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SCS 46: A Result about O(X)

Gerhard Gierz
University of California, Riverside, CA 92521, gerhard.gierz@ucr.edu

Jimmie D. Lawson
Louisiana State University, Baton Rouge, LA 70803 USA, lawson@math.lsu.edu

Michael Mislove
Tulane University, New Orleans, LA USA, mislove@tulane.edu

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SCS MEMO

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Title: A Result About O(X)

References: 1. Hofmann--Lawson, Spectral theory for distributive continuous lattices, submitted.

2. Keimel--Mislove, The lattice of open subsets of a topological space, SCS Memo Dec. 15, 1976.

For a Hausdorff space X, Day and Kelley were the first to notice that the topology O(X) is a continuous lattice if and only if X is locally compact. The question of when O(X) is a continuous lattice for more general spaces has been successfully attacked in (1), and that study has produced the spectral theory for distributive continuous lattices alluded to in the title. A concommitant question is the following: For a topological space X, if O(X) is a compact semilattice, then must O(X) be a continuous lattice? A partial answer was presented in (2), where it was shown that the answer is yes for Hausdorff spaces which are embeddable in first countable compacta. The purpose of this memo is to show that the answer remains true for all Hausdorff spaces; i.e., if X is a Hausdorff space and O(X) is a compact semilattice, then O(X) is a continuous lattice, and, consequently, X is locally compact.

We begin with the following lemma, which is also of independent interest:

<u>1. Lemma.</u> Let S be a compact semilattice, and let $\{x_i\}$ be a net in S. Let $x = \lim x_i$ exist in X, and let $y = \underline{\lim}\, x_i = \sup_i \inf_{j \geq i} x_j$. Then $y \leq x$, and there is an order arc connecting x and y.

<u>Proof.</u> For any index i, $\inf_{j \geq i} x_j \leq x_k$ for every $k \geq i$, and so $\inf_{j \geq i} x_j \leq$

$\lim x_i = x$. Thus $\sup_i \inf_{j \geq i} x_j \leq x$, and so $y \leq x$.

Recall that any compact Hausdorff space is a uniform space and that the neighborhoods of the diagonal form a base for the uniform structure. Also, for a compact monoid, the open Δ-ideals form a base for the uniform structure, where a Δ-ideal is a subset U of the product space with $U \cdot \Delta \cup \Delta \cdot U \subset U$.

Let U be any open Δ-ideal of S, and choose an open subset V of S with $x \in V$ and $V \times V \subset U$. Since $y = \underline{\lim}\, x_i$ and $U(y)$ is an open set containing y, there is some index i_1 with $\inf_{j \geq i} x_j \in U(y)$ for $i \geq i_1$; similarly $x = \lim x_i$ implies there is some index i_2 with $x_j \in V$ for $j \geq i_2$. If i is any index greater than i_1 and i_2, then $\inf_{j \geq i} x_j \in U(y)$, which is an open set, and this implies there is some finite subset of the indices greater than i, say $j_1, \ldots, j_n$ with $\inf_{k=1}^{n} x_{j_k} \in U(y)$. Note that $x_{j_k} \in V$ also holds for each $k = 1, \ldots, n$, and so $(x, x_{j_1}), (x_{j_1}, x_{j_2}), \ldots, (x_{j_{n-1}}, x_{j_n}) \in U$. Since U is a Δ-ideal, we conclude that (x, xx_{j_1}), $(xx_{j_1}, xx_{j_1}x_{j_2}), \ldots, (xx_{j_1} \ldots x_{j_{n-1}}, xx_{j_1} \ldots x_{j_n}) \in U$. Hence this forms a totally ordered U-chain from x to $xx_{j_1} \ldots x_{j_n}$, and since $(x_{j_1} \ldots x_{j_n}, y) \in U$ and $y \leq x$, we can extend this to a totally ordered U-chain from x to y.

Now, for each open Δ-ideal U of S, we have a finite totally ordered U-chain from x to y; moreover, if $V \subset U$, then the V-chain is also a U-chain. Thus, if we direct this family of finite totally ordered chains by the natural direction (inclusion) on the open Δ-ideals, then they must cluster to some limit set C in the space of closed subsets of S. Since each chain is totally ordered, so is C, and clearly x and y are in C. Finally, since the finite chains are residually U-connected (i.e., are residually a U-chain) for each U, we conclude that C, their limit is connected. This clearly is the order-arc we are seeking. □

2. Lemma. Let X be a topological space, and suppose that O(X) is a compact semilattice. The map $f : X \to O(X)$ by $f(x) = X \setminus \overline{\{x\}}$ is then an open mapping onto its image in O(X). Moreover, if $\{x_i\}$ is any net in X, then $\underline{\lim}\, f(x_i) = X \setminus \bigcap_i \overline{\{x_j : j \geq i\}}$.

