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SCS 48: Projective Limits in CL and Scott's Construction (Comp. III-3)

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The following pages present a first version of a new section of the compendium which is not contained in the DARMSTADT edition It had been suggested in Darmstadt that someone ought to provide a first draft of a third section in Chapter III which would exploit the material provided in Section 1 of that Chapter to give a systematic treatment of Scott's construction of the continuous lattices which are isomorphic to their own function spaces.

Some of this material will be a portion of Jaime Niño's dissertation.

If comments and suggestions are to be made they have to be made quickly if they are to affect the final entry into the compendium. We are closing on on the deadline of the timetable provided by Klaus Keimel.

It may be good to recall that it was also suggested that someone write a fourth section of Chapter III concerning free objects in CL. If someone has a draft it would be good to know about it so that we are not duplicating efforts.

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Section III - 3

Projective limits and Scott's construction

D. Scott's original motivation to consider continuous lattices had much to do with the construction of continuous lattices L which were naturally isomorphic to their own function spaces [L—>L] (see II-2.5). Such continuous lattices provide set theoretical models for the LAMBDA calculus of Church, Curry and Scott. Scott constructed such continuous lattices through suitable limit constructions. In this section we analyze the particular properties of projective limits in the category of continuous lattices, and we illuminate the general principle underlying Scott's construction.

We begin by recalling the concept of a projective limit. We are quite aware that projective limits (in the special sense in which we will use this word in a moment) are special cases of the more general concept of a limit in a category. We prefer to define, for the present record, only the particular kind of limit we will be using in the present section.

3.1. DEFINITION. 1) An inverse system (respectively, direct system) in a category $\underline{A}$ is a family $\{L_j, g_{jk} ; j,k \in J\}$ of objects L_j indexed by a directed set J, and of morphisms $g_{jk}: L_k \longrightarrow L_j$ (respectively, $g_{jk}: L_j \longrightarrow L_k$), one for each pair $j \le k$ in J, such that the relations $g_{ij}g_{jk} = g_{ik}$ hold for all $i \le j \le k$ (respectively, $g_{jk}g_{ij} = g_{ik}$ in the case of a direct system).

2) A cone (respectively, co-cone) ~~of~~ over an inverse (resp., direct) system is a collection $(L, g_j; j \in J)$ consisting of an object and maps $g_j: L \longrightarrow L_j$ (resp., $g_j: L_j \longrightarrow L$) such that the relations $g_{jk}g_k = g_j$ (resp, $g_k g_{jk} = g_j$) hold for $j \le k$.

Cone

$$\begin{tikzcd} & L \arrow[dl, "g_j"'] \arrow[dr, "g_k"] & & \\ \cdots L_j & & L_k \arrow[ll, "g_{jk}"] & \cdots \arrow[l] \end{tikzcd}$$

Co-cone

$$\begin{tikzcd} \cdots L_j \arrow[rr, "g_{jk}"] \arrow[dr, "g_j"'] & & L_k \cdots \arrow[dl, "g_k"] \\ & L & \end{tikzcd}$$

3) A limit cone of an inverse system is a cone $(L, g_j ; j \in J)$ such that for any cone $(L', g'_j ; j \in J)$ over the system there is a unique $\underline{A}$-morphism $g: L' \longrightarrow L$ such that $g_j g = g'_j$ for all $j \in J$. A colimit cone is defined dually.

The object L of a limit cone is called a projective limit of the system, written $\lim L_j$, and the maps g_j are called the limit maps. Dually, the object L of a colimit cone is called a direct limit of the system, written $\operatorname{colim} L_j$, and the g_j are called the colimit maps.

4) A strict projective system is an inverse system in which all maps g_{ij} are surjective (where we assume that we are in a concrete, i.e. set-based category). The projective limit of such a system is called a strict projective limit. ∎

We will work in such categories as $\underline{INF}^{\uparrow} = \underline{INF} \cap \underline{UPS}$ of 1.9 and its dual category $\underline{SUP}^{\circ}$ (see Theorem 1.10), or as $\underline{CL}$, and its dual category $\underline{CL}^{op}$. For mere convenience, we introduce the following convention:

3.2. NOTATION. If $g: S \longrightarrow T$ is a map in $INF^{\uparrow}$ we write $\hat{g}$ in place of $D(g)$. Thus $\hat{\ } : INF^{\uparrow} \longrightarrow (SUP^{\circ})^{op}$ is an equivalence of categories. (See 1.1 - 1.10.)

