therefore, a positive risk premium implies that higher returns occur when, (in terms of Brownian risk, $dZ_t > 0$), the agent's marginal utility is low, so the agent does not value this higher return at that time as much as the time when her marginal utility is high.

Now take the regime switching into consideration. In a 'bad' state of the economy, $\boldsymbol{\sigma_{i,t}}$, and $\boldsymbol{\sigma_t}$ are higher than in a 'good state'. If $d\mathbf{Z_t}$ is high, then $R_{j,t}$ will be higher than the situation without regime switching and the stochastic discount factor will be lower. This implies the situation is more risky when the market is in a recession. The agent demands a higher risk premium in such a recession than in a normal period. This could at least be an explanation of the equity premium puzzle, that is, the reason why there is a large equity premium is that the representative agent values a higher return less than its 'true value' since higher returns always occur when the agent's marginal utility is low.

6. Concluding Remarks

In a Markov regime switching framework, we have investigated the asset pricing problem in general equilibrium with a representative agent who maximizes a cost function. Based on assumptions of a CRRA utility function, we have derived a partial differential equation satisfied by the representative agent's cost function. A form of the solution of the partial differential equation has been given in general equilibrium with intermediate consumption. In the case when the representative agent does not have intermediate consumption, we have found an explicit solution of the cost function. A closed-form expression for the riskless rate has been derived. We have also provided a partial differential equation satisfied by any contingent claim written on any risky asset in the market. The stochastic discount factor has been investigated in our framework. Based on the stochastic discount factor, we have suggested an explanation for the equity premium puzzle.

Some possible extensions of our paper include considering an n-state economy instead of only a two-state one. We can investigate different utility functions for the representative agent, and consider transaction costs, and even taxes.

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