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LARGE-TIME BEHAVIOR OF NON-SYMMETRIC FOKKER-PLANCK TYPE EQUATIONS

ANTON ARNOLD, ERIC CARLEN, AND QIANGCHANG JU

Abstract. Large time asymptotics of the solutions to non-symmetric Fokker-Planck type equations are studied by extending the entropy method to this case. We present a modified Bakry-Emery criterion that yields covergence of the solution to the steady state in relative entropy with an explicit exponential rate. In parallel it also implies a logarithmic Sobolev inequality w.r.t. the steady state measure. Explicit examples illustrate that skew-symmetric perturbations in the Fokker Planck operator can "help" to improve the constant in such a logarithmic Sobolev inequality.

1. Introduction

In this paper we consider the large-time behavior of the Cauchy problem for linear *Fokker-Planck type equations* (advection-diffusion equations) for probability densities $\rho(x,t)$:

$$\rho_t = \mathcal{L}\rho := \operatorname{div}(\mathbf{D}[\nabla\rho + \rho(\nabla\phi + F)]), \quad x \in I\!R^n,\ t > 0, \tag{1.1a}$$

$$\rho(t=0) = \rho_I \in L^1_+(I\!R^n), \tag{1.1b}$$

with the confinement potential $\phi = \phi(x)$ satisfying $e^{-\phi} \in L^1(I\!R^n)$, and the symmetric, locally uniformly positive definite diffusion matrix $\mathbf{D} = \mathbf{D}(x) = (d^{ij}(x))$. Due to the divergence form we obviously have the conservation property

$$\int_{I\!R^n} \rho(x,t)dx = \int_{I\!R^n} \rho_I(x)dx, \tag{1.2}$$

and without restriction of generality we shall always assume

$$\int_{I\!R^n} \rho_I(x)dx = \int_{I\!R^n} e^{-\phi(x)}dx = 1.$$

Now suppose that the given vector field $F(x)$ and the scalar potential $\phi(x)$ satisfy

$$\operatorname{div}(\mathbf{D}e^{-\phi}F) = 0 \qquad \text{on } I\!R^n. \tag{1.3}$$

Then the unique normalized steady state of (1.1a) is $\rho_\infty = e^{-\phi} \in L^1_+(I\!R^n)$. Because of (1.3), $\mathcal{L}_{SS}\rho := \operatorname{div}(\mathbf{D}\rho F)$ is the skew-symmetric part of the operator $\mathcal{L}$ in $L^2(I\!R^n; \rho_\infty^{-1}(dx))$ acting on ρ, and this skew-symmetric part annihilates the steady state ρ_∞. Hence, the steady state is independent of F.

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In the sequel we shall assume that the data ϕ, $\mathbf{D}$, F, and ρ_I are sufficiently regular (for example, $\phi \in W^{2,\infty}_{loc}(\mathbb{R}^n;\mathbb{R})$, $d^{ij} \in W^{2,\infty}_{loc}(\mathbb{R}^n;\mathbb{R})$, $i,j = 1,\dots,n$, and $F = (F_i(x)) \in W^{1,\infty}_{loc}(\mathbb{R}^n;\mathbb{R}^n)$) such that (1.1) has a unique solution $\rho \in C([0,\infty), L^1_+(\mathbb{R}^n))$, and $\rho \in C([0,\infty), L^2(\mathbb{R}^n;\rho_\infty^{-1}(dx)))$ if $\rho_I \in L^2(\mathbb{R}^n;\rho_\infty^{-1}(dx))$. We remark that (by a simple minimum principle) $\rho_I(x) \ge 0$ implies $\rho(x,t) > 0$ for all $x \in \mathbb{R}^n, t > 0$.

Simple examples of (1.1a) include the (symmetric) *Fokker-Planck equation* [18, 22]

$$\rho_t = \mathrm{div}(\nabla\rho + x\rho), \quad x \in \mathbb{R}^n,\, t > 0,$$

where $\phi(x) = |x|^2/2 + const$, $\mathbf{D} = \mathbf{I}$ ($\mathbf{I}$ being the identity matrix), and $F = 0$. As $t \to \infty$ its solution converges with an exponential rate towards the Gaussian steady state

$$\rho_\infty(x) = (2\pi)^{-n/2} e^{-|x|^2/2}.$$

An important example for a non-symmetric equation is the *quantum-kinetic Wigner-Fokker-Planck* equation (cf. [3, 19]) with the quadratic confinement potential $V(y) = |y|^2/2$:

$$\begin{aligned} &w_t + v\cdot\nabla_y w - y\cdot\nabla_v w \qquad (1.4)\\ &\quad = D_{pp}\Delta_v w + 2\gamma \mathrm{div}_v(vw) + 2D_{pq}\mathrm{div}_y(\nabla_v w) + D_{qq}\Delta_y w, \quad y,v \in \mathbb{R}^n, t > 0. \end{aligned}$$

(1.4) can indeed be recast in the form of (1.1a) (see [19] for details). It describes the evolution of the *Wigner function* $w(y,v,t)$ with the position variable y and velocity v, and (y,v) plays here the role of x in (1.1a). Under simple (and physically necessary) assumptions on the r.h.s. of (1.4), $w(t)$ also converges exponentially to the unique steady state w_∞.

Other examples of non-symmetric Fokker-Planck equations appear in the modelling of polymeric fluid flows, where $\rho(x,t)$, $x \in \mathbb{R}^n$ describes the distribution of polymeric chains of length and orientation given by x. In a given homogeneous sheer flow $u(x) = \mathbf{F}\cdot x$ the (scaled) evolution equation reads (cf. [14] for details)

$$\rho_t = \frac{1}{2}\mathrm{div}(\nabla\rho + \rho(\nabla\phi - 2\mathbf{F}\cdot x)). \qquad (1.5)$$

In this paper we are interested in the possibly exponential decay rate of $\rho(t)$ towards ρ_∞ in *relative entropy*, i.e.

$$e(\rho|\rho_\infty) := \int_{\mathbb{R}^n} \rho \ln\frac{\rho}{\rho_\infty} dx \ge 0. \qquad (1.6)$$

This exponential convergence is closely related to the hypercontractivity of the semigroup generated by $\mathcal{L}$ and to the validity of a *logarithmic Sobolev inequality* (LSI) w.r.t. the steady state measure ρ_∞ (cf. [12, 13, 4]). In the case $\mathbf{D}(x) \equiv \mathbf{I}$ this inequality would read, if it holds,

$$\int_{\mathbb{R}^n} f^2\ln f^2\rho_\infty dx - \left(\int_{\mathbb{R}^n} f^2\rho_\infty dx\right)\ln\left(\int_{\mathbb{R}^n} f^2\rho_\infty dx\right) \le C\int_{\mathbb{R}^n}|\nabla f|^2\rho_\infty dx \qquad (1.7)$$

for some fixed $C < \infty$ and all $f \in L^2(\mathbb{R}^n;\rho_\infty(dx))$. Notice that ϕ enters the inequality through ρ_∞, but that F does not. The question to be addressed here is whether it is ever advantageous to consider a non-reversible evolution (i.e., one with $F \neq 0$) when attempting to establish the validity of (1.7) through the *entropy method* of [5, 7, 4]. Perhaps surprisingly, the answer is yes.

To be more specific we shall now briefly outline this idea for the simplest case when $\mathbf{D} = \mathbf{I}$, following the preliminary note [2]: Consider the symmetric part (in $L^2(\mathbb{R}^n; \rho_\infty^{-1} dx)$) of the Fokker-Planck operator in (1.1a), i.e.

$$\mathcal{L}_S \rho := \mathrm{div}\left(e^{-\phi}\nabla \frac{\rho}{e^{-\phi}}\right)$$

and assume that the potential $\phi(x)$ is uniformly convex, i.e. it satisfies a *Bakry-Emery condition* (BEC):

(A1)

$$\exists\, \lambda_1 > 0 \text{ such that } \quad \frac{\partial^2 \phi}{\partial x^2} = \Big(\frac{\partial^2 \phi(x)}{\partial x_i \partial x_j}\Big)_{i,j=1,\ldots,n} \geq \lambda_1 \mathbf{I} \quad \forall x \in \mathbb{R}^n$$

(in the sense of positive definite matrices). Then it is well known that $\rho_S(t)$, the solution of $\rho_t = \mathcal{L}_S \rho$ converges to ρ_∞ in relative entropy with an exponential rate of (at least) $2\lambda_1$, and the LSI (1.7) holds with $C = 2/\lambda_1$ (cf. [5, 7, 4]). Moreover, (1.3) implies that also $\rho(t)$, the solution to the non-symmetric Fokker-Planck equation (1.1) converges to ρ_∞ in relative entropy with rate of (at least) $2\lambda_1$ (cf. [4]).

On the other hand, consider now $\mathcal{L}_{SS}$ with $F(x) \neq 0$ as a *skew-symmetric perturbation* of $\mathcal{L}_S$ and assume that (ϕ, F) satisfy a *generalized Bakry-Emery condition* (GBEC):

(A2)

$$\exists\, \lambda_2 > 0 \text{ such that } \quad \frac{\partial^2 \phi}{\partial x^2} - \frac{1}{2}\left(\frac{\partial F}{\partial x} + (\frac{\partial F}{\partial x})^\top\right) \geq \lambda_2 \mathbf{I} \quad \forall x \in \mathbb{R}^n\,,$$

where $\left(\frac{\partial F}{\partial x}\right)_{i,j} = \frac{\partial F_i}{\partial x_j}$ denotes the Jacobian of F. As we shall show, the relative entropy of $\rho(t)$ then decays exponentially with rate (at least) $2\lambda_2$, and the LSI (1.7) then holds with $C = 2/\lambda_2$. And in some cases the perturbation F gives rise to a 'better' constant $\lambda_2 > \lambda_1$, hence 'improving' (1.7).

The goal of this paper is threefold: to understand the large-time behavior of non-symmetric Fokker-Planck equations (with applications like (1.4), (1.5)), to analyze skew-symmetric perturbations $\mathcal{L}_{SS}$ in order to possibly improve the LSI. And, finally, our analysis furnishes a proof of the entropy method for symmetric Fokker-Planck equations with full diffusion matrices $\mathbf{D}$ (which was not included in [4]).