We are ready for the first result:

3.3. THEOREM. Let $\{L_j, g_{jk}; j \in J\}$ be an inverse system in $INF^{\uparrow}$, and let $(L, g_j; j \in J)$ be a cone over this system in $\underline{INF}^{\uparrow}$. Then the following statements are equivalent:

(1) $(L, g_j ; j \in J)$ is a limit cone of $\{L_j, g_{jk}; j,k \in J\}$ in $INF^{\uparrow}$.

(2) $(\hat{L}, \hat{g}_j ; j \in J)$ is a colimit co-cone of $\{\hat{L}_j, \hat{g}_{jk}; j,k \in J\}$ in $\underline{UPS}$.

Remark. It is important to notice that in condition (2) the universal property for the colimit is satisfied for the category $\underline{UPS}$ which is much larger than the category $\underline{SUP}^{\circ}$ which is dual to $INF^{\uparrow}$.

Proof (2) $\Rightarrow$ (1): Since all maps g_{jk} and g_j are in $\underline{SUP}^{\circ}$ by 1.10

then L is in particular a colimit of the system of the $\widehat{L}_{jk}$ in <u>SUP</u>^{o}. Then (1) follows by simple dualizing.

(1)=>(2):

Proof. We need an explicit description of the upper adjoint $\hat{g}_i : L_i \to L$ of g_i. For this purpose we fix i and take an arbitrary $j \in J$ which we also fix temporarily. For any $k \geq i,j$ we have a function $g_{jk}\hat{g}_{ik} : L_i \longrightarrow L_j$. We claim the family $\{g_{jk}\hat{g}_{ik} : k \geq i,j\}$ is monotone in $[L_i \longrightarrow L_j]$: Consider $i,j \leq k \leq k'$. Then $g_{jk'}\hat{g}_{ik'} = (g_{jk}g_{kk'})(\hat{g}_{kk'}\hat{g}_{ik}) \geq g_{jk}\hat{g}_{ik}$, since $g_{kk'}\hat{g}_{kk'} \geq 1$ by 0-3.6. We let $f_j : L_i \longrightarrow L_j$ be the directed sup $f_j = \sup\{g_{jk}\hat{g}_{ik} : i,j \leq k\}$ and claim that for each $j \leq j'$ we have $f_j = g_{jj'}f_{j'}$: Indeed

$g_{jj'}f_{j'}(x) = g_{jj'}(\sup\{g_{j'k}\hat{g}_{ik}(x) : i,j' \leq k\}) =$

$\sup\{g_{jj'}g_{j'k}\hat{g}_{ik}(x) : i,j' \leq k\}$ (since $g_{jj'}$ is Scott continuous and the sup is directed)

$= \sup\{g_{jk}\hat{g}_{ik}(x) : i,j \leq k\}$ (since $g_{jj'}g_{j'k} = g_{jk}$ and the sup is directed)

$= f_j(x)$, as was asserted. Thus $(L_i, f_j; J)$ is a cone over the inverse system $\{L_k, g_{kk'}; i,j \leq k,k' \in J\}$ in <u>UPS</u>. Now $(L, g_k; i,j \leq k \in J)$ is a limit cone of this system in INF$\uparrow$, since the set $\{k : i,j \leq k \in J\}$ is cofinal in J; but then it is also a limit cone in <u>UPS</u>, since the forgetful functor from <u>INF</u>$\uparrow$ to <u>UPS</u> preserves limits. Hence there is a unique <u>UPS</u>-map $g_i^! : L_i \longrightarrow L$ with $f_j = g_j g_i^!$ for all $j \in J$.

But now $g_i g_i^! = f_i = \sup\{g_{ik}\hat{g}_{ik} : i \leq k\} \geq 1$, since $g_{ik}g_{ik} \geq 1$ by 0-3.6; and *for all $j \in J$ we have*

$g_j g_i^! g_i(x) \overset{f_i g_i(x)=}{=} \sup\{g_{jk}\hat{g}_{ik}g_i(x) : i,j \leq k\}$

$= \sup\{g_{jk}\hat{g}_{ik}g_{ik}g_k(x) : i,j \leq k\}$

$\leq \sup \{g_{jk}g_k(x)\colon j \leq k\}$ (since $\hat{g}_{ik}g_{ik} \leq 1$ by 0-3.6 and $\{k\colon i,j \leq k\}$ is cofinal in $\{k\colon j \leq k\}$)

$= \sup \{g_j(x)\colon j \leq k\} = g_j(x)$.

