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SCS 56: On a Question of O. Wyler

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TOPIC On a question of O. Wyler

- REFERENCE
1. G. Gierz et al.: A Compendium of Continuous Lattices
 2. Hofmann K.H. and I. Watkins: A review of a theorem of Dixmier's, SCS-Memo 49, 11-30-78
 3. Wyler O.: On continuous lattices as topological algebras, Preprint
 4. --- : Algebraic theories of continuous lattices

In his paper [3,4], O. Wyler raises the following question: Let L be a lattice with 1 ; then the ideal lattice $\text{Id } L$ is an algebraic lattice with an isolated greatest element. The set $\text{Spec } \text{Id } L$ of all proper prime ideals is then Lawson-closed, hence compact in the Lawson topology which equals the patch topology. We write $\text{Spec}_p L$ when we consider the spectrum with the patch topology. Let $\Gamma(\text{Spec}_p \text{Id } L)$ be the continuous lattice of all closed subsets with respect to union. We define a map $D : \Gamma(\text{Spec}_p \text{Id } L) \rightarrow \text{Id } L$ by $D(C) = \bigcap C$ for any closed set C of prime ideals. Wyler notes that D is surjective iff L is distributive and asks:

QUESTION. When is D injective?

He observes:

REMARK. If L is a Boolean algebra, then D is injective.

In this Note we want to indicate that, for distributive lattices, the converse also holds.

In the following let S be any complete lattice and $\text{Spec } S$ the set of all prime elements $p \neq 1$ of S . There is a natural map from the powerset of $\text{Spec } S$ into S :

$$k: \mathcal{P}(\text{Spec } S) \longrightarrow S$$

$$A \longmapsto \inf A .$$

Conversely, there is a map from S onto the collection $\Gamma(\text{Spec } S)$ of all hull kernel closed subsets of S :

$$h: S \longrightarrow \Gamma(\text{Spec } S) \subseteq \mathcal{P}(\text{Spec } S)$$

$$a \longmapsto \uparrow a \cap \text{Spec } S .$$

LEMMA 1. $k(A) = k(B)$ iff $h(k(A)) = h(k(B))$ i.e. iff A and B have the same hull kernel closure. \square

CONSEQUENCE. Let λ be a topology on $\text{Spec } S$ finer than the hull kernel topology and $\Gamma_\lambda(\text{Spec } S)$ the collection of λ -closed sets:

- (1) The restriction of k to $\Gamma_\lambda(\text{Spec } S)$ is injective iff λ coincides with the hull kernel topology.
- (2) If λ is T_1 and if the restriction of k to $\Gamma_\lambda(\text{Spec } S)$ is injective, then the hull kernel topology is T_1 which implies that $\text{Spec } S$ is an antichain. \square

LEMMA 2. Let S be a continuous lattice such that $\text{Spec } S \cup \{1\}$ is Lawson-closed. Then the Lawson topology coincides with the hull kernel topology on $\text{Spec } S$ iff $\text{Spec } S$ is an antichain.

Proof. \implies is clear as the Lawson topology is Hausdorff on a continuous lattice. \Leftarrow : It suffices to show that, for every a in S , the set $\text{Spec } S \setminus \uparrow a$ is hull kernel closed. For this we let $b = \inf A$ where $A = \text{Spec } S \setminus \uparrow a$, and we show that $h(b) = A$. Indeed, let $q \in \text{Spec } S$ be such that $q \not\geq b$. By THE LEMMA there is an element s in the Lawson-closure of A with $q \geq s$. We have $s \in \text{Spec } S$, as $\text{Spec } S \cup \{1\}$ is Lawson-closed. Thus $q = s$, as $\text{Spec } S$ is supposed to be an antichain. As $S \setminus \uparrow a$ is Lawson-closed, we finally ^{have} $q = s \in A$. \square

In a distributive continuous lattice, $\text{Spec } S \cup \{1\}$ is Lawson-closed iff \ll is multiplicative. Thus we have:

PROPOSITION. For a distributive continuous lattice S , the following are equivalent:

- (1) \ll is multiplicative and every prime is maximal.
- (2) $\text{Spec } S \cup \{1\}$ is compact in the Lawson topology and $k : \Gamma_\lambda(\text{Spec } S) \rightarrow S$ is injective.
- (3) $S = O(X)$ for some locally compact Hausdorff space X . \square

COROLLARY. For a distributive lattice L , the following are equivalent:

- (1) $D : \Gamma(\text{Spec}_p \text{Id } L) \rightarrow \text{Id } L$ is injective.
- (2) L is quasi-Boolean, i.e. every principal ideal of L is Boolean.

Indeed, a distributive lattice is quasi-Boolean iff every prime ideal is maximal.