This paper is organized as follows. We begin Section 2 by introducing the class of entropies with which we work. We then proceed to calculate the second derivative of the entropy in the presence of the skew-symmetric term, and derive the generalized Bakry–Emery condition. To get an estimate on the *initial* entropy in terms of the initial entropy production by the Bakry-Emery method, we need to know that the *final* entropy is zero. For this we need a theorem asserting that the entropy vanishes in the large time limit. We prove this in Theorem 2.5 for regular initial data. This part of the proof is somewhat technically involved, but once we have it, even for regular densities, the rest is straight-forward: We then use the results obtained up to this point to obtain a LSI for regular densities. Once this is extended by simple closure, the fact that the entropy vanishes in the large time limit, exponentially fast, then follows easily for general initial data.

In Section 3 we discuss several examples in which the skew-symmetric term plays a crucial role in establishing the LSI.

2. Entropy Method for Non-symmetric Fokker-Planck Equations

2.1. Admissible relative entropies. As a generalization of the logarithmic entropy (1.6) we now introduce the relative entropies that we shall use in the sequel.

Definition 2.1. Let $\psi \in C[0,\infty) \cap C^4(0,\infty)$ satisfy the conditions

$$\psi(1) = \psi'(1) = 0, \tag{2.1a}$$

$$\psi'' > 0, \quad \text{on } I\!R^+, \tag{2.1b}$$

$$(\psi''')^2 \le \frac{1}{2}\psi''\psi^{IV} \quad \text{on } I\!R^+. \tag{2.1c}$$

Let $\rho_1,\, \rho_2 \in L^1_+(I\!R^n)$ with $\int \rho_1 dx = \int \rho_2 dx = 1$ and $\rho_1/\rho_2 \in I\!R^+_0$ $\rho_2(dx)$–a.e. Then

$$e_\psi(\rho_1|\rho_2) := \int_{I\!R^n} \psi(\frac{\rho_1}{\rho_2})\, \rho_2(dx) \ge 0 \tag{2.2}$$

is called an *admissible relative entropy* (of ρ_1 with respect to ρ_2) with *generating function* ψ.

Our class of generating functions ψ coincides with those considered in [5] (up to the normalizations (2.1a)). The most typical examples of admissible relative entropies are the *physical relative entropy* (1.6) generated by

$$\psi_1(\sigma) := \sigma \ln \sigma - \sigma + 1, \quad \sigma \ge 0,$$

and the *p-entropies* (or *Tsallis relative entropies* [23]) generated by

$$\psi_p(\sigma) := \frac{\sigma^p - p\sigma}{p-1} + 1, \quad \sigma \ge 0; \quad 1 < p \le 2. \tag{2.3}$$

For $p = 2$ we have $e_{\psi_2}(\rho_1|\rho_2) = \|\rho_1 - \rho_2\|^2_{L^2(I\!R^n, \rho_2^{-1}(dx))}$.

The well-known Csiszár-Kullback inequality [10, 16, 24, 4] shows that the relative entropies (2.2) are a 'measure' for the distance between two normalized $L^1_+(I\!R^n)$-functions ρ_1, ρ_2:

$$\frac{1}{2}\|\rho_1 - \rho_2\|^2_{L^1(I\!R^n)} \le \frac{1}{\eta_2} e_\psi(\rho_1|\rho_2), \tag{2.4}$$

with the notation $\eta_2 := \psi''(1)$.

We remark that for each admissible relative entropy e_ψ, there exists a quadratic *superentropy* e_φ such that

$$0 \le \psi(\sigma) \le \eta_2(\sigma - 1)^2 =: \varphi(\sigma)\,, \quad \sigma > 0\,,$$

and hence $e_\psi(\rho_1|\rho_2) \le e_\varphi(\rho_1|\rho_2)$ (cf. Lemma 2.6 in [4]).

2.2. Generalized Bakry-Emery condition and Ricci tensor. As in (A1) and (A2), a Bakry-Emery condition (BEC) on the coefficients $(\phi, \mathbf{D}, F)$ of the Fokker-Planck operator $\mathcal{L}$ will be our main assumption for deriving exponential decay of the relative entropy. For the subsequent analysis it is convenient to rewrite (1.1). We set

$$\mu := \rho/\rho_\infty,$$

which satisfies the IVP

$$\begin{aligned} \mu_t &= \tilde{\mathcal{L}}\mu := \rho_\infty^{-1}\mathrm{div}(\rho_\infty \mathbf{D}\nabla\mu) + F^\top \mathbf{D}\nabla\mu \\ &= \mathrm{div}(\mathbf{D}\nabla\mu) - (\nabla\phi - F)^\top \mathbf{D}\nabla\mu, \quad x \in I\!R^n,\, t>0, \\ \mu_I &= \rho_I/\rho_\infty \in L^1(I\!R^n, \rho_\infty(dx)). \end{aligned} \tag{2.5}$$

Condition (A1) is a special case of the well-known *Bakry-Emery condition* for logarithmic Sobolev-inequalities [5, 6, 7, 4]. In order to extend the approach of Bakry and Emery to non-symmetric Fokker-Planck equations we shall now introduce a new *generalized Bakry-Emery condition* (GBEC). For understanding the BEC in the case of general (symmetric and uniformly positive definite) diffusion matrices $\mathbf{D}(x)$ we shall need some notions from basic differential geometry (see, e.g. [8], §7, 8). Therefore we consider the Riemannian manifold $\mathcal{M}= (I\!R^n; \mathbf{D}^{-1})$, with $\mathbf{D}(x)^{-1} =: (d_{ij}(x))$ as covariant metric tensor.

The *Ricci tensor of a symmetric Fokker-Planck operator* was defined in [7] (cf. also [4]). Here we shall extend this definition to non-symmetric Fokker-Planck operators that involve a vector field $F = (F_1, \dots, F_n)^\top$. The Fokker–Planck operator in (2.5) can be decomposed as

$$\tilde{\mathcal{L}} = \Delta^{\mathbf{D}} + \mathcal{X},$$

where

$$\Delta^{\mathbf{D}}\mu := (\det \mathbf{D})^{\frac{1}{2}} \,\mathrm{div}\left[(\det \mathbf{D})^{-\frac{1}{2}} \mathbf{D}\nabla\mu\right]$$

is the Laplace–Beltrami operator on $\mathcal{M}$ (cf. [9], §1). And

$$\mathcal{X} := \sum_{i=1}^n X^i \frac{\partial}{\partial x_i}$$

is a vector field (or, equivalently, a directional derivative) on $\mathcal{M}$, with the components

$$X^i(x) := -\sum_{j=1}^n d^{ij} \left[\frac{\partial}{\partial x_j}\left(\phi(x) - \frac{1}{2}\ln\det \mathbf{D}(x)\right) - F_j\right]. \tag{2.6}$$

The Christoffel symbols are defined as the elements of the 3-tensor:

$$\Gamma_{ij}^l := \frac{1}{2}\sum_{k=1}^n d^{kl}\left(\frac{\partial d_{jk}}{\partial x_i} + \frac{\partial d_{ki}}{\partial x_j} - \frac{\partial d_{ij}}{\partial x_k}\right). \tag{2.7}$$

The *Riemann curvature tensor* of $\mathcal{M}$ then reads

$$R_{ki}{}^l{}_j := \frac{\partial}{\partial x_i}\Gamma_{jk}^l - \frac{\partial}{\partial x_j}\Gamma_{ik}^l + \sum_{m=1}^n \Gamma_{im}^l \Gamma_{jk}^m - \sum_{m=1}^n \Gamma_{jm}^l \Gamma_{ik}^m \tag{2.8}$$

and the (symmetric) *Ricci-tensor* of $\mathcal{M}$ is (cf. [20], §C.3)

$$\rho_{ij} := \sum_{k=1}^{n} R_{ik}{}^{k}{}_{j}. \tag{2.9}$$

The covariant derivative of a vector field $\mathcal{X} = (X^1, \dots, X^n)$ is given by

$$\nabla_i X^j := \frac{\partial X^j}{\partial x_i} + \sum_{k=1}^{n} \Gamma^j_{ik} X^k. \tag{2.10}$$

We define the symmetric covariant derivative (2-tensor) of $\mathcal{X}$ as

$$\left(\nabla^S \mathcal{X}\right)_{ij} := \frac{1}{2} \sum_{l=1}^{n} (d_{jl} \nabla_i X^l + d_{il} \nabla_j X^l). \tag{2.11}$$

We now define the *Ricci tensor of a non-symmetric Fokker-Planck operator* as

$$Ric^{ij}(x) := \sum_{k,l=1}^{n} d^{ik} d^{jl} \left[\rho_{kl} - \left(\nabla^S \mathcal{X}\right)_{kl}(x)\right] \tag{2.12}$$

with the components of $\mathcal{X}$ defined in (2.6) (cf. [7, 4] for the symmetric counterpart).

Our GBEC for a general symmetric, positive definite diffusion matrix now reads:

(A3) $\exists\, \lambda_3 > 0$ such that $\mathbf{Ric}(x) \ge \lambda_3 \mathbf{D}(x) \quad \forall x \in I\!R^n$

(in the sense of positive definite matrices). From the explicit form of **Ric** (see (2.13) below) one easily sees that (A3) reduces to (A2) for $\mathbf{D}(x) \equiv \mathbf{I}$. And in the case of a scalar diffusion (i.e. $\mathbf{D}(x) = D(x)\mathbf{I}$) it reads:

(A4) $\exists\, \lambda_3 > 0$ such that

$$\begin{aligned}
\left(\frac{1}{2} - \frac{n}{4}\right) \frac{1}{D} \nabla D \otimes \nabla D + \frac{1}{2} \left[\Delta D - \nabla D \cdot (\nabla\phi - F)\right] \mathbf{I} \\
+ \frac{D}{2} \left(\frac{\partial(\nabla\phi - F)}{\partial x} + \left(\frac{\partial(\nabla\phi - F)}{\partial x}\right)^\top \right) \\
+ \frac{\nabla D \otimes (\nabla\phi - F) + (\nabla\phi - F) \otimes \nabla D}{2} - \frac{\partial^2 D}{\partial x^2} \quad \ge \quad \lambda_3 \mathbf{I}
\end{aligned}$$

$\forall x \in I\!R^n$.