Since this relation holds for all limit maps g_j and the limit maps separate the points of the projective limit we conclude $g_i'g_i \leq 1$. But the validity of the relations $g_ig_i' \geq 1$ and $g_i'g_i \leq 1$ implies $g_i' = \hat{g}_i$ by 0-3.6. Therefore we have shown

(1) $\quad g_j\hat{g}_i = \sup \{g_{jk}\hat{g}_{ik} : i,j \leq k \in J\}$ for all $i,j \in J$.

and this relation expresses $\hat{g}_i$ in terms of the original data(and the limit maps).

Now we prove the claim on the colimit property. Let therefore $(S,d_j;\ j\in J)$ be an co-cone under the direct system $\{L_j, \hat{g}_{jk};\ j,k \in J\}$. We define a function $d\colon L \longrightarrow S$ by

(2) $\quad d(x) = \sup \{d_j(g_j(x))\colon j \in J\}$.

We first notice that h is in <u>UPS</u> since all the d_j and g_j are and $[L \longrightarrow S]$ is closed under sups.

Now let $i \in I$ and $x \in L_i$. Then $d\hat{g}_i(x) = \sup\{d_jg_j\hat{g}_i(x) : j\in J\}$ (by (2)) $= \sup_j \{d_j \sup\{ g_{jk}\hat{g}_{ik}(x)\colon i,j \leq k\}\}$ (by (1)) $= \sup d_jg_{jk}\hat{g}_{ik}(x) : j,k \in J$ with $i,j \leq k\}$ (since $d_j \in$ <u>UPS</u>).

But $j \leq k$ implies $d_j = d_k\hat{g}_{jk}$, and so $d_jg_{jk} = d_k\hat{g}_{jk}g_{jk} \leq d_k$, since $\hat{g}_{jk}g_{jk} \leq 1$ by 0-3.6. Therefore $d_jg_{jk}\hat{g}_{ik} \leq d_k\hat{g}_{ik} = d_i$, whence $dg_i(x) \leq d_i(x)$. But $d_i(x) = \sup \{d_kg_{kk}\hat{g}_{ik}(x)\colon i \leq k\}$ $\leq \sup\{ d_jg_{jk}\hat{g}_{ik}(x)\colon i,j \leq k\} = dg_i(x)$. Hence $dg_i(x) = d_i(x)$, and since d is the desired fill-in map for the colimit, It is clearly unique ~~determined~~. Thus we have shown that $(L,\hat{g}_j\colon j\in J)$ is a colimit cone in <u>UPS</u> as was claimed.□

From the proof of 3.3 we extract the following information which is of independent interest:

3.4 <u>COROLLARY</u>. Under the circumstances of Theorem 3 3, the colimit maps $\hat{g}_i : L_i \longrightarrow L$ are determined by the formula:

(1) $g_j \hat{g}_i = \sup \{g_{jk}\, \hat{g}_{ik} : i,j \leq k \text{ in } J\}$.

If $(S, d_j;\ j \in J)$ is a co-cone under the direct system $\{L_j, g_{jk};\ j,k \in J\}$ and $d: S \longrightarrow L$ the fill -in map guaranteed by the colimit porperty, then d is given by the formula

(2) $d = \sup \{d_j g_j : j \in J\}$.

Furthermore, one has the important formula

(3) $\sup \hat{g}_j g_j = 1_L$.

Proof We proved (1) and (2) in the proof of 3.3 and (3) will be an immediate consequence of the following slightly more general result □

3 5 <u>COROLLARY</u>. Let $\{L_j, g_{jk};\ j,k \in J\}$ be an inverse system with limit cone $(L, g_j : j \in J)$ in $\mathrm{INF}^{\uparrow}$. Let $(L', g'_j ;\ j \in J)$ be a con over the system and let $g: L' \longrightarrow L$ be the canonical map of 3.13 Then the following statements are equivalent:

(1) g is injective.
(2) $\hat{g} g = 1_{L'}$.
(3) $\sup \hat{g}'_j g'_j = 1_{L'}$.

Proof. (1)<=> (2) by 0-3.7.

