SCS 54: CL-projective Limits of Distributive Continuous Lattices are Distributive

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The spectral theory of C*-algebras led us to consider the category H\(L\) of continuous Heyting algebras with CL
-morphisms as maps. Thus H\(L\) is a full subcategory of CL. The category H\(L\) is clearly closed in CL under the formation of arbitrary products, but it just as clearly fails to be closed under the simplest forms of finite limits, namely, subobjects: The non-distributive five element lattice

\[
\begin{array}{ccc}
\text{0} & \text{1} & \text{2} \\
\text{3} & \text{4} & \text{5}
\end{array}
\]

is obviously a subobject of 3\(^2\), where 3 is the 3-element chain. Thus H\(L\) is not complete in CL. I believe it has escaped our attention that nevertheless H\(L\) is closed in CL under the formation of projective limits. I want to point this out here and state:

**THEOREM.** Within the category CL of continuous lattices and maps preserving arbitrary infs and directed sups, the full subcategory H\(L\) of all continuous Heyting algebras is closed under the formation of (arbitrary products and) projective limits.

Proof. a) The category S of all semilattices with identity and identity preserving morphisms has the property that the full subcategory S\(d\) of distributive semilattices is closed under the formation of direct limits in S. This was shown by Gaskill, loc.cit., and I gave an independent proof in the paper on AFC*-algebras. By the duality of the category S with the category AL of algebraic lattices and CL -maps (Compendium p.184) and the fact that an algebraic lattice is an algebraic Heyting algebra iff the sup-subsemilattice K(L) is a distributive semilattice this observation shows that the assertion of the theorem holds for algebraic lattices. (In fact in the AFC*-paper it is pointed out, among other things, that an algebraic lattice is a Heyting algebra iff in AL it is a projective limit of finite distributive lattices.)

b) The ideal functor Id: CL \(\rightarrow\) AL preserves projective limits by Theorem IV-3.23 on p.221 of the compendium. Now let \((L^\alpha, f^\alpha_j; j, k \in J)\) be an inverse system in H\(L\), and set L = \(\lim_{\alpha} L^\alpha\) in CL. We must show that L is distributive. Now (Id \(L^\alpha_1, Id f^\alpha_{jk}, j, k \in J)\) is an inverse system in AL. Since all \(L^\alpha_1\) are distributive, so are the Id \(L^\alpha_j\). By part a) above, \(\lim_{\alpha} Id L^\alpha_j\) is distributive. But since Id preserves projective limits we have \(Id L \cong \lim_{\alpha} Id L^\alpha_j\). Thus Id L and therefore L is distributive. Q.e.d.

In the light of the functorial spectral theory on H\(L\) which was detailed in the SCS-memo of 30-5-79 of myself and Watkins, where we show the equivalence of H\(L\) with the category LOC of locally quasicompact sober spaces and proper multi-valued maps mapping points to closed sets, it is then clear that the category LOC has projective limits. The techniques given in that memo suffice to calculate them explicitly once one knows the theorem above.