With tedious calculations (see the Appendix), the explicit form of the GBEC reads:

$\exists\, \lambda_3 > 0$ such that

$$\begin{aligned}
& U^\top \Big[\frac{1}{2} \mathrm{Tr} \left(\mathbf{D} \frac{\partial^2}{\partial x^2} \right) \mathbf{D} + \frac{1}{2} (\nabla^\top \mathbf{D} \nabla) \mathbf{D} - \mathbf{D} \left(\frac{\partial^2}{\partial x^2} \mathbf{D} \right) + \mathbf{D} \left(\frac{\partial^2 \phi}{\partial x^2} \right) \mathbf{D} \\
& \quad - \frac{1}{2} \mathbf{D} \left(\frac{\partial F}{\partial x} + \left(\frac{\partial F}{\partial x}\right)^\top \right) \mathbf{D} - \frac{1}{2} \left((\nabla\phi - F)^\top \mathbf{D} \nabla \right) \mathbf{D} \Big] U \qquad (2.13) \\
& \quad + \frac{1}{2} \left(U^\top \mathbf{D} \mathbf{E} (\nabla\phi - F) + (\nabla\phi - F)^\top \mathbf{E}^\top \mathbf{D} U \right) \\
& \quad - \frac{1}{4} \mathrm{Tr} \left(\mathbf{E}^\top + \mathbf{D}\, \mathbf{E}\, \mathbf{D}^{-1} - \mathcal{N}(\mathbf{D}) \mathbf{D}^{-1} \right)^2 \\
& \ge \quad \lambda_3 U^\top \mathbf{D} U
\end{aligned}$$

$\forall x \in I\!R^n$ and any vector field $U : I\!R^n \to I\!R^n$. Here we used the matrix $\mathbf{E} = (e_i^j) := (\partial_i d^{jk})U_k$ (i is the 'first index' in e_i^j). And $\mathcal{N} := U^\top \mathbf{D}\nabla$ is a scalar differential operator, which acts elementwise when applied to a matrix. The expression $\frac{\partial^2}{\partial x^2}\mathbf{D}$ denotes the (formal) matrix product between the Hessian operator and the matrix $\mathbf{D}$, i.e. $\partial_{ij}\big(d^{jk}\big)$.

2.3. Exponential decay of the entropy dissipation and the relative entropy. In this section, we shall first obtain the exponential decay of the entropy dissipation. As in [4], we consider the entropy dissipation

$$I_\psi(\rho(t)|\rho_\infty) := \frac{d}{dt} e_\psi(\rho(t)|\rho_\infty) \tag{2.14}$$

and the entropy dissipation rate

$$R_\psi(\rho(t)|\rho_\infty) := \frac{d}{dt} I_\psi(\rho(t)|\rho_\infty). \tag{2.15}$$

Eq. (2.14) is referred to as entropy equation. To facilitate the computations we rewrite (1.1a) in the following form:

$$\rho_t = \operatorname{div}(\mathbf{D}(\rho_\infty U + \rho F)) \tag{2.16}$$

with the notation $U = \nabla(\frac{\rho}{\rho_\infty})$. Differentiating the relative entropy $e_\psi(\rho(t)|\rho_\infty)$ w.r.t. time gives

$$I_\psi(\rho(t)|\rho_\infty) = \int_{I\!R^n} \psi'\Big(\frac{\rho}{\rho_\infty}\Big)\rho_t \, dx. \tag{2.17}$$

By using (2.16) we obtain after an integration by parts

$$I_\psi(\rho(t)|\rho_\infty) = -\int_{I\!R^n} \psi''\Big(\frac{\rho}{\rho_\infty}\Big) U^\top \mathbf{D} U \rho_\infty \, dx + T \tag{2.18}$$

where $T := \int_{I\!R^n} \psi'\Big(\frac{\rho}{\rho_\infty}\Big)\operatorname{div}(\mathbf{D}F\rho)\, dx$. In terms of (1.3), we get

$$\operatorname{div}(\mathbf{D}F\rho) = U^\top \mathbf{D} F \rho_\infty \, , \tag{2.19}$$

from which we have

$$\begin{aligned} T &= \int_{I\!R^n} \psi'\Big(\frac{\rho}{\rho_\infty}\Big) U^\top \mathbf{D} F \rho_\infty \, dx \\ &= \int_{I\!R^n} \nabla^\top \psi\Big(\frac{\rho}{\rho_\infty}\Big) \mathbf{D} F \rho_\infty \, dx \\ &= 0 \end{aligned}$$

by again using (1.3). Therefore

$$I_\psi(\rho(t)|\rho_\infty) = -\int_{I\!R^n} \psi''\Big(\frac{\rho}{\rho_\infty}\Big) U^\top \mathbf{D} U \rho_\infty \, dx \le 0, \tag{2.20}$$

due to the strict convexity of ψ and the positivity of $\mathbf{D}$.

Now, we compute (2.15):

$$\begin{aligned} R_\psi(\rho(t)|\rho_\infty) &= -\int_{I\!R^n} \psi'''\Big(\frac{\rho}{\rho_\infty}\Big)\rho_t U^\top \mathbf{D} U\,dx \\ &- 2\int_{I\!R^n} \psi''\Big(\frac{\rho}{\rho_\infty}\Big) U^\top \mathbf{D} U_t \rho_\infty dx, \\ &= R_1 + R_2, \end{aligned} \tag{2.21}$$

where

$$R_1 := -\int_{I\!R^n} \psi'''\Big(\frac{\rho}{\rho_\infty}\Big)\rho_t U^\top \mathbf{D} U\,dx$$

and

$$R_2 := -2\int_{I\!R^n} \psi''\Big(\frac{\rho}{\rho_\infty}\Big) U^\top \mathbf{D} U_t \rho_\infty dx.$$

With (2.16) we get

$$\begin{aligned} R_1 &= -\int_{I\!R^n} \psi'''\Big(\frac{\rho}{\rho_\infty}\Big)\mathrm{div}(\mathbf{D}U\rho_\infty) U^\top \mathbf{D} U\,dx \\ &\quad -\int_{I\!R^n} \psi'''\Big(\frac{\rho}{\rho_\infty}\Big)\mathrm{div}(\mathbf{D}F\rho) U^\top \mathbf{D} U\,dx \\ &= \tilde{R}_1 + T_3, \end{aligned} \tag{2.22}$$

where

$$\tilde{R}_1 := -\int_{I\!R^n} \psi'''\Big(\frac{\rho}{\rho_\infty}\Big)\mathrm{div}(\mathbf{D}U\rho_\infty) U^\top \mathbf{D} U\,dx$$

and

$$T_3 := -\int_{I\!R^n} \psi''\Big(\frac{\rho}{\rho_\infty}\Big)\mathrm{div}(\mathbf{D}F\rho) U^\top \mathbf{D} U\,dx.$$

Using (2.19) and an integration by parts lead to

$$\begin{aligned} T_3 &= -\int_{I\!R^n} \psi'''\Big(\frac{\rho}{\rho_\infty}\Big)\nabla^\top\Big(\frac{\rho}{\rho_\infty}\Big)\mathbf{D}F(U^\top \mathbf{D}U)\rho_\infty\,dx \\ &= -\int_{I\!R^n} \nabla^\top\psi''\Big(\frac{\rho}{\rho_\infty}\Big)\mathbf{D}F(U^\top \mathbf{D}U)\rho_\infty\,dx \\ &= \int_{I\!R^n} \psi''\Big(\frac{\rho}{\rho_\infty}\Big)\mathrm{div}[\mathbf{D}F(U^\top \mathbf{D}U)\rho_\infty]dx \\ &= \int_{I\!R^n} \psi''\Big(\frac{\rho}{\rho_\infty}\Big)\nabla^\top(U^\top \mathbf{D}U)\mathbf{D}F\rho_\infty\,dx. \end{aligned}$$

From (2.16) and (2.19), it follows that

$$U_t = \nabla\Big(\frac{1}{\rho_\infty}\mathrm{div}(\mathbf{D}U\rho_\infty) + U^\top \mathbf{D}F\Big).$$

Then

$$\begin{aligned} R_2 &= -2\int_{I\!R^n} \psi''\Big(\frac{\rho}{\rho_\infty}\Big) U^\top \mathbf{D}\nabla[e^\phi \mathrm{div}(\mathbf{D}e^{-\phi}U)]\rho_\infty\,dx \\ &\quad -2\int_{I\!R^n} \psi''\Big(\frac{\rho}{\rho_\infty}\Big) U^\top \mathbf{D}\nabla(U^\top \mathbf{D}F)\rho_\infty\,dx \\ &= \tilde{R}_2 + T_4, \end{aligned} \tag{2.23}$$

where

$$\tilde{R}_2 := -2 \int_{I\!R^n} \psi''\Big(\frac{\rho}{\rho_\infty}\Big) U^\top \mathbf{D} \nabla [e^\phi \mathrm{div}(\mathbf{D} e^{-\phi} U)] \rho_\infty \, dx$$

and

$$T_4 := -2 \int_{I\!R^n} \psi''\Big(\frac{\rho}{\rho_\infty}\Big) U^\top \mathbf{D} \nabla (U^\top \mathbf{D} F) \rho_\infty \, dx.$$

Clearly, the computations which lead to (2.20) and (2.21) are formal. However, they can easily be justified for initial data $\rho_I \in L^2(I\!R^n; \rho_\infty^{-1}(dx)) \subset L^1(I\!R^n)$ and for entropy generators without singularity at $\sigma = 0$ by taking into account the semigroup property of the evolution in $L^2(I\!R^n; \rho_\infty^{-1}(dx))$, and the fact that $\rho > 0$ on $I\!R^n, t > 0$. General admissible entropies can easily be dealt with by a local cut-off at $\sigma = 0$.

