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SCS 59: Sober Quotients

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TOPIC: Sober quotients

It is not easy to guarantee that a T_o quotient space of a sober space is sober. For instance, it the quotient map $X \longrightarrow Y$ is two-to-one, that is not enough. (Construct ω, with open set = upper set, by sticking together adjacent 2' s. If you want it exactly two-to-one , add a 1.)

<u>LEMMA.</u> *If $S \subset X$, if X is sober, and if the quotient space Y of X in which S is pinched to a point is T_o, then Y is sober.*

Proof. Let $f: X \longrightarrow Y$ be the quotient map. If $C \subset Y$ is irreducible closed, $f^{-1}(C)$ is closed; and if it is irreducible, it has a dense point which gives a dense point of C. So suppose $f^{-1}(C)$ reducible, having two disjoint, non-empty relatively open sets. It can't have two such sets that are f-saturated, for their images would be disjoint relatively open. (This depends on C being closed, so f $f^{-1}(C)$ is a quotient map.) In particular, $S \in C$, $S \subset f^{-1}(C)$. If $f^{-1}(C) \subset S^-$, then C has a dense point, viz.,S.

If not, we have a non-empty relatively open $W = f^{-1}(C) \setminus S^-$. There are not two disjoint relatively open sets meeting W, since subsets of W are f- saturated. Hence W^- is irreducible closed and has a dense point w. Now W^- meets S; otherwise W^-, S^- would be f-saturated closed proper subsets covering $f^{-1}(C)$, and $f(W^-)$, $f(S^-)$ would reduce C. Then in C, $\{f(W)\}^-$ contains $f(W^-)$, S, S^- ; This is all of C.