Proof. Fix an open subset U of X. Then $\uparrow U = \{V \in O(X) : U \subset V\}$ is a closed subset of O(X), and so its complement $O(X) \setminus \uparrow U = \{V \in O(X) : U \not\subset V\}$ is an open subset of O(X). Now, $f(U) = \{f(x) : x \in U\} = \{f(x) : U \not\subset f(x)\}$ $= f(X) \cap (O(X) \setminus \uparrow U)$ is open in f(X), and so f is open onto its image.

Suppose that $\{x_i\}$ is any net in X. Then $\underline{\lim}\, f(x_i) = \sup_i \inf_{j \geq i} f(x_j)$ $= \bigcup_i \left(\bigcap_{j \geq i} X \setminus \overline{\{x_j\}}\right)^\circ = \bigcup_i X \setminus \overline{\{x_j : j \geq i\}} = X \setminus \bigcap_i \overline{\{x_j : j \geq i\}}$, as claimed. □

3. Theorem. For a Hausdorff space X, O(X) is a compact semilattice if and only if O(X) is a continuous lattice, and this holds precisely when X is locally compact.

Proof. As we remarked at the beginning, it is well known that O(X) is a continuous lattice if and only if X is locally compact for Hausdorff spaces. Moreover any continuous lattice is a compact semilattice in the CL-topology. Thus we only need to show that X is locally compact if O(X) is a compact semilattice and X is Hausdorff.

Let $f : X \to O(X)$ by $f(x) = X \setminus \overline{\{x\}} = X \setminus \{x\}$, since X is Hausdorff. Consider $\overline{f(X)}$ in O(X). If $V \in \overline{f(X)}$, then there is a net $\{x_i\}$ in X with $V = \lim f(x_i)$. There are two possibilities:

1. $\{x_i\}$ has no cluster points in X. Then $\bigcap_i \overline{\{x_j : j \geq i\}} = \emptyset$, and so $\underline{\lim}\, f(x_i) = X$. Since $\underline{\lim}\, f(x_i) \leq \lim f(x_i) = V$, we conclude that $V = X$.

2. Suppose that $\{x_i\}$ has a cluster point x in X. Then there is a subnet $\{x_k\}$ with $x = \lim x_k$, and so $\{x\} = \bigcap_k \overline{\{x_m : m \geq k\}}$, and so V is an open subset

of X containing $X\setminus\{x\}$. Lemma 1 then implies that there is an order-arc in $O(X)$ from V to $X\setminus\{x\}$, which clearly implies $V = X\setminus\{x\}$, since the latter is a co-atom of $O(X)$.

We draw two conclusions from the above discussion. First, $\overline{f(X)} = f(X) \cup \{X\}$. Second. we see from the latter portion of the argument that, for a net $\{x_i\}$ in X with $\lim x_i = x$ in X, then $\lim f(x_i) = f(x)$; i.e., f is continuous. Lemma 2 implies that f is open onto $f(X)$, and f is clearly one-to-one since X is Hausdorff. Finally, since $f(X)$ is open in its closure, $f(X)$ is locally compact in the relative topology, and so X is also locally compact, being homeomorphic to $f(X)$. □

It is clear that the full strength of the Hausdorff hypothesis is utilized in this proof; without it, one cannot conclude in part 2 of the proof of the Theorem that $V = X\setminus\{x\}$ if the net $\{x_i\}$ has a cluster point in X. Thus the proof that f is continuous is lacking without X Hausdorff. The general question, if $O(X)$ is a compact semilattice must it be a continuous lattice, is thus open for general spaces.

Now, for any topology $O(X)$, the sets $X\setminus\overline{\{x\}}$ are primes in $O(X)$, and so the question referred to above is related to the following question:

If S is a compact semilattice such that the primes order-generate (i.e., every element is the infimum of the primes above it), must S be a continuous lattice?

Clearly an affirmative answer to this question would also settle the question about $O(X)$ in the affirmative; we show that in fact the questions are equivalent:

<u>Theorem.</u> The following are equivalent:

a. Each compact semilattice in which the primes order-generate is a continuous lattice.

b. Each topology $O(X)$ which is a compact semilattice is a continuous lattice.

Proof. We have already noted that a implies b. Conversely, suppose b holds, and let S be a compact semilattice in which the primes order-generate. Let P be the set of primes of S, other than 1. We topologize P with the hull-kernel topology; i.e., a base for the closed subsets of P are the sets of the form $P \cap \uparrow x$, where x ranges over S. As is shown in (1), it then follows that each closed subset of P in this topology is of the form $P \cap \uparrow x$, some $x \in S$. Thus, the open sets of P are of the form $P \setminus \uparrow x$, as x ranges over S. Thus the map $f : S \to O(P)$ by $f(x) = P \setminus \uparrow x$ is an algebraic isomorphism of S onto $O(P)$. Since S is a compact semilattice, it follows that $O(P)$ is also. According to b, $O(P)$ must be a continuous lattice. Thus S is a continuous lattice. □