Remark 2.2. Following the approach from [4] we have to give a meaning to $I_\psi(\rho|\rho_\infty)$ even when ρ becomes zero (which may be the case at the initial state). For positive and differentiable functions $\mu = \mu(x)$ we have

$$\psi''(\mu)(\nabla \mu)^\top \mathbf{D} \nabla \mu = \left(\nabla \int_1^\mu \sqrt{\psi''(s)} \, ds \right)^\top \mathbf{D} \nabla \int_1^\mu \sqrt{\psi''(s)} \, ds \,. \tag{2.24}$$

Hence, we set for $\rho \geq 0$

$$I_\psi(\rho|\rho_\infty) := -\int_{I\!R^n} (\nabla w)^\top \mathbf{D} \nabla w \rho_\infty(dx), \quad w := h_\psi \left(\frac{\rho}{\rho_\infty} \right), \tag{2.25}$$

if $w \in H^1_{loc}(I\!R^n)$ with

$$h_\psi(\mu) := \int_1^\mu \sqrt{\psi''(s)} \, ds, \quad \mu > 0. \tag{2.26}$$

As shown is [4], h_ψ is Hölder continuous with exponent $1/2$ locally at $\mu = 0$.

We now return to proving the exponential decay of $e_\psi(\rho(t)|\rho_\infty)$ under additional assumptions on ϕ, F, and $\mathbf{D}$. At first we shall derive an exponential decay rate for the entropy dissipation I_ψ by using the special form of the entropy dissipation rate (2.21).

Lemma 2.3. *Let the initial condition $\rho_I \in L^2(I\!R^n; \rho_\infty^{-1}(dx))$ satisfy $|I_\psi(\rho_I|\rho_\infty)| < \infty$ for the admissible entropy e_ψ. Assume that the coefficients $\phi(x)$, $F(x)$, and $\mathbf{D}(x)$ of (1.1a) satisfy the condition (A3). Then the entropy dissipation converges to 0 exponentially:*

$$|I_\psi(\rho(t)|\rho_\infty)| \leq e^{-2\lambda_3 t} |I_\psi(\rho_I|\rho_\infty)|, \quad t > 0. \tag{2.27}$$

Proof. An integration by parts (which can be justified as mentioned above) yields

$$\begin{aligned}
\tilde{R}_1 &= \int_{\mathbb{R}^n} \psi^{IV}(e^\phi\rho)(U^\top \mathbf{D}U)^2 e^{-\phi}\,dx \\
&\quad +2\int_{\mathbb{R}^n} \psi'''(e^\phi\rho)e^{-\phi}(\mathbf{D}U)^\top \frac{\partial U}{\partial x}(\mathbf{D}U)dx \\
&\quad +\int_{\mathbb{R}^n} \psi'''(e^\phi\rho)e^{-\phi}\Big(U_iU_j\frac{\partial d^{ij}}{\partial x_k}d^{kl}U_l\Big)\,dx \\
&= \int_{\mathbb{R}^n} \psi^{IV}(e^\phi\rho)(U^\top \mathbf{D}U)^2 e^{-\phi}\,dx \\
&\quad +\int_{\mathbb{R}^n} \psi'''(e^\phi\rho)e^{-\phi}U^\top\Big[\mathbf{D}\frac{\partial U}{\partial x}\mathbf{D}+\frac{\partial(\mathbf{D}U)}{\partial x}\mathbf{D}\Big]U\,dx\,.
\end{aligned}$$

Here and in the sequel we use the *Einstein summation convention* for double indices. Also we abbreviate $\frac{\partial}{\partial x_i}$ by ∂_i.

Motivated by the scalar diffusion case (cf. [4]), we introduce

$$\begin{aligned}
S_1 &:= \int_{\mathbb{R}^n} \psi'''(e^\phi\rho)e^{-\phi}\Big[2(\mathbf{D}U)^\top\frac{\partial U}{\partial x}(\mathbf{D}U)+U_iU_j\frac{\partial d^{ij}}{\partial x_k}d^{kl}U_l\Big]dx \\
&= \int_{\mathbb{R}^n} \psi'''(e^\phi\rho)e^{-\phi}U^\top\Big[\mathbf{D}\frac{\partial U}{\partial x}\mathbf{D}+\mathbf{D}\frac{\partial(\mathbf{D}U)}{\partial x}\Big]U\,dx \qquad (2.28)\\
&= \int_{\mathbb{R}^n} \psi'''(e^\phi\rho)e^{-\phi}\nabla^\top(U^\top\mathbf{D}U)\mathbf{D}U dx \\
&= -\int_{\mathbb{R}^n} \psi''(e^\phi\rho)\mathrm{div}(\mathbf{D}\nabla(U^\top\mathbf{D}U)\rho_\infty)dx \\
&= -\int_{\mathbb{R}^n} \psi''(e^\phi\rho)[\mathrm{div}(\mathbf{D}\nabla(U^\top\mathbf{D}U))\rho_\infty-\nabla^\top\phi\mathbf{D}\nabla(U^\top\mathbf{D}U)\rho_\infty]dx.
\end{aligned}$$

In a cumbersome calculation we obtain

$$\begin{aligned}
S_2 &:= \tilde{R}_2 - S_1 \\
&= 2\int_{\mathbb{R}^n} \psi''(e^\phi\rho)e^{-\phi}\Bigg[U^\top\Big(\mathbf{D}(\frac{\partial(\mathbf{D}\nabla\phi)}{\partial x})^\top-\mathbf{D}(\frac{\partial^2}{\partial x^2}\mathbf{D}) \\
&\quad -\frac{1}{2}(\nabla\phi^\top\mathbf{D}\nabla)\mathbf{D}+\frac{1}{2}\mathrm{Tr}(\mathbf{D}\frac{\partial^2}{\partial x^2})\mathbf{D}+\frac{1}{2}(\nabla^\top\mathbf{D}\nabla)\mathbf{D}\Big)U \\
&\quad -\frac{1}{4}\mathrm{Tr}\Big(\mathbf{E}^\top+\mathbf{D}\mathbf{E}\mathbf{D}^{-1}-\mathcal{N}(\mathbf{D})\mathbf{D}^{-1}\Big)^2\Bigg]dx \\
&\quad +2\int_{\mathbb{R}^n} \psi''(e^\phi\rho)e^{-\phi}\mathrm{Tr}\Big[\mathbf{D}\frac{\partial U}{\partial x}+\frac{1}{2}\mathbf{E}^\top+\frac{1}{2}\mathbf{D}\mathbf{E}\mathbf{D}^{-1}-\frac{1}{2}\mathcal{N}(\mathbf{D})\mathbf{D}^{-1}\Big]^2dx \\
&=: T_1+T_2.
\end{aligned}$$

Next we rewrite T_3 and T_4 as

$$T_3 = \int_{\mathbb{R}^n} \psi''(e^\phi\rho)e^{-\phi}\left[F^\top\mathbf{D}\frac{\partial U}{\partial x}\mathbf{D}U+F^\top\mathbf{D}(\frac{\partial(\mathbf{D}U)}{\partial x})^\top U\right]dx,$$

and

$$T_4 = -2\int_{\mathbb{R}^n} \psi''(e^\phi\rho)e^{-\phi}\left[U^\top \mathbf{D}\frac{\partial F}{\partial x}\mathbf{D}U + U^\top\mathbf{D}(\frac{\partial(\mathbf{D}U)}{\partial x})^\top F\right]dx\,,$$

where we used $\frac{\partial U}{\partial x} = (\frac{\partial U}{\partial x})^\top$, since U is a gradient.

Using the fact that $\frac{\partial(\mathbf{D}U)}{\partial x} = \mathbf{E} + \frac{\partial U}{\partial x}\mathbf{D}$, we have

$$\begin{aligned}
T_3 + T_4 &= 2\int_{\mathbb{R}^n} \psi''(e^\phi\rho)e^{-\phi}\left[-U^\top\mathbf{D}\frac{\partial F}{\partial x}\mathbf{D}U + U^\top(\frac{1}{2}\mathbf{E}^\top\mathbf{D} - \mathbf{D}\mathbf{E})F\right]dx \\
&= 2\int_{\mathbb{R}^n} \psi''(e^\phi\rho)e^{-\phi}\Big[-\frac{1}{2}U^\top\mathbf{D}\left(\frac{\partial F}{\partial x} + (\frac{\partial F}{\partial x})^\top\right)\mathbf{D}U \\
&\quad + U^\top((\frac{1}{2}F^\top\mathbf{D}\nabla)\mathbf{D})U - \frac{U^\top\mathbf{D}\,\mathbf{E}F + F^\top\mathbf{E}^\top\,\mathbf{D}U}{2}\Big]dx.
\end{aligned}$$

Then

$$\begin{aligned}
T_1 + T_3 + T_4 &= 2\int_{\mathbb{R}^n} \psi''(e^\phi\rho)e^{-\phi}\Bigg\{U^\top\Bigg[\frac{1}{2}\mathrm{Tr}\left(\mathbf{D}\frac{\partial^2}{\partial x^2}\right)\mathbf{D} + \frac{1}{2}(\nabla^\top\mathbf{D}\nabla)\mathbf{D} \\
&\quad -\mathbf{D}\left(\frac{\partial^2}{\partial x^2}\mathbf{D}\right) + \mathbf{D}\left(\frac{\partial^2\phi}{\partial x^2}\right)\mathbf{D} - \frac{1}{2}\mathbf{D}\left(\frac{\partial F}{\partial x} + (\frac{\partial F}{\partial x})^\top\right)\mathbf{D} \\
&\quad -\frac{1}{2}\left((\nabla\phi - F)^\top\mathbf{D}\nabla\right)\mathbf{D}\Bigg]U \\
&\quad +\frac{1}{2}\left(U^\top\mathbf{D}\mathbf{E}(\nabla\phi - F) + (\nabla\phi - F)^\top\mathbf{E}^\top\mathbf{D}U\right) \\
&\quad -\frac{1}{4}\mathrm{Tr}\left(\mathbf{E}^\top + \mathbf{D}\,\mathbf{E}\,\mathbf{D}^{-1} - \mathcal{N}(\mathbf{D})\mathbf{D}^{-1}\right)^2\Bigg\}dx.
\end{aligned}$$

Condition (A3) (or (2.13)) leads to the estimates:

$$T_1 + T_3 + T_4 \ge 2\lambda_3\int_{\mathbb{R}^n}\psi''(e^\phi\rho)e^{-\phi}U^\top\mathbf{D}U\,dx.$$

