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SCS 61: The Category CD of Completely Distributive lattices and Their Free Objects

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TOPIC: The category CD of completely distributive lattices and their free objects

REFERENCES: The usual scriptures, and
Nino, J., Dissertation Tulane 1981; Hoffmann, R.-E., SCS 9-10-81;
Hofmann, K.H. SCS-Memo 11-27-79.

This is roughly the content of a lecture given at the Tulane miniworkshop on 11-21-81. Much of it was discussed in the Tulane continuous lattice seminar in the fall of 81 with Mike Mislove and Eric Partridge.

Section 1. The category CD.

I must begin with a definition of the category of completely distributive lattices, and I want to use this occasion for proselytizing for them and to propagandize the demand for a complete exposition of everything known about completely distributive lattices and their natural maps. The writing of a monograph would now be timely, interesting and rewarding. (I might even do it myself, but not alone. I guess I am looking for takers.)

1.1. DEFINITION. The objects of the category CD of completely distributive lattices are complete lattices in which the identity

$$\text{(cd)} \qquad \inf_{i\in I} \sup_{j\in J} a_{ij} = \sup_{f\in J^I} \inf_{i\in I} a_{i,f(i)}$$

holds for all families $(a_{ij})_{i\in I, j\in J}$. The morphisms of CD are the maps preserving all infs and all sups. □

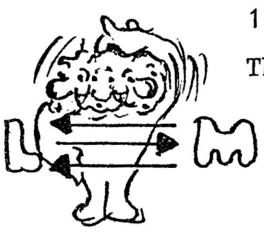
REMARKS. The identity (cd) is called *complete distributivity*. It always holds simultaneously with its opposite identity. Completely distributive lattices therefore share with many other classical objects of lattice theory the feature of being preserved under the passage $L \mapsto L^{op}$ to the opposite lattice which is so strikingly absent in the more general continuous lattices.

The choice of morphisms is dictated by (cd): The operations (infinitary, to be sure) entering into the defining equation must be preserved. This choice of morphisms makes the category CD, in the words of the Old Testament, a full subcategory of INF∩SUP.

The category CD is obviously a variety which is closed under formation of arbitrary products, subalgebras and quotients. Cartesian products are the categorical products; equalizers are formed as in the category of sets. The category is also cocomplete, but one must fight off the temptation to believe that the coproduct of, say, two CD-objects is the cartesian product with any of the obvious injections as coprojections. It looks suspiciously as though this was the case, but a closer inspection reveals that neither of these injections will preserve both 0 and 1 while every CD morphism must respect them. Coproducts are more complicated. More about that later.

That free objects exist is perhaps not obvious, considering, as J.D.Lawson pointed out, that the category of complete lattices has no free objects. But they do exist and we will analyze their structure.

Let us record a few quick throughts concerning the definition.



1.2. PROPOSITION. i) Let $f: L \longrightarrow M$ be a monotone map between CD -objects. Then the following statements are equivalent

(1) f is a CD -morphism.

(2) f has an upper adjoint $g: M \longrightarrow L$ and a lower adjoint $d: M \longrightarrow L$.

ii) If (1) and (2) are satisfied, then the following statements hold:

(a) g preserves Spec, and $g|\text{Spec } M: (\text{Spec } M, \gg) \longrightarrow (\text{Spec } L, \gg)$ preserves directed sups (i.e. is Scott continuous)

(b) d preserves Cospec, and $d|\text{Cospec } M: \text{Cospec } M \longrightarrow \text{Cospec } L$ preserves directed sups (i.e. is Scott continuous), where Cospec carries the induced order.

iii) The following two conditions are equivalent:

(3) f preserves Spec.

(4) d preserves finite infs, i.e. d is a cHa-map.

Under these circumstances, $f|\text{Spec } L: (\text{Spec } L, \gg) \longrightarrow (\text{Spec } M, \gg)$ is upper (!) adjoint of $g|\text{Spec } M$.

iv) The following two conditions are equivalent:

(5) f preserves Cospec.

