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# SCS 43: Locally Quasicompact Sober Spaces are Baire Spaces

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TOPIC Locally quasicompact sober spaces are Baire spaces

REFERENCES Lawmann I (K.H.Hofmann and J.D.Lawson, Irreducibility, Semigroup F.13(76/77)

Lawmann II ( " " " , The spectral theory of continuous lattices, on the referee's table)

LEMMA 1. Let L be a continuous latticd and V a Scott-open subset such that for some sequence $a_0 > a_1 > a_2 > \ldots$ one has

(HYP) $Va_n \subseteq V$ for $n = 0,1,\ldots$

Then for all $v \in V$ there is an open filter $U \subseteq V$ such that $va_n \in U$, n 0,1,...

(Remark. For the applications in this memo, v=1 would suffice.)

Proof. Let $v \in V$ be given and set $b_0 = 1$. Suppose that we found elements $b_k$, $k=0,\ldots,n$ such that

(i) $vb_k \in V$ for $k = 0,\ldots,n$,

(ii) $b_k \ll va_{k-1}b_{k-1}$ for $k=1,\ldots,n$.

(For n=0, nothing is assumed in lieu of (ii)). We construct $b_{n+1}$: By (i) we have $vb_n \in V$, hence by (HYP) we have (1) $va_nb_n \in V$. Let D denote the directed set $\Downarrow va_nb_n$. Then (2) $\sup D = va_nb_n$ since L is continuous. In particular, L is meet-continuous, and thus $\sup vD = v \sup D = v^2a_nb_n = va_nb_n \in V$ by (2) and (1). As V is Scott-open, $vD \cap V \neq \emptyset$, i.e. there is a $b_{n+1} \in D$ with (3) $vb_{n+1} \in V$ and $b_{n+1} \in \Downarrow va_nb_n$, i.e. (4) $b_{n+1} \ll va_nb_n$. Then (3) and (4) make (i) and (ii) valid with n+1 in place of n. By induction we thus have (i) and (ii) for all k=0,1,... (resp., k=1,2,...). Then (ii) implies $b_{n+1} \ll va_nb_n \leq b_n$, hence

(5) $b_{n+1} \ll b_n$ for all n. By (i) we note $b_k \geq vb_k \in V$, whence

(6) $\uparrow b_n \in V$ for all n. By (ii) we note $va_n \geq va_nb_n \geq b_{n+1} \in V$, whence

(7) $va_n \in \uparrow b_{n+1}$ for all n. Now set $U = \bigcup_{n=0}^{\infty} \uparrow b_n$. Then U is a filter as an ascending union of filters. By (5) it is an open filter, and by (6) $U \subseteq V$. But (7) shows $va_n \in U$ for all n. ∎

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LEMMA 2. Let $L$ be a continuous lattice and $A \subseteq L$ a countable set such that $a \in A$, $x \neq 0$ implies $xa \neq 0$. Then $\mathrm{IRR}\, L \subseteq \uparrow xA$ implies $x = 0$.

Proof. Fix $x \neq 0$ and write $A = \{x_0, x_1, \ldots\}$ and set $a_n = x_0 \ldots x_n$. Then $xa_n \neq 0$ for all $n$ (by induction). Apply Lemma 1 with $V = L\setminus\{0\}$, $v = x$, and find an open filter $U$ such that $0 \notin U$, $xa_n \in U$ for all $n$. Now let $p$ be a maximal element in $L \setminus U$. By Lawmann I, $p \in \mathrm{IRR}\, L$. BUT

$p \in L\setminus U \subseteq L \setminus \bigcup \uparrow xa_n \subseteq L \setminus \bigcup \uparrow xx_n = L \setminus \uparrow xA$. □

LEMMA 3. Let $L$ be a continuous lattice and $A \subseteq L$. Then the following are equivalent: (1) $\uparrow A \cap \mathrm{Spec}\, L$ is nowhere dense in $\mathrm{Spec}\, L$.

(2) $\mathrm{Spec}\, L \not\subseteq \uparrow x \cup \uparrow A$ for all $x$ with $\mathrm{Spec}\, L \not\subseteq \uparrow x$.

(3) $\mathrm{Spec}\, L \not\subseteq \uparrow xA$ for all $x$ with $\mathrm{Spec}\, L \not\subseteq \uparrow x$.

If $L$ is also distributive and $A = \{a\}$, then these are also equivalent to (4) $0 \neq xa$, for all $x \neq 0$.