(2)=>(3): $\sup \hat{g}'_j g'_j = \sup \hat{g}\hat{g}_j g'_j$ (since $g'_j = g_j g$)

$= \hat{g}(\sup \hat{g}_j g'_j)$ (since $\hat{g} \in$ <u>UPS</u>) $= \hat{g}(\sup \hat{g}_j g_j g) = \hat{g}(\sup \hat{g}_j g_j) g$ (since sup is calculated pointwise) $= \hat{g} g$ (since $\sup \hat{g}_j g_j = 1_L$ by 3.4.(2) with $d_j = \hat{g}_j$, $d = 1_L$) $= 1_{L'}$ by (2).

(3) =>(2): $\hat{g} g = (\sup \hat{g}_j g_j) g$ (by 3.4.(2) with $d_j = \hat{g}'_j$ and $\hat{g} = d$)

$= \sup \hat{g}'_j g_j g = \sup \hat{g}'_j g'_j$ (since $g'_j = g_j g$) $= 1_{L'}$, by (3). □

Note that in particular we have:

3.5. COROLLARY. For the limit maps g_j of a projective limit we have

$$\sup \hat{g}_j g_j = 1. \square$$

We now address the question when the map g in Corollary 3.5 is surjective.

3.6. PROPOSITION. Under the conditions of 3.5 the following conditions are equivalent:

(1) g is surjective.

(2) $\operatorname{im} g_j \subseteq \operatorname{im} g'_j$ for all j.

Proof. (1)=>(2): $\operatorname{im} g'_j = g_j g(L') = \operatorname{im} g_j$ if g is surjective.

(2)=>(1): By (2), all sets $g'^{-1}_j g_j(y)$ are non-empty for any $y \in L$. If $j \leq k$, then $u \in g'^{-1}_k g_k(y)$ implies $g'_k(u) = g_k(y)$ and so $g'_j(u) = g_{jk} g'_k(u) = g_{jk} g_k(y) = g_j(y)$, i.e. $u \in g'^{-1}_j g_j(y)$. Thus the family $\{g'^{-1}_j g_j(y) : j \in J\}$ is a filter basis in L'. By II-5.8, and II-5.9 these sets are closed in $\Lambda L'$, and $\Lambda L'$ is quasi-compact by II-5.9. Hence there is an element x in the intersection of the filter basis. Then $g_j g(x) = g'_j(x) = g_j(y)$ for all $j \in J$, whence $g(x) = y$. ∎

3.7. PROPOSITION. Under the conditions of 3.5. assume that all L_j are continuous lattices. Then L is a continuous lattice. If all g_{jk} are also surjective, then all g_j are surjective, too. More generally, $\operatorname{im} g_j = \bigcap_{j \leq k} \operatorname{im} g_{jk}$ in this case.

Proof. Since CL is closed under product and subalgebras (I-2.7), the category CL is complete and L is a continuous lattice. We now consider the Lawson topologies on L_j and L, which are compact by II-5.10. All maps g_{jk} and g_j are continuous by II-5.8. It is a well-known fact that for an inverse system of compact spaces and continuous maps one has $\operatorname{im} g_j = \bigcap_{j \leq k} \operatorname{im} g_{jk}$ for all j. ∎

We now return for a moment to the general category theoretical setting and recall what it means that a functor preserves projective limits:

3.8. DEFINITION. Let $\underline{A}$ and $\underline{B}$ be complete categories A functor $F: \underline{A} \longrightarrow \underline{B}$ is said to preserve propjective limits [resp., in the case of concrete categories, strict projective limits] iff the following condition is satisfied:

Let $(L, G_j;\ j \in J)$ be a limit cone of an inverse system [resp a strict projective system (3.1.4)] $\{L_j, g_{jk};\ j,k \in J\}$ in $\underline{A}$, and let $(\ T,\ h_k;\ j \in J\ )$ be the limit cone of the image inverse system $(FL_J, F_{gjk};\ j,k \in J\}$ in $\underline{B}$. Let $f: FL \longrightarrow T$ be the natural map guaranteed by 3.1 3 Then f is an isomorphism

In short: $F(\lim L_j) = \lim FL_j$

Notice that the preservation of strict projective limits is a weaker property than the preservation of projective limits (in case we are dealing, as we always are, with concrete categories).

For the purposes of the construction we are about to begin it is convenient to have a special notation:

3.9. DEFINITION. A retro-functor of a category $\underline{A}$ is a pair (F,p) consisting of a self functor $F:\underline{A} \longrightarrow \underline{A}$ of $\underline{A}$ together with an epic natural transformation $p_L: FL \longrightarrow L$.

