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SCS 41: An Exercise on the Spectrum of Function Spaces

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SEMINAR ON CONTINUITY IN SEMILATTICES (SCS)

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TOPIC: An exercise on the spectrum of function spaces

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On the occasion of a stop-over in Paris, D. Scott was shown the recent preprint by Lawmann on the spectral theory of continuous Heyting algebras (as distributive continuous lattices will undoubtedly be called in the Compendium and from here on forward). He began immediately to mull over his favorite subject in the area; function spaces and continuous lattices. He argued as follows: If L and M are continuous lattices, so is [L-->M] (the space of all Scott continuous functions from L to M with pointwise lattice operations). In fact this is a sublattice of M^L (closed even under arbitrary sups), so if M is distributive, then the function space is a distributive lattice. What is its spectrum (which, as we know from Lawmann, determines it completely)?

Some remarks regarding this question are communicated in the following for later inclusion in the compendium.

PROPOSITION. Let X be a locally quasicompact sober space and D a continuous Heyting algebra. Then the primes of Top(X,SD) (where SD denotes D with the Scott topology) are precisely the functions $f_{(x,p)}$ which are defined by $f_{(x,p)}(y) = \begin{cases} p & \text{for } y \in \overline{\{x\}} \\ 1 & \text{otherwise} \end{cases}$, ~~The function~~ $x \in X$, $p \in$ PRIME D.

The function $(x,p) \longmapsto f_{(x,p)} : X \times \text{Spec } D \longrightarrow \text{Spec Top}(X,SD)$

is a homeomorphism relative to the hull kernel topologies on the spectra.

REMARKS. i) The hypothesis that X be sober is no restriction of generality since Top(X,SD) = Top($\check{X}$,SD), where $\check{X}$ is the sobrification of X. (Lawmann, 2.8)

ii) The distributivity of D is only essential to secure the non-emptyness of the spectra. The statement in itself remains valid without it.

We procedd by steps and derive in fact a more general statement.

LEMMA 1. Let X be a topological space and L a continuous lattice. Let F be the complete lattice Top(X,SL) under pointwise operations (see Scott LNM 274, p.112). For f ε F the following statements are equivalent: (1) f ε IRR F . (2) There is a prime U ε O(X) and a pεIRR L with ~~fxxxxxxx~~

$f = ch_U \vee const_p$ (where ch_A is the characteristic function of $A \subseteq X$)

Proof. (2)=>(1). Suppose $ab = f$. Let $A = a^{-1}(\downarrow p) = a^{-1}(p)$ and $B = b^{-1}(\downarrow p) = b^{-1}(p)$. Then A and B are closed and since p is irreducible, $A \cup B = X \setminus U$. Since U is prime, $A = X \setminus U$ or $B = X \setminus U$, and so $a = f$ or $b = f$.

(1)=>(2). Suppose $t \in f(X)$, $t < 1$. Take an arbitrary $s \ll t$ in L and set $U = f^{-1}(\Uparrow s)$. Then U is open and the two functions $a = f \vee ch_U$ and $b = f \vee const_s$ are in F. If $x \in U$, then $s \ll f(x)$ and so $a(x) = f(x) \vee 1 = f(x)$ and $b(x) = f(x) \vee s = f(x)$, i.e. $(ab)(x) = f(x)$. If $x \notin U$, then $a(x) = f(x) \vee 0 = f(x)$ and $b(x) = f(x) \vee s$, i.e. $(ab)(x) = f(x)(f(x) \vee s) = f(x)$. Hence $ab = f$. Since $f \in$ IRR F we have $a = f$ or $b = f$. But $a = f$ means $ch_U \leq f$, and this means that $s \ll f(x)$ implies $f(x) = 1$, but this is not possible since there is an x with $f(x) = t$, i.e. with $s \ll f(x) < 1$. Hence we conclude $b = f$ which means $s \leq f(x)$ for all $x \in X$. Since $s \ll t$ was arbitrary and L is continuous, we conclude $t \leq f(x)$ for all X. We have so far shown either that $f(X) = \{1\}$ (in which case (2) holds) or that f takes at most two different values 1 and p. Since it is continuous, this means that f is of the form $ch_U \vee const\ p$ with the open set $U = f^{-1}(1)$. -If $p = vw$ in L set $a = ch_U \vee const_v$ and $b = ch_U \vee const_w$ in F and note $ab = f$. Since f is irreducible, $f = a$ or $f = b$, i.e. $p = v$ or $p = w$ follows. Thus $p \in$ IRR L. - If $U = V \cap W$ set $a = ch_V \vee const_p$ and $b = ch_W \vee const_p$. In calculating $ab = f$ do not use the distributivity of L, since we do not assume it, but make a simple case distinction. The irreducibility of f implies $f = a$ or $f = b$, i.e. $V = U$ or $W = U$. Thus $U \in$ IRR $O(X)$ = PRIME $O(X)$. §