(6) g preserves finite sups. □

Since the insights of Lawson and Hoffmann we appreciate the fact that on the object level there is a bijection between CD-objects and continuous posets. What sort of maps one should consider on the continuous poset level has begun to crystallize only recently, for instance in Jaime Niño's dissertation. We saw in 1.2.a (and b) that the adjoints of CD-maps produce Scott continuous maps on the spectra (and cospectra). But not every Scott continuous map between continuous posets arises in this fashion. Indeed let $g:T \longrightarrow S$ be a Scott continuous map between continuous posets. If we let $L = \sigma(S)$ and $T = \sigma(T)$ be the respective Scott topologies, then L and M are CD-objects and the map $f:L \longrightarrow M$ given by $f(U) = g^{-1}(U)$ is at any rate a cHa-map. By 1.2.i, this f is a CD-morphism iff it has a lower adjoint.

Let us pause for a moment and consider the situation more generally on the level of general topology. If $g:Y \longrightarrow X$ is a continuous map between topological spaces, we generate a cHa-map $f:O(X) \longrightarrow O(Y)$ via $f(U) = g^{-1}(U)$. The map f has a lower adjoint $d:O(Y) \longrightarrow O(X)$ iff every open set V of Y determines an open set d(V) in X so that $d(V) \subseteq U$ for an open set U of X iff $V \subseteq f(U) = g^{-1}(U)$ iff $g(V) \subseteq U$. Thus $d(V) = \bigcap\{U \in O(X): g(V) \subseteq U\}$ = sat g(V) where the saturation sat A of a subset A of X is the intersection of the filter of (open) neighborhoods of A. We have in fact shown the following

1.3. LEMMA. The cHa-map $O(g):O(X) \longrightarrow O(Y)$ induced by a continuous function $g:Y \longrightarrow X$ via $O(g)(U) = g^{-1}(U)$ has a lower adjoint if and only if the saturation sat g(V) is open in X for each open set V in Y. □

REMARK. If X is T_1 this occurs precisely when g is open. (Indeed sat A=A for all sets A in a T_1 space.)

For easy reference we choose the following nomenclature:

1.4. DEFINITION . A function $g:Y \longrightarrow X$ between topological spaces is called *quasi-open* iff sat g(V) is open in X for each open set V in Y. □

Since the Scott continuous and quasi-open maps between continuous posets are precisely those arising from CD-morphisms by restricting adjoints to spectra we declare:

1.5. DEFINITION. A map between up-complete posets $g:T \longrightarrow S$ is called a *comorphism* iff it is Scott continuous and Scott quasi-open, i.e. iff it preserves directed sups and $\uparrow c(V)$ is Scott open in S for each Scott open set V in T. □

We then have the following remark:

1.6. PROPOSITION. The category CD of completely distributive lattices is dually equivalent to the category of continuous posets and comorphisms between them. □

We recall an ancient knowledge of the scriptures of the old covenant: If $F: S \longrightarrow T$ is an upper adjoint of $c:T \longrightarrow S$, then F is Scott continuous iff $F^{-1}V$ is Scott open for each Scott open set V; but $F^{-1}V = \uparrow c(V)$, and so F *is Scott continuous iff* c *is Scott quasi-open.*

In particular, if F is an INF-map between complete lattices, then it is in fact an $\underline{INF}^{\uparrow}$ -map iff its upper adjoint is a comorphism. This leads us to the following definition:

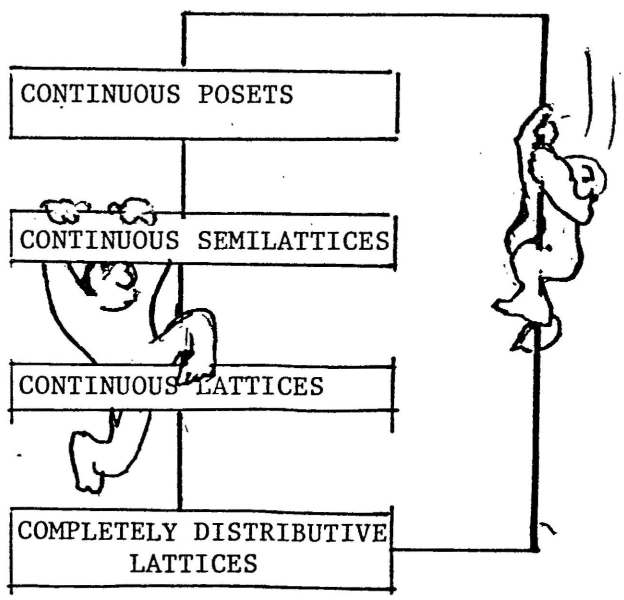
1.7. DEFINITION. A map $F:S \longrightarrow T$ between up-complete posets is called a *morphism* iff it has a lower adjoint $c:T \longrightarrow S$ which is a comorphism. iff it has a lower adjoint and is Scott continuous. □