All in all we have by using (2.28)

$$\begin{aligned}
R_1 + R_2 &= (\tilde{R}_1 + S_1 + T_2) + (T_1 + T_3 + T_4) \\
&\ge \int_{\mathbb{R}^n}\Bigg\{\psi^{IV}(e^\phi\rho)(U^\top\mathbf{D}U)^2 + 2\psi'''(e^\phi\rho)U^\top\Big(\mathbf{D}\,\frac{\partial U}{\partial x}\,\mathbf{D} + \frac{\partial(\mathbf{D}U)}{\partial x}\mathbf{D}\Big)U \\
&\quad + 2\psi''(e^\phi\rho)e^{-\phi}\mathrm{Tr}\Big[\mathbf{D}\frac{\partial U}{\partial x} + \frac{1}{2}\mathbf{E}^\top + \frac{1}{2}\mathbf{D}\,\mathbf{E}\,\mathbf{D}^{-1} - \frac{1}{2}\mathcal{N}(\mathbf{D})\mathbf{D}^{-1}\Big]^2\Bigg\}e^{-\phi}\,dx \\
&\quad + 2\lambda_3\int_{\mathbb{R}^n}\psi''(e^\phi\rho)e^{-\phi}U^\top\mathbf{D}U\,dx.
\end{aligned}$$

The first integral can be written as

$$\int_{\mathbb{R}^n}\mathrm{Tr}(\mathbf{X}\mathbf{Y})e^{-\phi}dx,$$

where $\mathbf{X}$ and $\mathbf{Y}$ are the 2×2-matrices

$$\mathbf{X} = \begin{pmatrix} 2\psi''(e^\phi\rho) & 2\psi'''(e^\phi\rho) \\ 2\psi'''(e^\phi\rho) & \psi^{IV}(e^\phi\rho) \end{pmatrix}$$

and, resp.,

$$\mathbf{Y} = \begin{pmatrix} \alpha & \frac{1}{2}U^\top\Big(\mathbf{D}\frac{\partial U}{\partial x}\mathbf{D} + \frac{\partial(\mathbf{D}U)}{\partial x}\mathbf{D}\Big)U \\ \frac{1}{2}U^\top\Big(\mathbf{D}\frac{\partial U}{\partial x}\mathbf{D} + \frac{\partial(\mathbf{D}U)}{\partial x}\mathbf{D}\Big)U & (U^\top\mathbf{D}U)^2 \end{pmatrix},$$

with

$$\alpha = \mathrm{Tr}\Big[\mathbf{D}\frac{\partial U}{\partial x} + \frac{1}{2}\mathbf{E}^\top + \frac{1}{2}\mathbf{D}\,\mathbf{E}\,\mathbf{D}^{-1} - \frac{1}{2}\mathcal{N}(\mathbf{D})\mathbf{D}^{-1}\Big]^2.$$

$\mathbf{X}$ is non-negative definite since ψ generates an admissible entropy (cf. Definition 2.1). Next, we will show that $\mathbf{Y}$ is also non-negative definite. To this end, we introduce the symmetric matrices $\mathbf{Z}$ and $\mathbf{W}$ as follows:

$$\begin{aligned} \mathbf{Z} \;&:=\; \sqrt{\mathbf{D}}\,\frac{\partial U}{\partial x}\,\sqrt{\mathbf{D}} + \frac{1}{2}\sqrt{\mathbf{D}}^{-1}\,\mathbf{E}^\top\,\sqrt{\mathbf{D}} \\ &\quad + \frac{1}{2}\sqrt{\mathbf{D}}\,\mathbf{E}\,\sqrt{\mathbf{D}}^{-1} - \frac{1}{2}\sqrt{\mathbf{D}}^{-1}\,\mathcal{N}(\mathbf{D})\,\sqrt{\mathbf{D}}^{-1} \end{aligned}$$

and

$$\mathbf{W} := \sqrt{\mathbf{D}}\,(U\otimes U)\,\sqrt{\mathbf{D}}.$$

Using the cyclicity of the trace, we prove

$$\alpha = \mathrm{Tr}\mathbf{Z}^2,$$

$$\begin{aligned} \mathrm{Tr}\mathbf{W}^2 &= \mathrm{Tr}[\sqrt{\mathbf{D}}(U\otimes U)\mathbf{D}(U\otimes U)\sqrt{\mathbf{D}}] \\ &= \mathrm{Tr}[\mathbf{D}(U\otimes U)\mathbf{D}(U\otimes U)] \\ &= (U^\top\mathbf{D}U)^2, \end{aligned}$$

and

$$\begin{aligned} \mathrm{Tr}(\mathbf{W}\,\mathbf{Z}) &= \mathrm{Tr}(\mathbf{Z}\,\mathbf{W}) \\ &= \mathrm{Tr}\left[\Big(\mathbf{D}\,\frac{\partial U}{\partial x}\,\mathbf{D} + \frac{1}{2}\mathbf{E}^\top\mathbf{D} + \frac{1}{2}\mathbf{D}\,\mathbf{E} - \frac{1}{2}\mathcal{N}(\mathbf{D})\Big)U\otimes U\right] \\ &= U^\top\left[\mathbf{D}\,\frac{\partial U}{\partial x}\,\mathbf{D} + \frac{1}{2}\mathbf{E}^\top\mathbf{D} + \frac{1}{2}\mathbf{D}\,\mathbf{E} - \frac{1}{2}\mathcal{N}(\mathbf{D})\right]U \\ &= U^\top\left[\mathbf{D}\,\frac{\partial U}{\partial x}\,\mathbf{D} + \frac{1}{2}\mathbf{E}^\top\mathbf{D}\right]U \\ &= \frac{1}{2}U^\top\left[\mathbf{D}\frac{\partial U}{\partial x}\mathbf{D} + \frac{\partial(\mathbf{D}U)}{\partial x}\mathbf{D}\right]U. \end{aligned}$$

Then, it follows that

$$\mathbf{Y} = \begin{pmatrix} \mathrm{Tr}\mathbf{Z}^2 & \mathrm{Tr}(\mathbf{Z}\,\mathbf{W}) \\ \mathrm{Tr}(\mathbf{W}\,\mathbf{Z}) & \mathrm{Tr}\mathbf{W}^2 \end{pmatrix}.$$

Since

$$\begin{pmatrix} \mathbf{Z^2} & \mathbf{Z\,W} \\ \mathbf{Z\,W} & \mathbf{W^2} \end{pmatrix} \geq \mathbf{0}$$

by the positivity of partial traces (we include a proof in Lemma 2.4 below for completeness), $\mathbf{Y}$ is nonnegative. Thus

$$\int_{\mathbb{R}^n} \mathrm{Tr}(\mathbf{XY}) e^{-\phi} dx \geq 0$$

and we have for the entropy dissipation rate (2.21):

$$R_\psi(\rho(t)|\rho_\infty) \geq 2\lambda_3 \int_{\mathbb{R}^n} \psi''(e^\phi \rho) e^{-\phi} U^\top \mathbf{D} U \, dx = -2\lambda_3 I_\psi(\rho(t)|\rho_\infty).$$

The assertion now follows from

$$\frac{d}{dt}|I_\psi(\rho(t)|\rho_\infty)| \leq -2\lambda_3 |I_\psi(\rho(t)|\rho_\infty)|. \tag{2.29}$$

□

Lemma 2.4. *Let* $\mathbf{P} = \mathbf{P}^\top \geq \mathbf{0}$*, and*

$$\mathbf{P} = \begin{pmatrix} \mathbf{P}_{11} & \mathbf{P}_{12} \\ \mathbf{P}_{21} & \mathbf{P}_{22} \end{pmatrix},$$

where $\mathbf{P}_{ij}$*,* $i, j = 1, 2$ *are* $n \times n$ *matrices. Then*

$$\mathbf{Q} := \begin{pmatrix} \mathrm{Tr}\mathbf{P}_{11} & \mathrm{Tr}\mathbf{P}_{12} \\ \mathrm{Tr}\mathbf{P}_{21} & \mathrm{Tr}\mathbf{P}_{22} \end{pmatrix} \geq 0.$$

Proof. Let $\mathbf{I}_j := (I_{kl})_{2\times 2n}$, $j = 1, \ldots, n$, where $I_{1j} = I_{2,n+j} = 1$; the other elements are 0. Then we have

$$\mathbf{I}_j \mathbf{P} \mathbf{I}_j^\top = \begin{pmatrix} p_{jj} & p_{j,n+j} \\ p_{n+j,j} & p_{n+j,n+j} \end{pmatrix} \geq 0.$$

Hence, $\mathbf{Q} = \sum_{j=1}^n \mathbf{I}_j \mathbf{P} \mathbf{I}_j^\top \geq 0$. □

Next, we shall derive the exponential decay of the relative entropy. For this purpose, we first show the convergence of $\rho(t)$ to ρ_∞ in relative entropy (without a rate, for the moment). We remark that the analogous result for the symmetric Fokker-Planck equation was obtained in [4], §2.1 using spectral theory. Specifically, $\sigma(\mathcal{L}_S) \subset \mathbb{R}_0^-$ when considering $\mathcal{L}_S$ in $L^2(\mathbb{R}^n; \rho_\infty^{-1}(dx))$. Hence, $\|\rho(t)\|_{L^2(\mathbb{R}^n;\rho_\infty^{-1}(dx))}$ is monotonically decaying also for the non-symmetric Fokker-Planck equation (1.1). And we have the apriori estimate

$$\|\rho(t)\|_{L^2(\mathbb{R}^n;\rho_\infty^{-1}(dx))} \leq \|\rho_I\|_{L^2(\mathbb{R}^n;\rho_\infty^{-1}(dx))}, \quad t \geq 0. \tag{2.30}$$

In contrast, we shall derive it here from the decay of the entropy dissipation:

Theorem 2.5. *Let* $\rho_I/\sqrt{\rho_\infty} \in L^2(\mathbb{R}^n)$*,* $|I_{\psi_1}(\rho_I|\rho_\infty)| < \infty$*, and let the coefficients* $\phi(x)$*,* $F(x)$*, and* $\mathbf{D}(x)$ *satisfy condition (A3). Then*