When dealing with concrete categories we will insist that p is surjective.

3.10 CONSTRUCTION. Let (F,p) be a retro-functor of a complete category $\underline{A}$ and let $\widetilde{F}L$ be the projective limit of the inverse system

$$L \xleftarrow{p_1} FL \xleftarrow{Fp_L} F^2L \xleftarrow{F^2p_L} F^3L \ \ldots\ldots$$

Let $\widetilde{p}_L: \widetilde{F}L \longrightarrow L$ be the limit map from the limit cone.

Then $\widetilde{F}:\underline{A} \longrightarrow \underline{A}$ is a self functor of $\underline{A}$ and $\widetilde{p}_L:FL \longrightarrow L$ is a natural transformation If $\underline{A}$ is a concrete category, F preserves surjectives, and if the limit maps of any strict projective limit are surjective, then $(\widetilde{F},\widetilde{p})$ is a retro-functor, By 3.1.3, there is a natural map $f_L : F(\lim F^n L) = F\tilde{F}L \longrightarrow \lim F^n L = \tilde{F}L$ filling in the diagram

$$
(1)\qquad
\begin{tikzcd}
 & FL \arrow[d, equal] & F(FL) \arrow[l, "Fp_L"'] \arrow[d, equal] & F(F^2L) \arrow[l, "F(Fp_L)"'] \arrow[d, equal] & \cdots \arrow[l, "F(F^2p_L)"'] & F(\lim F^nL) = F\tilde{F}L \arrow[d, "f_L"] \\
L & FL \arrow[l, "p_L"] & F^2L \arrow[l, "Fp_L"] & F^3L \arrow[l, "F^2p_L"] & \cdots \arrow[l, "F^3p_L"] & \lim F^nL = \tilde{F}L .
\end{tikzcd}
$$

We have two commuting squares

$$
(2)\qquad
\begin{tikzcd}
FL \arrow[d, "p_L"'] & F\tilde{F}L \arrow[l, "F\tilde{p}_L"'] \arrow[d, shift right, "p_{\tilde{F}L}"'] \arrow[d, shift left, "f_L"] \\
L & \tilde{F}L \arrow[l, "\tilde{p}_L"]
\end{tikzcd}
\quad ;
$$

in particular, $\tilde{p}_L$ coequalizes f_L and $p_{\tilde{F}L}$.

If F preserves projective limits, then $f_L : F\tilde{F}L \longrightarrow \tilde{F}L$ is an isomorphism.

If F preserves surjective maps and strict projective limits, then f_L is an isomorphism too.

Proof The assertions are straightforward from the definitions $\square$

3.11 DEFINITION. If (F,p) is a retro-functor of $\underline{A}$, we say that (F,p) is the associated retro-functor, and we call $f_L : F\tilde{F}L \longrightarrow \tilde{F}L$ the associated morphism. $\square$

We need a rather technical condition.

3.12. DEFINITION. We say that a self functor $F: \mathrm{INF}^{\uparrow} \longrightarrow \mathrm{INF}^{\uparrow}$ [or $F: \underline{CL} \to \underline{CL}$] Scott continuous is adapted provided that there exists a natural/function $\pi_S: [S \longrightarrow S] \longrightarrow [FS \longrightarrow FS]$ [resp. CL] such that $\pi_L(1) = 1$ for all $g, h: S \longrightarrow T$ in $\mathrm{INF}^{\uparrow}$ we have $(Fh)^{\wedge} Fg = \pi_S(\hat{h}g)$. □

The relevance of this condition becomes apparent in the following result:

3.13. PROPOSITION. Let F be an adapted self-functor of CL which preserves surjectivity of CL-functions. Then F preserves strict projective limits of continuous lattices.

Proof. Let $\{L_j, g_{jk}; j,k \in J\}$ be an inverse system of continuous lattices with surjective maps g_{jk}. Then the limit maps $g_j: L \longrightarrow L_j$ are surjective by 3.7. By hypothesis all Fg_j are surjective. Hence the natural map $f: FL \longrightarrow \lim FL_k$ is surjective by 3.6.