From the old testament we know that a map $F:S \longrightarrow T$ between continuous lattices is a morphism in the sense of 1.7 iff it is a CL -morphism, i.e., an $\underline{INF}^{\uparrow}$ - morphism. We may think of a morphism between up-complete posets as the pair (F,c) of adjoint maps. The category of continuous posets and morphisms between them contains CL as a full subcategory and is a bit smaller than the dual category of the category of continuous posets and comorphisms between them. To be a bit more precise:

1.8. PROPOSITION. The subcategory $\underline{CD}_{Spec}$ of CD with the same objects as CD and all CD -morphisms between them which preserve spectra is equivalent to the category of all continuous posets and all morphisms between them.

The functor which implements the equivalence simply associates with a CD-object its spectrum and with a Spec-preserving morphism its restriction to the spectra. Recall that a CD-morphism $f:L \longrightarrow M$ is a $\underline{CD}_{Spec}$-morphism iff the lower adjoint d of f is a cHa-map (see 1.2.iii.)

I point to these relations between completely distributive lattices and continuous posets in order to advocate the feasibility of a systematic collection of information on completely distributive lattices, accepting as given the current interest in continuous semilattices and posets and their applications. The case for completely distributive lattices does not rest on their rich structure and symmetry alone but in their hierarchical position within a whole chain of classes of posets that have caught our attention. It is not untypical for conditions in lattice theory that the chain is closed. This is illustrated in the following Erné -diagram



Section 2. Freedom.

We recall from the old testament that we denote with $\sigma(L)$ the Scott topology of a poset L. It will be convenient for us to relize the opposite lattice $\sigma(L)^{op}$ concretely:

2.1. NOTATION. For a poset let $\gamma(L)$ be the complete lattice of all Scott closed subsets of L. □

Thus $A \in \gamma(L)$ iff $\downarrow A = A$ and $\sup D \in A$ for every directed set $D \subseteq A$ with a sup. (We will consider up-complete posets anyway.) Clearly, $\gamma(L) \cong \sigma(L)^{op}$.

2.2. DEFINITION. If L is a poset and M a complete lattice, then for any function $g: L \longrightarrow M$ we define $g^*: \gamma(L) \longrightarrow M$ by $g^*(A) = \sup g(A)$. □

2.3. PROPOSITION. Let $g: L \longrightarrow M$ be a Scott continuous map between posets such that M is a complete lattice. Then

$$g^*: \gamma(L) \longrightarrow M, \quad g^*(A) = \sup g(A),$$

has an upper adjoint

$$u : M \longrightarrow \gamma(L), \quad u(m) = g^{-1}(\downarrow m).$$

Proof. Since g is Scott continuous, the function u is well-defined. Now let $A \in \gamma(L)$ and $m \in M$. Then $g^*(A) \leqslant m$ iff $\sup g(A) \leqslant m$ iff $(\forall a \in A)\ g(a) \leqslant m$ iff $(\forall a \in A)\ a \in g^{-1}(\downarrow m)$ iff $A \subseteq g^{-1}(\downarrow m) = u(m)$. □

2.4. COROLLARY. Under the circumstances of 2.3, the function g^* is in SUP . □

REMARK. If L is just a set, and 2^L the lattice of all subsets of L, then for any function $g: L \longrightarrow M$ into a complete lattice, the functions $g^*: 2^L \longrightarrow M$ and $u: M \longrightarrow 2^L$ are well defined by the formulae given in 2.3, and g^* is a lower adjoint of u by the same proof. In particular, the extention g^* still preserves arbitrary sups.

2.5. PROPOSITION. Let $g: L \longrightarrow M$ be a map between posets with a lower adjoint and suppose that M is completely distributive. Then $g^*: \gamma(L) \longrightarrow M$ is in INF .

REMARK. The claim persists for the extension $g^*: \lambda(L) \longrightarrow M$ to the lattice $\lambda(L)$ of all lower sets $A = \downarrow A$ of L.

Proof. Let $\{A_j : j \in J\}$ be a family in $\lambda(L)$. Since g^* is monotone, the relation

$$\text{(i)} \qquad g^*\left(\bigcap_{j \in J} A_j\right) \leqslant \inf_j g^*(A_j) = \inf_{j \in J} \sup g(A_j)$$

is automatic, and we must prove the reverse inequality.