(a) $e_{\psi_p}(\rho(t)|\rho_\infty) \to 0$ *as* $t \to \infty$ *for* $1 \leq p < 2$*.*
(b) *If, additionally,* $e_{\psi_{2+\varepsilon}}(\rho_I|\rho_\infty) < \infty$ *for some* $\varepsilon > 0$*, then* $e_{\psi_2}(\rho(t)|\rho_\infty) \to 0$*.*

(c) *Let e_ψ be any admissible relative entropy, and e_φ its quadratic superentropy with $|I_\varphi(\rho_I|\rho_\infty)| < \infty$. Then $e_\psi(\rho(t)|\rho_\infty) \to 0$.*

Proof. First we establish this result for the (logarithmic) physical relative entropy $e(\rho|\rho_\infty)$. Its entropy dissipation satisfies

$$\begin{aligned} |I(\rho(t)|\rho_\infty)| &= \int_{\mathbb{R}^n} \frac{{\rho_\infty}^2}{\rho}\Big(\nabla^\top \frac{\rho}{\rho_\infty}\mathbf{D}\nabla\frac{\rho}{\rho_\infty}\Big)dx \\ &= 4\int_{\mathbb{R}^n}\Big(\nabla^\top\sqrt{\frac{\rho}{\rho_\infty}}\mathbf{D}\nabla\sqrt{\frac{\rho}{\rho_\infty}}\Big)\rho_\infty dx. \end{aligned}$$

Since $\mathbf{D}(x)$ is locally uniformly strictly positive definite, $\rho_\infty > 0$ and $\rho_\infty \in L^\infty_{loc}(\mathbb{R}^n)$, Lemma 2.3 implies that

$$\nabla\sqrt{\frac{\rho(t)}{\rho_\infty}} \overset{t\to\infty}{\longrightarrow} 0 \qquad \text{in} \quad L^2(\Omega)$$

for any bounded domain $\Omega \subset \mathbb{R}^n$. By a well-known result on Beppo-Levi spaces (cf. [11], [21] p. 49, or Lemma III.2 of [1]) we have for bounded Lipschitz domains Ω and an arbitrary sequence $t_k \to \infty$:

$$\sqrt{\frac{\rho_k}{\rho_\infty}} - c_k(\Omega) \overset{k\to\infty}{\longrightarrow} 0 \qquad \text{in} \quad L^2(\Omega), \tag{2.31}$$

with the notation

$$\rho_k := \rho(t_k) \quad \text{and} \quad c_k(\Omega) := \int_\Omega \sqrt{\rho_k/\rho_\infty}dx/\text{vol}(\Omega).$$

For any Ω fixed, we have

$$\Big\|\sqrt{\frac{\rho_k}{\rho_\infty}}\Big\|^2_{L^2(\Omega)} = \int_\Omega \frac{\rho_k}{\rho_\infty}dx \le C(\Omega)\int_{\mathbb{R}^n}\rho_k dx = C(\Omega).$$

Thus, $c_k(\Omega)$ is uniformly bounded with respect to k for Ω fixed. Since $\rho_\infty \in L^\infty(\Omega)$, (2.31) implies that

$$\sqrt{\rho_k} - c_k(\Omega)\sqrt{\rho_\infty} \overset{k\to\infty}{\longrightarrow} 0 \qquad \text{in} \quad L^2(\Omega).$$

Because

$$\begin{aligned} &\|\rho_k - c_k^2(\Omega)\rho_\infty\|_{L^1(\Omega)} \\ \le\ &\|\sqrt{\rho_k} - c_k(\Omega)\sqrt{\rho_\infty}\|_{L^2(\Omega)}(\|\sqrt{\rho_k}\|_{L^2(\Omega)} + c_k(\Omega)\|\sqrt{\rho_\infty}\|_{L^2(\Omega)}), \end{aligned}$$

we have

$$\rho_k - c_k^2(\Omega)\rho_\infty \overset{k\to\infty}{\longrightarrow} 0 \qquad \text{in} \quad L^1(\Omega). \tag{2.32}$$

Due to the uniform boundedness of $c_k(\Omega)$, there exists a subsequence (still denoted by $\{c_k(\Omega)\}$) such that

$$c_k(\Omega) \overset{k\to\infty}{\longrightarrow} c(\Omega).$$

Now we choose the domain sequence $\Omega_N := B_N(0) \subset \mathbb{R}^n$. And take the diagonal subsequence of all $\{c_k(\Omega_N)\}$ such that for any N fixed, we have

$$c_k(\Omega_N) \overset{k\to\infty}{\longrightarrow} c_N(\Omega_N)\,. \tag{2.33}$$

In view of (2.32) and (2.33), we obtain

$$\rho_k \overset{k\to\infty}{\longrightarrow} c_N^2(\Omega_N)\rho_\infty \qquad \text{in} \quad L^1(\Omega_N). \tag{2.34}$$

Since $\rho_\infty > 0$ in $I\!R^n$, we conclude that $c_N(\Omega_N) = c$ for all N. Using (2.30) and the Hölder inequality we have

$$\int_{\Omega_N^c} \rho_k\, dx \le \|\rho_k\|_{L^2(I\!R^n;\rho_\infty^{-1}(dx))} \left(\int_{\Omega_N^c} e^{-\phi} dx\right)^{1/2} \overset{N\to\infty}{\longrightarrow} 0, \quad \text{uniformly in } k \in I\!N. \tag{2.35}$$

Thus

$$\rho_k \overset{k\to\infty}{\longrightarrow} c^2\rho_\infty \qquad \text{in} \quad L^1(I\!R^n).$$

Due to (1.2), we deduce that $c = 1$ and hence

$$\rho_k \overset{k\to\infty}{\longrightarrow} \rho_\infty \qquad \text{in} \quad L^1(I\!R^n). \tag{2.36}$$

Therefore

$$\mu_k := \frac{\rho_k}{\rho_\infty} \to 1 \quad \text{in measure}$$

(in the measure space $(I\!R^n, \rho_\infty(dx))$). The three assertions of the Lemma will now be discussed separately.

Part (a):

In order to apply Vitali's convergence theorem we rewrite

$$e_{\psi_p}(\rho_k|\rho_\infty) = \frac{1}{p-1}\left[\|\mu_k\|^p_{L^p(I\!R^n;\rho_\infty(dx))} - 1\right], \quad 1 < p < 2.$$

Proceeding as for (2.35) we obtain $\forall\, \Omega \subset I\!R^n$:

$$\left|\int_\Omega \mu_k^p\, \rho_\infty dx\right| \le \|\rho_k\|^p_{L^2(I\!R^n;\rho_\infty^{-1}(dx))} \left(\int_\Omega \rho_\infty dx\right)^{1-p/2}. \tag{2.37}$$

And this yields both the uniform integrability of $\{\mu_k^p\}$ and the uniform decay of its 'tails'. Thus, Vitali's theorem yields

$$\mu_k \overset{k\to\infty}{\longrightarrow} 1 \quad \text{in } L^p(I\!R^n; \rho_\infty(dx)),$$

and hence $e_{\psi_p}(\rho_k|\rho_\infty) \to 0$.

For the logarithmic entropy the result follows from $\psi_1(\sigma) \le \psi_p(\sigma),\ \sigma \ge 0$.

Part (b):

From (2.20) we obtain the apriori estimate for the $(2+\varepsilon)$–entropy:

$$\frac{1}{1+\varepsilon}\left[\|\mu(t)\|^{2+\varepsilon}_{L^{2+\varepsilon}(I\!R^n;\rho_\infty(dx))} - 1\right]$$
$$= e_{\psi_{2+\varepsilon}}(\rho(t)|\rho_\infty) \le e_{\psi_{2+\varepsilon}}(\rho_I|\rho_\infty), \quad t \ge 0.$$

Now, estimating $\int_\Omega \mu_k^2\, \rho_\infty dx$ analogously to (2.37) proves the assertion.

Part (c):

Here we consider the decay of the quadratic superentropy e_φ that satisfies

$$0 \le e_\psi(\rho(t)|\rho_\infty) \le e_\varphi(\rho(t)|\rho_\infty) := \eta_2\, \|\rho(t) - \rho_\infty\|^2_{L^2(I\!R^n;\rho_\infty^{-1}(dx))}. \tag{2.38}$$

From (2.20) its entropy dissipation satisfies

$$|I_\varphi(\rho(t)|\rho_\infty)| = \eta_2 \int_{\mathbb{R}^n} \Big(\nabla^\top \frac{\rho(t)}{\rho_\infty} \mathbf{D} \nabla \frac{\rho(t)}{\rho_\infty}\Big) \rho_\infty dx.$$

A similar analysis as before yields that for any bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$ and an arbitrary sequence $t_k \to \infty$, it holds:

$$\frac{\rho_k}{\rho_\infty} - d_k(\Omega) \stackrel{k\to\infty}{\longrightarrow} 0 \qquad \text{in} \quad L^2(\Omega)$$

with $d_k(\Omega) := \int_\Omega \frac{\rho_k}{\rho_\infty} dx/\text{vol}(\Omega)$. Since

$$\int_\Omega \Big[\frac{\rho_k}{\rho_\infty}\Big]^2 dx \le C(\Omega)$$

(because of (2.30)), $d_k(\Omega)$ is also uniformly bounded with respect to k for Ω fixed. Now we can take the diagonal subsequence of all $d_k(\Omega_N)$ such that for any N fixed, we have

$$d_k(\Omega_N) \stackrel{k\to\infty}{\longrightarrow} d_N \tag{2.39}$$

and

$$\frac{\rho_k}{\rho_\infty} \stackrel{k\to\infty}{\longrightarrow} d_N(\Omega_N) \qquad \text{in} \quad L^2(\Omega_N). \tag{2.40}$$

From the previous analysis we know that $d_N(\Omega_N) = c_N^2(\Omega_N)$ and $d_N = 1$ for all N. Since $\rho_\infty > 0$, (2.40) implies

$$\int_{\Omega_N} |\frac{\rho_k}{\rho_\infty} - 1|^2 \rho_\infty \, dx \stackrel{k\to\infty}{\longrightarrow} 0. \tag{2.41}$$