On the other hand we calculate

$$\sup(Fg_j)^{\wedge}(Fg_j) = \sup \pi_L(\hat{g}_j g_j) = \pi_L(\sup \hat{g}_j g_j) = \pi_L(1) \text{ (by 3.5)}$$

$= 1$. Then f is injective by 3.4. □

This allows us to conclude the following central result:

3.14. THEOREM. Let (F,p) be a retro-functor of CL and suppose that F is adapted and preserves surjectivity of CL-maps. Then the associated retrofunctor $(\tilde{F}, \tilde{p})$ exists, and the associated map $f_L: F\tilde{F}L \longrightarrow \tilde{F}L$ is an isomorphism.

Proof. Since p is surjective and F preserves surjectivity, all maps in the inverse system

$$L \xleftarrow{p_L} FL \xleftarrow{Fp_L} F^2L \xleftarrow{F^2p_L} F^3L \ldots$$

are surjective. Hence $\tilde{F}L$ is a strict projective limit and all limit maps, in particular $\tilde{p}_L: \tilde{F}L \longrightarrow L$ are surjective. By 3.13, the map f_L is an isomorphism. □

Following Scott we associate with each complete lattice L the complete lattice

$H(L) = [L \longrightarrow L]$ (see II-2.5). If $g:S \longrightarrow T$ is in INF we define a function $H(g):H(S) \longrightarrow H(T)$ by $H(g)(\varphi) = g\varphi\hat{g}$; note that $g\varphi\hat{g}$ is indeed Scott continuous and so $H(g)$ is well defined. Clearly $H(1) = 1$ and $H(g)H(g') = H(gg')$, and so H is functorial. We now claim that $H(g)$ has a lower adjoint $H(g)^\wedge : H(T) \longrightarrow H(S)$. Indeed if we set $H(g)^\wedge(\psi) = \hat{g}\psi g$,then $H(g)^\wedge H(g)(\varphi) = \hat{g}g\varphi\hat{g}g \leq \varphi$ and $H(g)H(g)^\wedge(\psi) = g\hat{g}\psi g\hat{g} \geq \psi$ by 0-3.6, which shows by 0-3.6 that $H(g)^\wedge$ is the desired adjoint. In particular, $H(g)$ preserves arbitrary infs by 0-3.3. Finally, if g is Scott continuous, then so is $H(g)$, since sups are calculated pointwise and g preserves directed sups. We have

3.15.LEMMA. There is a retro-functor (H,p) of $INF^\uparrow$ such that $H(L) = [L \longrightarrow L]$ and $H(g) = g\varphi\hat{g}$; also

$p_L(g) = \min g(L)$. If we let $\pi_S:[S \longrightarrow S] \longrightarrow [HS \longrightarrow HS]$ be defined by $\pi_S(g)(\varphi) = g\varphi g$, then π_S preserves directed sups and

$$(Hg)^\wedge(Hg) = \pi_S(\hat{g}g).$$

Moreover, H maps CL into itself and preserves the surjectivity of morphisms.

Proof. If we define $p_L:H(L) \longrightarrow L$ by $p_L(g) = \mathrm{mil}\ p(L)$, then p_L is a surjective $INF^\uparrow$-morphism whose lower adjoint associates with an element $x \in L$ the constant function $L \longrightarrow L$ with value x. We have $(Hg)^\wedge(Hg)(\varphi) = \hat{g}g\varphi\hat{g}g = \pi_S(\hat{g}g)(\varphi)$. It is straightforward to verify that π_S preserves directed sups. If L is a continuous lattice then so is $H(L) = [L \longrightarrow L]$ by II-2.8. In order to see that H preserves surjectivity, let $g:S \longrightarrow T$ be a surjective $INF^\uparrow$-map. Then take $\psi \in H(T)$ and set $\varphi = H(g)^\wedge(\psi)$. Then $H(g)(\varphi) = g\hat{g}\psi g\hat{g} = \psi$ since $g\hat{g} = 1$ by 0-3.7. ▯

3.16.NOTATION. We call H the Scott functor. ▯

By 3.14 we now retrieve Scott's original theorem:

3.17.THEOREM. For any continuous lattice L, the retrofunctor $(\tilde{H},\tilde{p})$ associated with the Scott functor exists and the associated map $f_L:H\tilde{H}L \longrightarrow \tilde{H}L$ is an isomorphism. In other words, if S is the continuous lattice $\tilde{H}L$, then there is a natural isomorphism $[S \longrightarrow S] \longrightarrow S$. Each element f of S may be considered as a ~~function~~ Scott continuous function $S \to S$ so that for $s \in S$ the element $f(s)$ is well-defined. $\square$

Notice that Scott's theorem could be ~~p~~ rephrased as saying, in short terms, ~~xxx~~ that every continuous lattice is functorially the quotient of a continuous lattice which is naturally isomorphic to its own function space.