By complete distributivity of M we have (see 1.1.(cd))

$$\text{(ii)} \quad \inf_{j \in J} \sup g(A_j) = \sup_{(a_k)_{k \in J}} \inf_{j \in J} g(a_j),\ (a_k)_{k \in J} \in \prod_J A_k .$$

Now let $d:M \to L$ be a lower adjoint of g. Then, setting $m = \inf_{j\in J} g(a_j)$, we note $m \leq g(a_j)$ for all $j \in J$, and this is equivalent to $d(m) \leq a_j$ for all $j \in J$. Since $A_j = \downarrow A_j$ for all j, we derive $d(m) \in A_j$ for all $j \in J$, i.e. $d(m) \in \bigcap_{j\in J} A_j$. In particular $m \leq gd(m) \in g(\bigcap_{j\in J} A_j)$, whence

$$\inf_{j\in J} g(a_j) = m \leq \sup g(\bigcap_{j\in J} A_j) = g^*(\bigcap_{j\in J} A_j)$$

for all $(a_k)_{k\in J} \in \prod_{j\in J} A_j$. Hence $\sup_{(a_k)_{k\in J}} \inf_{j\in J} g(a_j) \leq g^*(\bigcap_{j\in J} A_j)$,

which, in view of (ii) is the required reverse inequality to (i). □

As a spin-off we recover (?) the following characterisation of completely distributive lattices:

2.6. COROLLARY. Let M be a complete lattice. Then the following conditions are equivalent:

(1) M is completely distributive.

(2) For any family A_j of lower sets in M we have

$$\sup \bigcap_{j\in J} A_j = \inf_{j\in J} \sup A_j.$$

(3) The function $id^* : \lambda(M) \to M$, $id^*(A) = \sup A$, preserves all infs.

(3') The function $id^* : \lambda(M) \to M$ is in INF ∩ SUP.

(4) The function $id^* : \lambda(M) \to M$ has a lower adjoint.

(5) For each $m \in M$ we have $m = \sup \Downarrow m$ (where $x \lll y$ iff for any subset P of M the relation $y \leq \sup P$ implies $x \leq p$ for some $p \in P$).

Proof. (1) ⇒ (2) : Apply 2.4 to $id:M \to M$.

(2) ⇒ (1) : Exercise from 1.1.

(2) ⇔ (3) : By definition of id*.

(3) ⇔ (3'): Due to the fact that the function id* is always in SUP (see Remark following 2.4).

(3) ⇔ (4) : Recall the theory of the Wunderwaffe'adjoint'.

(4) ⇔ (5) : Sketch: The lower adjoint of id*, if it exists, must be the function $s:M \to \lambda(M)$ given by $s(m) = \bigcap\{A \in \lambda(M): m \leq \sup A\}$. This function can, of course, be always defined; it is the desired adjoint iff $m = id^*(s(m)) = \sup s(m)$ for all m. Take it from here. □

Our principal aim was information on the operation *. We collect the essence from 2.3 and 2.5:

2.7. THEOREM. Let $g:L \to M$ be a map between posets with a lower adjoint and assume that M is a completely distributive lattice. Then

$g^*: \lambda(L) \to M$, $g^*(A) = \sup g(A)$, is in SUP ∩ INF.

If g is Scott continuous, then $g^*: \gamma(L) \to M$ is in SUP ∩ INF. □

2.8. <u>COROLLARY</u> . Let $g: L \longrightarrow M$ be a morphism of up-complete posets in the sense of Definition 1.7. If M is a completely distributive lattice, them

$$g^*: \gamma(L) \longrightarrow M$$

is a morphism from $\underline{\mathrm{INF}} \cap \underline{\mathrm{SUP}}$. □

2.9. <u>LEMMA.</u> Let S be an up-complete poset. Then the function

$$\eta_S: S \longrightarrow \gamma(S), \quad \eta_S(s) = \downarrow s,$$

is Scott continuous, and the following statements are equivalent:

(1) η_S is a morphism of up-complete posets à la 1.7 (i.e. η_S has a lower adjoint).

(2) S is a complete lattice.