Proof. Write $X = \mathrm{Spec}\, L$. Then $\uparrow A \cap X$ is nowhere dense in $X$ iff every non-empty open set in $X$ meets the complement $X \setminus \uparrow A$, and since the closed sets of $X$ are precisely the sets $\uparrow x \cap X$, this is equivalent to saying that for all $x$ with $X \setminus \uparrow x \neq \emptyset$ we have $\emptyset \neq (X \setminus \uparrow x) \cap (X \setminus \uparrow A) = X \setminus (\uparrow x \cup \uparrow A)$, which shows the equivalence of (1) and (2). But $p \in X \setminus (\uparrow x \cup \uparrow A)$ iff $x \not\leq p$ and $a \not\leq p$ for all $a \in A$ iff $xa \not\leq p$ for all $a \in A$, since $p$ is prime, and this means that $p \in X \setminus \uparrow xA$. Thus (2) and (3) are equivalent. If $L$ is distributive, then $\mathrm{Spec}\, L \not\subseteq \uparrow x$ iff $x \neq 0$ by Lawmann I, and $\mathrm{Spec}\, L \not\subseteq \uparrow xa$ iff $xa \neq 0$. Thus (3) and (4) are equivalent. □

RECALL. A topological space $X$ is a Baire space iff for all closed subsets $Y \neq \emptyset$ of $X$ and each sequence $Y_1, Y_2, \ldots$ of subspaces of $Y$ which are (closed and) nowhere dense in $Y$, we have $\bigcup Y_n \neq Y$. □

THEOREM 4. Every locally quasicompact sober space is a Baire space.

Proof. Let $X \neq \emptyset$ be locally quasicompact sober. By Lawmann II we may assume $X = \mathrm{Spec}\, L$ for a continuous Heyting algebra $L$. Each closed subset of $X$ is of the form $\uparrow x \cap X$ $= \uparrow x \cap \mathrm{IRR}\, L = \mathrm{IRR} \uparrow x$ $= \mathrm{Spec} \uparrow x$, hence is itself the spectrum of a continuous Heyting algebra. Thus it suffices to show that for any sequence $X_n$ of nowhere dense closed sets in $X$ we have $X \neq \bigcup X_n$.

Now $X_n = X \cap \uparrow x_n$ with $xx_n \neq 0$ for all $x \neq 0$ by Lemma 3. Since L is distributive, IRR $L \setminus \{1\} = X$ (see Lawmann II). Apply Lemma 2 with $A = \{x_0, x_1, \ldots\}$ and find $X \not\subseteq \uparrow A = \bigcup \uparrow x_n$, i.e. $X \not\subseteq \bigcup X_n$. ∎

EXAMPLE 5. 1. Let $X = \mathbb{N}$ with upper sets open. Then X is locally quasicompact $T_0$, but is not a Baire space.

2. Let X be the set of all ordinals less than the first uncountable one with upper sets open. Then X is a locally quasicompact $T_0$ Baire space which is not sober. ∎

Remark. Both examples are 1st countable. For the second, $\check{X}$ (the sobrification) is not.

PROPOSITION 6. Let L be a continuous Heyting algebra and X an order generating subspace of Spec L. (Note. By Lawmann II, <u>every</u> core compact space is of this form.) Now suppose that L satisfies the following countability hypothesis:

(COUNT) For each prime $p \neq 1$ the space $\uparrow p$ is first countable in p (w.r.t. the Lawson topology).

Then the following statements are equivalent:

(1) X is sober. (2) X = Spec L. (3) X is a Baire space.

Proof. (1) ⇔ (2) by Lawmann II and (2) ⇒ (3) by Theorem 4.

Suppose not (2). Then we find a $p \in \operatorname{Spec} L \setminus X$, and $Y = \uparrow p \cap X$ is a closed subspace of X. Note that by Lemma (3) we have

(i) for all $v > p$, ($xv > p$ for $x > p$ and thus) $\uparrow v \cap Y$ is nowhere dense closed in Y.

Let $V = \uparrow p \setminus \{p\}$; since p is prime, V is a filter. Since X is order generating, (ii) $p = \inf(\uparrow p \cap X) = \inf Y$ whence $Y \neq \emptyset$. By the definition of the Lawson topology, p now has a neighborhood basis in $\uparrow p$ sets of the form $\uparrow p \setminus (\uparrow v_1 \cup \ldots \cup \uparrow v_n)$ with $v_k \in V$. Since p is prime, however, we can say that (iii) in $\uparrow p$ the point p has a neighborhood basis of sets of the form $\uparrow p \setminus \uparrow v$, $v \in V$. By (COUNT), this becomes:

(iv) in $\uparrow p$ the point p has a basis of neighborhoods of the form $\uparrow p \setminus \uparrow v_n$ with a sequence $v_0 \geq v_1 \geq \ldots$ in V.

The Lawson topology is Hausdorff, whence $\bigcap (\uparrow p \setminus \uparrow v_n) = \{p\}$, and thus

(v) $V \subseteq \bigcup \uparrow v_n$. By (i), the closed sets $Y_n = \uparrow v_n \cap Y$ are nowhere dense in Y, but $\emptyset \neq Y = \bigcup Y_n$ by (ii) and (v). Thus X is not a Baire space, i.e. we proved not (3). ∎

Remark. The fact that X is order generating was also used in (1) ⇒ (2).

Clearly (COUNT) is satisfied if L is metrizable (which is tantamount to saying that L contains a countable subset C such that $(\forall x,y)\ x \ll y \Rightarrow (\exists c)$ $c \in C$ and $x \leq c \ll y$). Further, (COUNT) is equivalent to: "SpecL is first countable" (i.e. "$\check{X}$ is first countable"). This is satisfied if SpecL is second countable; but O(SpecL) ≅ O (X) (Lawmann II); hence the latter means that X is 2nd countable.

Consequence: For 2nd countable X, sobriety ⟺ Bairity.

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