Now we consider the functor $\mathrm{Id}\colon \underline{CL} \to \underline{CL}$ (see 1.18 and 1.19). Then (Id, r), $r(I) = \sup I$ is a retrofunctor with surjective r by I-2.1. We define $\pi_S\colon [S \to S] \to [\mathrm{Id}\, S \to \mathrm{Id}\, S]$ by $\pi_S(g)(I) = \downarrow g(I)$. Then $\pi_S(g)$ preserves directed sups and satisfies $\pi_S(1) = 1$. Moreover, by 1.18 and 1.19 we have $(\mathrm{Id}\, g)^\wedge (\mathrm{Id}\, g)(I) = \downarrow \hat{g}(\downarrow g(I)) = \downarrow \hat{g}g(I)$ (by 0-1.11) $= \pi_S(\hat{g}g)(I)$. Furthermore, the functor Id preserves surjectivity: Indeed if g is surjective, then $g\hat{g} = 1$ by 0-3.7 and thus $(\mathrm{Id}\, g)(\mathrm{Id}\, g)^\wedge(I) = \downarrow g\hat{g}(I) = \pi_S(g\hat{g})(I) = \pi_S(1)(I) = I$.

Now we have the following theorem from 3.14:

3.18. THEOREM. The retro-functor (Id, r) of $\underline{CL}$ has an associated retro-functor $(\widetilde{\mathrm{Id}}, \tilde{r})$ with a surjective $\underline{CL}$-map $\tilde{r}\colon \widetilde{\mathrm{Id}}\, L \to L$ such that the associated map $f_L\colon \mathrm{Id}\, \widetilde{\mathrm{Id}}\, L \to \widetilde{\mathrm{Id}}\, L$ is an isomorphism. In orther words, if S is the continuous lattic $\widetilde{\mathrm{Id}}\, L$, then there is a natural isomorphism $\mathrm{Id}\, S \to S$. Each element I of S may be considered as an ideal of S so that for $s \in S$ the ~~element~~ relation $s \in I$ is well defined. $\square$

Notice that this theorem could be rephrased by saying, ~~that~~ in short terms, that every continuous lattice is functorially the quotient of an arithmetic lattice which is naturally isomorphic to its own ideal lattice.

6

The constructions in 3.17 and 3.18 appear to yield rather big continuous lattices. We record, however, that in terms of weights the increase in size is not so exorbitant in the case of Scott's construction. The ideal construction may be substantial, though.

3.19. PROPOSITION. Let L be a continuous lattice. then

(1) $w(\tilde{H}(L)) = \max(\aleph_0, w(L))$.

(2) $w(\widetilde{\mathrm{Id}}\ L)$ ~~xxmaxxxxxxxxx~~ $\leq \exp^{\aleph_0}$ card S , where $\exp x = 2^x$ for a cardinal x and $\exp^{\aleph_0} x = \sup \exp^n x$.

Proof. (1) By II-8.13 we have $w(S) = wH(S)$ for any infinite continuous lattice. Since $\tilde{H}(S)$ is a subalgebra of a countable product of continuous lattice of weight $w(S)$, we conclude that $w(\tilde{H}(S)) = w(S)$ for any infinite continuous lattice S by II-8.14. If S is finite, then $w(\tilde{H}(S)) = \aleph_0$.

(2) For every continuous lattice S we have $w(\mathrm{Id}\ S) =$ card $(K(\mathrm{Id}\ S))$ (by II-8.4) = card S. Now card Id S $\leq$ exp card S where $\exp x = 2^x$ for a cardinal x. Thus $w(\mathrm{Id}^n\ S) \leq \exp^{n-1}$ card S If we write $\exp^{\aleph_0} x = \sup \exp^n x$, we obtain, as before, $w(\widetilde{\mathrm{Id}}\ L) = \exp^{\aleph_0}$ card S. ☐

EXERCISES

3.14 <u>EXERCISE.</u> An adapted functor $F:\mathrm{INF}^{\uparrow} \longrightarrow \mathrm{INF}^{\uparrow}$ preserves injectivity of maps.