Proof. Since S is up-complete, η_S preserves directed sups, hence is Scott continuous. Condition (1) says precisely that for each $A \in \gamma(S)$ there is a smallest element $a \in S$ such that $A \subseteq \downarrow a$, i.e. that each $A \in \gamma(S)$ has a sup. An arbitrary subset of S, however, has a sup iff its Scott closure has a sup and the two agree. Hence (1) and (2) are equivalent. □

2.1o. <u>LEMMA</u> . Let $f: S \longrightarrow T$ be a morphism of up-complete posets in the sense of 1.7. Define $\gamma(f): \gamma(S) \longrightarrow \gamma(T)$ by $\gamma(f)(A) = (\downarrow f(A))^-$. Then $\gamma(f)$ has the upper adjoint $B \mapsto f^{-1}(B): \gamma(T) \longrightarrow \gamma(S)$, whence it is in <u>SUP</u>. Furthermore, we have $\gamma(f) = (\eta_T f)^*$ and $\eta_T f = \gamma(f)\eta_S$, and these conclusions hold:

(i) If T is a complete lattice, then $\gamma(f) \in \underline{\mathrm{SUP}} \cap \underline{\mathrm{INF}}$.

(ii) If S is a complete lattice, then for every $\underline{\mathrm{SUP}} \cap \underline{\mathrm{INF}}$ -morphism $F: \gamma(S) \longrightarrow \gamma(T)$ the map $g = F\eta_S: S \longrightarrow \gamma(T)$ is a morphism of up-complete posets such that $g^* = F$.

Diagram:

$$\begin{array}{ccc} S & \xrightarrow{\ \eta_S\ } & \gamma(S) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \gamma(f)} \\ T & \xrightarrow[\ \eta_T\ ]{} & \gamma(T). \end{array}$$

Proof. We have $A \subseteq f^{-1}B$ for $A \in \gamma(S)$ and $B \in \gamma(T)$ iff $f(A) \subseteq B$ iff $(\downarrow f(A))^- \subseteq B$, since B is Scott continuous. This proves the first claim. Furthermore we have $\gamma(f)(A) = (\downarrow f(A))^- = (\bigcup\{\downarrow f(a): a \in A\})^- =$

$\sup \{(\eta_T f)(a): a \in A\} = \sup (\eta_T f)(A) = (\eta_T f)^*(A)$. Also, $\eta_T f(s) = \downarrow f(s) = \downarrow f(\downarrow s)$, since f is monotone, and this set is closed as principal ideal. Thus it agrees with $(\downarrow f(\downarrow s))^- = \gamma(f)\eta_S(s)$.

If T is a complete lattice, then η_T is a morphism by 2.9, and so $f\eta_T$ is a morphism, whence $\gamma(f) = (f\eta_T)^*$ is in $\underline{\mathrm{SUP}} \cap \underline{\mathrm{INF}}$ by 2.8.

If S is a complete lattice, and F is as in (ii), then $g = F\eta_S$ is a morphism of up-complete posets, since η_S is one by 2.9. Also $g^*(A) = \sup g(A) = \sup F\eta_S(A) = F(\sup \eta_S(A)) = F((\bigcup\{\downarrow a : a \in A\})^-) = F(A^-) = F(A)$. □

2.11. NOTATION. Let CP denote the category of continuous posets with morphisms in the sense of 1.7 between them. The category CL is then a full subcategory of CP and the category CL of continuous lattices and CL -morphisms contains the category CD as a subcategory which is not full. The forgetful functor from CD to CL and to CP will be denoted $|\ \ |$.

After 1.8, the categories CP and CD_{Spec} are equivalent. □

2.12. THEOREM. Let S be a continuous poset, M a completely distributive lattice, and $g: S \longrightarrow |M|$ a CP-morphism. Then there is a unique CD -morphism $g^*: \gamma(S) \longrightarrow M$ such that $g = |g^*|\eta_S$. The function $\alpha_{SM} = (g \mapsto g^*): CP(S,|M|) \longrightarrow CD(\gamma(S), M)$ is a natural injection.

The map $\eta_S: S \longrightarrow \gamma(S)$ is a Scott continuous map; it is a CP -morphism iff S is a continuous lattice. In this case, α_{SM} is a natural isomorphism.