The monotone decay of $\|\rho(t)\|_{L^2(\mathbb{R}^n;\rho_\infty^{-1}(dx))}$ and (2.30) imply that there exists a subsequence (still denoted by $\{\rho_k\}$) with

$$\rho_k \rightharpoonup \tilde\rho \ \text{ in } L^2(\mathbb{R}^n;\rho_\infty^{-1}(dx)) \quad \text{and } \|\rho_k\|_{L^2(\mathbb{R}^n;\rho_\infty^{-1}(dx))} \searrow \tau \ge 0.$$

(2.36) then implies $\tilde\rho = \rho_\infty$ and the weak lower semicontinuity of the norm yields

$$\tau \le \|\rho_\infty\|_{L^2(\mathbb{R}^n;\rho_\infty^{-1}(dx))} = \Big(\int \rho_\infty dx\Big)^{1/2} = 1.$$

Indeed, we have $\tau = 1$, since

$$\forall\, \varepsilon > 0: \quad \exists\, N = N(\varepsilon) \ \text{ with } \|\rho_\infty\|_{L^2(\Omega_N;\rho_\infty^{-1}(dx))} \ge 1 - \varepsilon.$$

The strong convergence (2.41) then implies

$$\|\rho_k\|_{L^2(\Omega_N;\rho_\infty^{-1}(dx))} \to \|\rho_\infty\|_{L^2(\Omega_N;\rho_\infty^{-1}(dx))} \ge 1 - \varepsilon,$$

and hence

$$\tau = \lim_{k\to\infty} \|\rho_k\|_{L^2(\Omega_N;\rho_\infty^{-1}(dx))} \ge 1 - \varepsilon.$$

Weak convergence of ρ_k and convergence of its norms then imply

$$0 \le e_\psi(\rho(t)|\rho_\infty) \le e_\varphi(\rho(t)|\rho_\infty) = \eta_2 \, \|\rho(t) - \rho_\infty\|^2_{L^2(\mathbb{R}^n;\rho_\infty^{-1}(dx))} \stackrel{k\to\infty}{\longrightarrow} 0.$$

This proves the assertion. □

In the above theorem, the assumptions $\rho_I/\sqrt{\rho_\infty} \in L^2(I\!R^n)$ and $|I_{\psi_1}(\rho_I|\rho_\infty)| < \infty$ are perhaps unnaturally restrictive. However, once we have proved a logarithmic Sobolev inequality for ρ_I smooth with compact support, simple closure yields us the inequality in full generality, and then the conclusion of the Theorem follows immediately without this assumption. Thus, nothing is lost in making this assumption. We then obtain:

Theorem 2.6. *Let e_ψ be an admissible relative entropy and $e_\psi(\rho_I|\rho_\infty) < \infty$. Let the coefficients $\phi(x)$, $F(x)$, and $\mathbf{D}(x)$ satisfy condition (A3). Then the relative entropy converges to 0 exponentially:*

$$e_\psi(\rho(t)|\rho_\infty) \leq e^{-2\lambda_3 t} e_\psi(\rho_I|\rho_\infty), \quad t > 0. \tag{2.42}$$

Moreover, the convex Sobolev inequality *(LSI for $\psi = \psi_1$)*

$$\int_{I\!R^n} \psi\left(\frac{\rho}{\rho_\infty}\right)\rho_\infty(dx) \leq \frac{1}{2\lambda_3}\int_{I\!R^n} \nabla^\top h_\psi\left(\frac{\rho}{\rho_\infty}\right)\mathbf{D}\nabla h_\psi\left(\frac{\rho}{\rho_\infty}\right)\rho_\infty(dx) \tag{2.43}$$

with h_ψ from (2.26) *holds*

$$\forall \rho \in L^1_+(I\!R^n) \quad \textit{with} \quad \int_{I\!R^n} \rho dx = \int_{I\!R^n} \rho_\infty dx. \tag{2.44}$$

This inequality, of course, does not require our usual normalization $\int \rho(x)dx = 1$. Note that $L^1_+(I\!R^n)$ in (2.44) can be replaced by $L^1(I\!R^n)$ if ψ is quadratic.

Proof. We proceed in two steps and first derive (2.42) for

$$\rho_I \in S := \{\rho \in L^2_+(I\!R^n, \rho_\infty^{-1}(dx)) \,\Big|\, |I_{\psi_1}(\rho|\rho_\infty)| + |I_\varphi(\rho|\rho_\infty)| < \infty\}.$$

From the Theorem 2.5(c) and Lemma 2.3 we then know that $e_\psi(\rho(t)|\rho_\infty) \to 0$ and $I_\psi(\rho(t)|\rho_\infty) \to 0$ as $t \to \infty$. Hence, integrating (2.29) (which also holds under condition (A3)) over (t,∞) gives

$$I_\psi(t) = \frac{d}{dt}e_\psi(t) \leq -2\lambda_3 e_\psi(t), \quad t \geq 0\,, \tag{2.45}$$

which proves the exponential entropy decay for sufficiently regular initial data.

In explicit terms (2.45) just is the convex Sobolev inequality (2.43) for all sufficiently regular ρ_I. Then, by simple closure (cf. the proof of Corollary 2.18 in [4]), we obtain this inequality in full generality. Once we have this, we no longer need Theroem 2.5 to prove $e_\psi(\rho(t)|\rho_\infty) \to 0$, and we obtain the full result. □

The desired L^1–convergence of $\rho(t)$ to ρ_∞ is now a direct consequence of Theorem 2.6 and the Csiszár-Kullback inequality (2.4):

Corollary 2.7. *Let e_ψ be an admissible relative entropy and $e_\psi(\rho_I|\rho_\infty) < \infty$. Let the coefficients $\phi(x)$, $F(x)$, and $\mathbf{D}(x)$ satisfy condition (A3). Then the solution of (1.1) satisfies*

$$\|\rho(t) - \rho_\infty\|_{L^1(I\!R^n)} \leq e^{-\lambda_3 t}\sqrt{\frac{2}{\eta_2}e_\psi(\rho_I|\rho_\infty)}, \quad t > 0, \tag{2.46}$$

with the notation $\eta_2 = \psi''(1)$.

3. Examples

In this section we shall construct examples to illustrate how the non-symmetric perturbation $\mathrm{div}(\mathbf{D}\rho F)$ can help to "improve" the constant in the LSI (1.7). For simplicity of the presentation we confine ourselves here to the case $\mathbf{D}(x) \equiv \mathbf{I}$. Assume that $\phi(x)$ is smooth on $\mathbb{R}^n$ and satisfies

$$\text{(i)} \quad \nabla\phi(0) = 0;\ \frac{\partial^2\phi}{\partial x^2}(x) > 0,\ \forall\, x \neq 0\,;$$

$$\text{(ii)} \quad \frac{\partial^2\phi}{\partial x^2} \geq \lambda\mathbf{I} > 0 \text{ on } \mathbb{R}^n\backslash B_\delta(0) \text{ for some (small) } \delta > 0\,;$$

$$\text{(iii)} \quad \frac{\partial^2\phi}{\partial x^2}(0) = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & & \cdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & \frac{\partial^2\phi}{\partial x_n^2}(0) \end{pmatrix},$$

where $\frac{\partial^2\phi}{\partial x_n^2}(0) > 0$. Clearly, this confinement potential $\phi(x)$ satisfies the BEC (A1) only with the convexity constant $\lambda_1 = 0$. Let $\rho_\infty = e^{-\phi(x)}$ be normalized on $\mathbb{R}^n$. Our goal is to find a vector field $F = (F_1(x), \ldots, F_n(x))^\top$ with $\mathrm{div}(\rho_\infty F) = 0$ such that the generalized Bakry-Emery condition (GBEC) holds, i.e.

(A2)

$$\exists\, \lambda_2 > 0 \text{ such that } \quad \frac{\partial^2\phi}{\partial x^2} - \frac{1}{2}\left(\frac{\partial F}{\partial x} + (\frac{\partial F}{\partial x})^\top\right) \geq \lambda_2\mathbf{I} \quad \forall x \in \mathbb{R}^n.$$

More precisely, we shall construct $F \in \mathrm{Lip}(\mathbb{R}^n)$ with $\mathrm{supp}F \subset [-L, L]^n$ and $L > 0$ sufficiently small, such that

$$\frac{\partial F}{\partial x}(0) = \begin{pmatrix} -1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \\ 0 & 0 & \cdots & -1 & 0 \\ * & * & \cdots & * & n-1 \end{pmatrix}, \tag{3.1}$$

where the derivatives $\partial_j F_n(0)$ are yet unspecified (we use the abbreviation $\partial_j := \frac{\partial}{\partial x_j}$). The first principal minors of

$$\mathbf{G} := \frac{\partial^2\phi}{\partial x^2}(0) - \frac{\varepsilon}{2}\left(\frac{\partial F}{\partial x} + (\frac{\partial F}{\partial x})^\top\right)(0)$$

are $\varepsilon, \ldots, \varepsilon^{n-1}$, and its determinant is of the form

$$\frac{\partial^2\phi}{\partial x_n^2}(0)\,\varepsilon^{n-1} + \mathcal{O}(\varepsilon^n)\,.$$

Then, for some $\varepsilon > 0$ sufficiently small, it holds: $\det\mathbf{G} > 0$ and $(\phi, \varepsilon F)$ clearly satisfies the GBEC (A2). We remark that F could be chosen as smooth as desired, by using easy modifications of the strategy below.

Now we shall construct two vector fields F and $J = (J_1(x), \ldots, J_{n-1}(x))^\top$ that satisfy

$$\rho_\infty F = \begin{pmatrix} \partial_n J_1 \\ \partial_n J_2 \\ \cdot \\ \cdot \\ \cdot \\ \partial_n J_{n-1} \\ -\partial_1 J_1 - \partial_2 J_2 - \cdots - \partial_{n-1} J_{n-1} \end{pmatrix}$$

and hence, $\mathrm{div}(\rho_\infty F) = 0$.

