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SCS 55: MC Direct Limits

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Two corollaries: Every variety of MC lattices is closed under classical direct limits; if the bonding maps $L_\alpha \to L_\beta$ are monic so are the limit maps $L_\alpha \to L$.

This responded to M. Barr's conjecture of the second corollary for MC distributive lattices: inverse limits of epimorphisms of locales project epically. A. Joyal has suggested an alternate proof of that, considering the MC distributive (local) lattices as sites with the canonical Grothendieck topology, taking a limit site, and sheafifying. This seems to be no good if the lattices aren't distributive.

The paper is being submitted to JPAA, preprints on request.

The theorem is, of course, a description of L. For the direct system $\{L_\alpha; f_{\alpha\beta}\}$ take the inverse system of coadjoints $\{L_\alpha; g_{\beta\alpha}\}$.

Theorem 0. An inverse limit of completely meet-preserving maps of MC lattices L_α is MC and is in the smallest variety containing all L_α.

Proof. The limit L is contained in the product P of all L_α and is $\{(x_\alpha): g_{\beta\alpha}(x_\beta) \equiv x_\alpha\}$.

There is a reflection $r: P \to L$, completely join-preserving. But consider $S = \{(x_\alpha): g_{\beta\alpha}(x_\beta) \geq x_\alpha\}$. A complete sublattice of P, and when all L_α are MC, r on S preserves finite meets. That is proved by transfinite smoothing of $(x_\alpha) = x \in S$ via $(x^1)_\alpha = \lim_\beta g_{\beta\alpha}(x_\beta)$.

Theorem 1. The inverse limit (semilattice) of $\{L_\alpha; g_{\beta\alpha}\}$ is the direct limit MC lattice.

Proof. We are really, implicitly, using the usual partial functions $i_0(x)$ for x in L_α, defined on all $\beta > \alpha$ by $i_0(x)\ (\beta) = f_{\alpha\beta}(x)$. Define $i_0^1: L_\alpha \to P$ by $(i_0^1)_\beta(x) = \lim_\gamma g_{\gamma\beta}\, f_{\alpha\gamma}\,(x)$. Trivially $i_0^1(L_\alpha) \subset S$; define $i_\alpha = r\, i_0^1$.

Now draw the cat's cradle composed of $\{L_\alpha\}$, $\{f_{\alpha\beta}\}$, $\{g_{\beta\alpha}\}$, $L \subset S \subset P$; note f's and g's compose (direct/inverse mapping systems), $f_{\alpha\beta}\, g_{\beta\alpha}(y) \leq y$, $g_{\beta\alpha}\, f_{\alpha\beta}(x) \geq x$. With no trouble, the i_α are morphisms (finite meet and all join) compatible with $f_{\alpha\beta}$, the images generate nicely, (x_α) in L being $\bigvee i_\alpha(x_\alpha) = \lim i_\alpha\,(x_\alpha)$, and compatible morphisms $h_\alpha: L_\alpha \to M$ factor through $h: L \to M$ defined $h((x_\alpha)) = \bigvee h_\alpha(x_\alpha)$.

If the $f_{\alpha\beta}$ were monic we have $g_{\beta\alpha}\ f_{\alpha\beta}$ = identity and $i_\alpha = i_0^1$, $(i_\alpha)_\alpha$ = identity.