Proof. The function g^* is a CD-map by 2.8. We have $g^*\eta_S(s) = g^*(\downarrow s) = \sup g(\downarrow s) = g(s)$ since g is monotone. If $g': \gamma(S) \longrightarrow M$ is any SUP -morphism with $g'\eta_S = g^*\eta_S$, then the equalizer of g' and g^* in $\gamma(S)$ is a SUP - subalgebra containing $\operatorname{im}\eta_S = \operatorname{Cospec}\gamma(S)$. Since $\gamma(S)$, as a completely distributive lattice, is order cogenerated by its cospectrum, this equalizer is all of $\gamma(S)$. Hence $g' = g^*$. Since g^* determines its restriction to the cospectrum, thne g^* determines g uniquely. Thus α_{SM} is injective. It is clearly natural. By 2.9, η_S is a CP-morphism iff S is a continuous lattice. If, in this case, we identify M with $\gamma(\operatorname{Cospec} M)$ (which we may!) then 2.10ii shows that α_{SM} is surjective. □

This Theorem calls for a commentary. By the skin of our teeth it fails to show that the forgetful functor $|\ \ |: CD \longrightarrow CP$ has a left adjoint. The standard universal property is satisfied. Only a very close look shows that the adjunction will generally fail due to the fact that the candidate for the front adjunction η_S is not always a morphism in CP. In fact it is one if an only if S is a continuous lattice. Perhaps one can tinker with the morphisms a bit and improve the situation, but I do not think that very much can be done. At any rate, the universal property we proved in 2.12 suffices to show that the forgetful functor $|\ \ |: CD \longrightarrow CL$ has a left adjoint:

2.13. THEOREM . The forgetful functor $| \ |$: CD —— CL has the functor γ: CL ⟶ CD as left adjoint which associates with each continuous lattice L the completely distributive lattice $\gamma(L)$ of all Scott closed sets under inclusion and with a CL -morphism f: S ⟶ T the map $\gamma(f)$ given by $\gamma(f)(A) = (\downarrow g(A))^- = g(A)^-$ (where $(\ )^-$ is Scott-closure).

A completely distributive lattice M is free (over CP) iff Cospec M is a continuous lattice in which case Cospec M is the freely generating continuous lattice. □

At this point all the free constructions over the category of compact spaces and, finally, over sets fall out. The category CL has the grounding functor Λ : CL ⟶ COMP in the category of compact spaces and continuous maps which associates with a continuous lattice the underlying space in the Lawson-topology and with a CL -morphism the induced continuous map (see the old testament). The old scriptures also provide the information that there is a left adjoint Γ : COMP ⟶ CL which associates with a compact Hausdorff space X the CL -object ΓX of all closed subsets with respect to reverse containment (so that $\Gamma X = O(X)$), and with $\Gamma(f)(A) = f(A)$, thus $\Gamma(f)$ is upper (!) adjoint to the map $B \mapsto f^{-1}B$ for any continuous map f: X ⟶ Y.

This gives, via composition of adjoints, the following result:

2.14. COROLLARY. The grounding functor Λ : CD ⟶ COMP which associates with a completely distributive lattice the underlying space with the Lawson - (and indeed interval-) topology has the functor $X \mapsto \gamma(\Gamma X)$ as left adjoint. The front adjunction is $\psi_X: X \to \gamma(\Gamma X)$, $\psi_X(x) = \{A = A^- \subseteq X : x \in A\}$ (If one interprets the free object as $\gamma(O(X))$, then the front adjunction associates with an $x \in X$ the set $\{U \in O(X) : x \notin U\} = O(x) \setminus U(x)$ where $U(x)$ is the filter of open neighborhoods of x.) □

The category COMP is grounded in the category SET of sets by the functor which associates with a compact space X the underlying set. Its left adjoint ß: SET ⟶ COMP associates with a set its Čech-Stone compactification. We recall that $\Gamma(\beta X) \cong O(\beta X) \cong 2^X$. As a consequence we have now identified the free completely distributive lattice tout court:

2.15. THEOREM. The free completely distributive lattice over a set X is $\gamma(2^X)$. The front adjunction is $x \mapsto \{A \in X : x \notin A\} : X \to \gamma(2^X)$

As a functor, the free construction $X \mapsto \gamma(2^X)$ transforms a function f:X ⟶ Y between sets to the CD -morphism $\gamma(2^f) = (\mathcal{A} \mapsto \{B \in 2^Y : f^{-1}(B) \subseteq A$ for some $A \in \mathcal{A}\}^-$: $\gamma(2^X) \longrightarrow \gamma(2^Y)$. □

The free bounded distributive lattice over a set X is the sublattice generated in $\lambda(2^X)$ by the principal ideals.(See literature on distributive lattices, e.g., Balbes and Dwinger.) This is a sublattice of the free completely distributive lattice $\lambda(2^X)$ over X and is sup-dense in it. For finite X, the two agree. The free completely distributive lattice over a set X is quite large:

2.16. <u>COROLLARY</u>. If X is an infinite set, then the cardinality of the free completely distributive lattice over X is $2^{(2^{\text{card } X})}$. □

Let us pause for some comments on the almost free construction $\eta_S : S \longrightarrow \gamma(S)$ for continuous posets. We recall that Rudolf-E. Hoffmann identified the largest essential extension $e_X : X \longrightarrow \mathcal{E}(X)$ of a T_o-space as a subspace of the lattice of all closed sets C(X) of X as $\mathcal{E}(X) = \{A \in C(X)$: A is a convergence set ,i.e. the set of limit points of a filter of open sets$\} \subseteq C(X)$, and $e_X(x) = \{x\}^-$. I do not exactly know how the topology is constructed on $\mathcal{E}(X)$ in terms of the lattice C(X), since Hoffmann weasels around that,too, by transporting a topology which Banaschewski introduced in a different construction. I would not be too surprised if it were the topology induced from the upper topology of C(X); at any rate <u>that</u> topology will make e_X an embedding. The lattice $\mathcal{E}(X)$ is closed in C(X) under arbitrary infs and directed sups,hence is an <u>INF</u>$^\uparrow$ -subalgebra of C(X).

The construction $\eta_S : S \longrightarrow \gamma(S)$ is a special case of $x \mapsto \{x\}^- : X \longrightarrow C(X)$, for a continuous poset S. Thus $\mathcal{E}(S)$ is an <u>INF</u>$^\uparrow$ -subalgebra, and hence a <u>CL</u> -subalgebra of $\gamma(S)$, since $\gamma(S)$ is completely distributive, hence continuous. The relevant topology on $\gamma(S)$ is the Scott topology which agrees here with the upper topology. It induces on $\mathcal{E}(S)$ the Scott topology which is the topology for the essential extention according to the Banaschewski-Hoffmann theory.

The members of the family of all <u>CL</u> -subalgebras containing im η_S = Cospec $\gamma(S)$ all share the property of being sup-generated by im η_S; each such member T is characterized by the lower adjoint $d_T : \gamma(S) \longrightarrow T$ to the <u>CL</u> - inclusion $i_T : T \longrightarrow \gamma(S)$, and by the Scott continuous closure operator $k_T = i_T d_T$ which induces the identity on Cospec $\gamma(S)$. Now let T be a <u>CL</u> -algebra containing Cospec $\gamma(S)$ and being contained in $\mathcal{E}(S)$. Then $d_T | \mathcal{E}(S) : \mathcal{E}(S) \to T$ is the lower adjoint of the inclusion $T \longrightarrow \mathcal{E}(S)$, and $(d_T | \mathcal{E}(S))\, e_S : S \longrightarrow T$ is an embedding for the Scott topologies. Since e_S is essential, $d_T | \mathcal{E}(S)$ is an embedding which is the case iff $T = \mathcal{E}(S)$. It follows that $\mathcal{E}(S)$ is the smallest member of the family in question.

Thus

2.17. PROPOSITION. For a completely distributive lattice M , the smallest CL -subobject L containing the cospectrum S = Cospec M of M is the essential hull of S when S and L are given the Scott topologies (both of which are induced by the Scott topology of M). And all essential hulls of continuous posets are so obtained. Further, M is free over S (as a CD-object is free over CP) iff S=L. □

The example of the ordinary square $M=[0,1]^2$ is instructive in this regard, because it shows the substantial difference between $L\cong\mathcal{E}(S)$ and $M\cong\gamma(S)$. It also shows that there are millions of CL-objects between $\mathcal{E}(S)$ and $\gamma(S)$, in every one of which S is sup-dense in the induced order.

The category CD contains a full subcategory AD of all algebraic completely distributive lattices. The objects were discussed in an earlier memo under the name of baHa (bialgebraic Heyting algebras).We should observe at this point that the free competely distributive lattice $\gamma(2^X)$ is algebraic. Indeed, an element $A\in\gamma(S)$ for a continuous poset S is compact iff $A = \downarrow F$ with a finite set $F\subseteq K(S)$. It follows that $\gamma(S)$ is algebraic iff S is an algebraic poset. Since 2^X is algebraic, it follows that $\gamma(2^X)$ is algebraic.