For $j = 1, \ldots, n-1$ we put

$$-F_j(x_1, \ldots, x_n) := \begin{cases} 0, & -L \le x_n \le -L/2; \\ f(x_j)\cos(\frac{x_n}{L}\pi)\prod_{\substack{k=1 \\ k\ne j}}^{n-1} \cos^2(\frac{x_k}{L}\frac{\pi}{2}), & -L/2 \le x_n \le L/2; \\ g_j(x_1, \ldots, x_{n-1})\sin(\frac{x_n}{L}2\pi), & L/2 \le x_n \le L, \end{cases}$$

with $L > 0$ to be chosen later. $f(s)$ is a smooth function on $I\!R$ with support in $[-L, L]$ and it satisfies

$$\begin{array}{ll} \text{(i)} & f(\pm L) = f'(\pm L) = f''(\pm L) = 0; \\ \text{(ii)} & f \ge 0, f'(0) = 1, f''(0) = 0. \end{array}$$

Further,

$$g_j(x_1, \ldots, x_{n-1}) := f(x_j)\prod_{\substack{k=1 \\ k\ne j}}^{n-1} \cos^2(\frac{x_k}{L}\frac{\pi}{2}) \frac{\int_{-L/2}^{L/2} \cos(\frac{x_n}{L}\pi)\rho_\infty(x_1, \ldots, x_n)\, dx_n}{\int_{L/2}^{L} \sin(\frac{x_n}{L}2\pi)\rho_\infty(x_1, \ldots, x_n)\, dx_n},$$

which implies

$$\int_{-L}^{L} F_j \rho_\infty \, dx_n = 0, \quad \forall\, (x_1, \ldots, x_{n-1}) \in I\!R^{n-1}, \quad \forall\, j = 1, \ldots, n-1. \tag{3.2}$$

Next we define for $j = 1, \ldots, n-1$:

$$J_j(x_1, \ldots, x_n) := \begin{cases} \int_{-L}^{x_n} F_j(x_1, \ldots, x_{n-1}, \tilde{x}_n)\rho_\infty(x_1, \ldots, x_{n-1}, \tilde{x}_n)\, d\tilde{x}_n, & |x_n| \le L; \\ 0, & |x_n| > L. \end{cases}$$

Due to (3.2) we have $J_j \in \mathrm{Lip}(I\!R^n)$. Finally we put

$$F_n := -\rho_\infty^{-1} \sum_{k=1}^{n-1} \partial_k J_k \,.$$

In order to verify (3.1), one easily finds for $j = 1, \ldots, n-1$:

$$\begin{array}{l} \partial_j F_j(0) = -f'(0) = -1; \\ \partial_k F_j(0) = 0, \quad k \ne j. \end{array}$$

In order to analyze $\partial_n F_n$ we use $\nabla\phi(0) = 0$ in $\mathrm{div}(\rho_\infty F) = 0$ and obtain

$$\rho_\infty(0)\sum_{j=1}^{n}\partial_j F_j(0) = 0.$$

Hence

$$\partial_n F_n(0) = -\sum_{j=1}^{n-1}\partial_j F_j(0) = n-1.$$

Thus, $\frac{\partial F}{\partial x}(0)$ is of form (3.1) and $(\phi, \varepsilon F)$ satisfies the GBEC (A2) for some (small) $\varepsilon > 0$.

Appendix: Calculation of the Ricci Tensor

The definition (2.12) of the *Ricci tensor* gives (using the Einstein summation convention)

$$\begin{aligned} U^\top \mathbf{Ric}(x) U &= U_i Ric^{ij} U_j \\ &= U_i d^{ik} d^{jl} \rho_{kl} U_j - U_i d^{ik} d^{jl} (\nabla^S \mathcal{X})_{kl} U_j \\ &=: W_1 + W_2. \end{aligned}$$

Using the definitions (2.7)-(2.9), after a long computation, we have

$$\begin{aligned} W_1 &= U_i d^{ik} d^{jl} R^p_{kpl} U_j \\ &= U_i d^{ik} d^{jl}\Big(\partial_p \Gamma^p_{lk} - \partial_l \Gamma^p_{pk} + \Gamma^p_{pm}\Gamma^m_{lk} - \Gamma^p_{lm}\Gamma^m_{pk}\Big) U_j \\ &= \frac{1}{2} U_i d^{qp}\partial_{pq} d^{ji} U_j - \frac{1}{2} U_i \Big(d^{ik}\partial_{kp} d^{pj} + d^{jl}\partial_{lp} d^{pi}\Big) U_j \\ &\quad + \frac{1}{2} U_i \partial_q d^{ji} \partial_p d^{pq} U_j - \frac{1}{4} U_i \Big(d^{ik} d_{mr} d_{sp} d^{jl} \partial_l d^{sm} \partial_k d^{rp} + 2 d^{qp} d_{lk} \partial_p d^{jl} \partial_q d^{ik} \\ &\quad + 2 d^{qp} d_{lk} \partial_p d^{jl} \partial_q d^{ik} + 2\partial_m d^{jp} \partial_p d^{im} \\ &\quad - 2 d^{ik} d_{ql} \partial_p d^{jl} \partial_k d^{qp} - 2 d^{jl} d_{kq} \partial_p d^{ik} \partial_l d^{qp}\Big) U_j \\ &\quad + \frac{1}{2} U_i \Big(d^{ik} d^{jl} \partial_l (d_{qp} \partial_k d^{qp}) + \frac{1}{4} d^{ik} d_{rp} \partial_m d^{rp} \partial_k d^{jm} \\ &\quad + \frac{1}{4} d^{jl} \partial_m d_{rp} \partial_l d^{im} - \frac{1}{4} d^{sm} d_{rp} \partial_m d^{rp} \partial_s d^{ji}\Big) U_j \\ &= U^\top \left[\frac{1}{2}\mathrm{Tr}\Big(\mathbf{D}\,\frac{\partial^2}{\partial x^2}\Big)\mathbf{D} + \frac{1}{2}(\nabla^\top \mathbf{D} \nabla)\mathbf{D} - \mathbf{D}\left(\frac{\partial^2}{\partial x^2}\mathbf{D}\right)\right] U \\ &\quad - \frac{1}{4}\mathrm{Tr}\Big(\mathbf{E}^\top + \mathbf{D}\,\mathbf{E}\,\mathbf{D}^{-1} - \mathcal{N}(\mathbf{D})\mathbf{D}^{-1}\Big)^2 \\ &\quad + \frac{1}{4} U^\top \mathbf{D}\Big(\frac{\partial(\mathbf{D}\nabla \ln(\det \mathbf{D}))}{\partial x} + (\frac{\partial(\mathbf{D}\nabla \ln(\det \mathbf{D}))}{\partial x})^\top\Big) U \\ &\quad - \frac{1}{4} U^\top \Big[(\nabla \ln(\det \mathbf{D}))^\top \mathbf{D} \nabla\Big] \mathbf{D} U\,, \end{aligned}$$

where we have used formulas such as

$$d^{qp}\partial_k d_{qp} = -\partial_k \ln(\det \mathbf{D}),$$
$$d^{ql}\partial_k d_{lm} = -d_{lm}\partial_k d^{ql}.$$

Next we compute W_2, which involves $\phi(x)$ and $F(x)$. We use (2.10) and (2.11) to obtain

$$\begin{aligned} W_2 &= -\frac{1}{2}U_i d^{ik}\Big(d_{lm}\nabla_k X^m + d_{km}\nabla_l X^m\Big)d^{jl}U_j \\ &= -\frac{1}{2}U_i d^{ik}\Big[d_{lm}\Big(\partial_k X^m + \Gamma^m_{kp}X^p\Big) + d_{km}\Big(\partial_l X^m + \Gamma^m_{lp}X^p\Big)\Big]d^{jl}U_j \\ &= -\frac{1}{2}U_i d^{ik}\big(d_{lm}\partial_k X^m + d_{km}\partial_l X^m\Big)d^{jl}U_j \\ &\quad -\frac{1}{2}U_i d^{ik}\Big(d_{lm}\Gamma^m_{kp}X^p + d_{km}\Gamma^m_{lp}X^p\Big)d^{jl}U_j \\ &=: V_1 + V_2. \end{aligned}$$

From (2.6), we have

$$\begin{aligned} V_1 &= \frac{1}{2}U_i d^{ik}\Big[d_{lm}\Big(\partial_k[d^{mq}(\partial_q(\phi - \frac{1}{2}\ln(\det \mathbf{D})) - F_q)]\Big) \\ &\quad + d_{km}\Big(\partial_l[d^{mq}(\partial_q(\phi - \frac{1}{2}\ln(\det \mathbf{D})) - F_q)]\Big)\Big]d^{jl}U_j \\ &= \frac{1}{2}\Big(U^\top \mathbf{D}\,\mathbf{E}(\nabla\phi - F) + (\nabla\phi - F)^\top \mathbf{E}^\top\,\mathbf{D}U\Big) \\ &\quad + U^\top \mathbf{D}\Big(\frac{\partial^2\phi}{\partial x^2} - \frac{1}{2}\left(\frac{\partial F}{\partial x} + (\frac{\partial F}{\partial x})^\top\right)\Big)\mathbf{D}U \\ &\quad - \frac{1}{4}U^\top \mathbf{D}\Big(\frac{\partial(\mathbf{D}\nabla \ln(\det \mathbf{D}))}{\partial x} + (\frac{\partial(\mathbf{D}\nabla \ln(\det \mathbf{D}))}{\partial x})^\top\Big)U. \end{aligned}$$

From (2.6) and (2.7), we have

$$\begin{aligned} V_2 &= -\frac{1}{2}U_i d^{pq}\partial_p d^{ij}\Big(\partial_q(\phi - \frac{1}{2}\ln(\det \mathbf{D})) - F_q\Big)U_j \\ &= -\frac{1}{2}\left[\nabla\big(\phi - \frac{1}{2}\ln(\det \mathbf{D})\big) - F\right]^\top \mathbf{DE}U \\ &= -\frac{1}{2}U^\top\Big[\Big((\nabla\phi - F)^\top \mathbf{D}\nabla\Big)\mathbf{D}\Big]U + \frac{1}{4}U^\top\Big((\nabla \ln(\det \mathbf{D}))^\top \mathbf{D}\nabla\Big)\mathbf{D}U. \end{aligned}$$

Hence, the GBEC (A3) can be written as

$$W_1 + W_2 = W_1 + V_1 + V_2 \geq \lambda_3 U^\top \mathbf{D}U,$$

which is exactly (2.13).

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