(Let $g:S \longrightarrow T$ be injective Then $\hat{g}g = 1_S$ by 0-3.7. Then $(Fg)\hat{}(Fg) = \pi_S(\hat{g}g) = \pi_S(1) = 1_{FS}$, and so Fg is injective by 0-3.7.)

3.15. <u>EXERCISE.</u> An adapted functor F (as in 3.14) which preserves the surjectivity of maps preserves images, i.e $F(\mathrm{im}\ g) \cong \mathrm{im}\ Fg$

(In $\mathrm{INF}^{\uparrow}$ every map has a unique (up to isomorphism) decomposition

$$\begin{tikzcd} S \arrow[rr, "f"] \arrow[dr, "\bar{f}"'] & & T \\ & \mathrm{im}\ f \arrow[ur, "\underline{f}"'] & \end{tikzcd}$$

(see 0-3.9); Apply F and observe that $F\bar{f}$ is surjective, $F\underline{f}$ is injective, so that one may write $(Ff)^{-} = F\bar{f}$, $(Ff)_{-} = F\underline{f}$ and $F(\mathrm{im}\ f) = \mathrm{im}\ (\bar{F}f)$.)

3.16 <u>EXERCISE.</u> Let $F: \underline{CL} \longrightarrow \underline{CL}$ be an adapted functor presering the surjectivity of maps and intersections of filtered families of subalgebras (i.e., projective limits with injective maps g_{jk}).
The F preserves arbitrary projective limits

(The injectivity of $f:FL \longrightarrow \lim FL_k$ follows as in 3.13. As to the surjectivity, observe

$$F(\mathrm{im}\ g_j) = F\Big(\bigcap_{j\leq k} \mathrm{im}\ g_{jk}\Big) = \bigcap_{j\leq k} F(\mathrm{im}\ g_{jk}) = \bigcap_{j\leq k} \mathrm{im}\ Fg_{jk}$$

$= \mathrm{im}\ h_j$, where $h_j:\lim L_k \longrightarrow L_j$ is the limit map Then 3.6 shows that g is surjective).

This may be used to show that Scott's functor H in fact preserves all projective limits in <u>CL</u>. By proving the surjectivity of the map $f:HL \longrightarrow \lim HL_k$ directly, one can show the stronger statement

3.17 EXERCISE. The Scott functor $H: \mathrm{INF}^{\uparrow} \longrightarrow \mathrm{INF}^{\uparrow}$ preserves projective limits.

EXERCISES

Proposition 3.1[illegible] is perfectly sufficient for the proof of the central theorem [illegible].14 But generalisations are possible

3.14 EXERCISE. Let F be an adapted self functor of $\mathrm{INF}^{|}$ which preserves surjectivity of maps and preserves intersections of filtered subalgebras Then F preserves arbitrary projective limits in $\mathrm{INF}^{|}$.

(As in [illegible].1[illegible] we only have to worry about the surjectivity of the map $f: FL \longrightarrow \lim FL_k$

NOTES

The basic construction which we have formulated in 3.14 in a general way, was introduced by D. Scott in [ ] for the construction of the continuous lattices obtained in 3 17, which are naturally isomorphic to their own self function space. This was a canonical solution for the question for a systematic way to construct set theoretical models for the lambda calculus of Church, Curry and Scott. This construction was one of Scott's motivations to introduce continuous lattices. It was also Scott who in [ ] observed for sequential projective limits the essence of theorem 3.[illegible] although in the present generality and in its precise formulation it had not been previously put down. Theorem 3 14 itself is new as is Theorem 3.18. Theorem 3 1[illegible] gives a solution to a question raised by R.E. Hoffmann in [ ] (Continuous posets and adjoint sequences. Semigroup Forum to appear). He analyzed precisely the question, when for a continuous lattice L the map r_L: Id L ⟶ L allows a finite sequence $r_0 = r_L, r_1, \ldots, r_p$ of morphisms such that r_{m+1} is lower adjoint to r_m (Example $r_1 : L \longrightarrow \mathrm{Id}\, L$, $r_1(x) = \downarrow x$, see I-2.1). Finite chains of this sort exist if L is of the form $\mathrm{Id}^n L$. The continuous lattices Id L give rise to infinite chains of lower adjoints For details we refer to Hoffmann's article.

At a later point we hope to discuss at greater length the applications and the raminfications of the ideas discussed in this section Theorem 3 17 will appear in the Tulane Dissertation of J. Nino.