Recall: If X is a sober space, then $U \in$ PRIME $O(X)$ iff $U = X \setminus \overline{\{x\}}$ for some $x \in X$; and this property is characteristic for sober spaces.

In Lawmann the topology generated on a lattice by the sets $L \setminus \uparrow x$ is called the INF-topology(p.52). We will do this here and consider on IRR L and IRR F the INF-topologies. If IRR = PRIME, this is precisely the hull kernel topology.

LEMMA 2. Let X be a sober space and L a continuous lattice. (If X is also locally compact, then $F = Top(X, SL)$ is a continuous lattice (see Isbell or Hofmann).) For $x \in X$ and $p \in$ IRR L let $f_{(x,p)} = ch_{X \setminus \overline{\{x\}}} \vee const_p$; then $(x,p) \longmapsto f_{(x,p)}$: $X \times (\text{IRR } L \setminus \{1\}) \xrightarrow{\beta} \text{IRR } F \setminus \{1\}$ is a bijection by Lemma 1. Claim: This map is a homeomorphism with the INF-topology on the irreducible spectra.

Proof. The generic closed sets of $S = \text{IRR } F \setminus \{1\}$ are of the form $\uparrow a \cap S$, $a \in F$. Now $\beta^{-1}(\uparrow a \cap S) = \{(x,p): x \in X,\ p \in \text{IRR } L,\ p \neq 1,\ a(x) \leq p\}$. We claim that the complement of this set is open in $X \times (\text{IRR } L \setminus \{1\})$. Indeed suppose $a(x) \not\in \downarrow p$. Pick an $s \in L$ with $s \notin \downarrow p$ and $s \ll a(x)$. Then $U \in a^{-1}(\Uparrow s)$ is an open neighborhood of x in X, and $V = (\text{IRR } L) \setminus \uparrow s$ is an open neighborhood of p in $\text{IRR } L \setminus \{1\}$. If $u \in U$ and $v \in V$, then $a(u) \not\leq v$ (since otherwise $s \ll a(u) \leq v \Rightarrow v \in \uparrow s$!). This proves the claim. Conversely, let A be closed in A and let $s < 1$ in L so that $\uparrow s \cap (\mathrm{IRR}\, L \setminus \{1\})$ is a generic closed set of $T = \mathrm{IRR}\ L \setminus \{1\}$. Define $a : X \longrightarrow L$ by

$a(x) = \begin{cases} s & \text{if } x \in A \\ 1 & \text{otherwise} \end{cases}$. Then a is continuous, i.e. $a \in F$. Moreover,

$a \leq \beta(x,p) = f_{(x,p)}$ iff $a(x) \leq p$ iff ($s \leq p$ for $x \in A$ and $1 \leq p$ for $x \notin A$); but $p < 1$, hence $a \leq \beta(x,p)$ iff $x \in A$ and $p \in \uparrow s$. Thus $A \times (\uparrow s \cap T) = \beta(\uparrow a \cap S)$ is the image of a generic closed set in the irreducible spectrum of F.§

We are now finished with the proof of the proposition, since the Lemma 2 is in fact a stronger statement (having nothing to do with core compactness or distributivity). If we return for a moment to distributivity, then we know by LAwmann that D= O(Y) for some locally quasicompact sober space. We can write D = Top(Y, S2) Then F becomes Top(X, STop(Y, S2)). In view of our Proposition and Lawmann we have shown that $F \cong O(X \times Y) = \mathrm{Top}(X \times Y, S2)$, and this is reasonable.