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ANALYSIS OF COMPLEX BROWNIAN MOTION

YUH-JIA LEE* AND KUANG-GHIEH YEN

Abstract. A theory of generalized functions based on the complex Brownian motion $\{Z(t) : t \in \mathbb{R}\}$, for which each $Z(t)$ is $N(0, |t|)$, is established on the probability space $(\mathcal{S}'_c, \mathcal{B}(\mathcal{S}'_c), \nu(dz))$, where $\mathcal{S}'$ is the dual of the Schwartz space $\mathcal{S}$, $\mathcal{S}'_c$, the complexification of $\mathcal{S}'$, identified as the product space $\mathcal{S}' \times \mathcal{S}'$, $\mathcal{B}(\mathcal{S}'_c)$ the Borel field of $\mathcal{S}' \times \mathcal{S}'$ and $\nu(dz)$ denotes the product measure $\mu_1(dx)\mu_1(dy)$. Using the representation of the complex Brownian motion

$$Z_t(x, y) = \frac{1}{\sqrt{2}} \left(\langle x, h_t \rangle + i\langle y, h_t \rangle \right),$$

where

$$h_t = \begin{cases} 1_{(0,t]}, & t > 0, \\ -1_{[t,0]}, & t < 0. \end{cases}$$

and employing the technique of white noise calculus initiated by Hida (see, e.g. [2] and [4]), we analyze functionals of complex Brownian motion. To define generalized complex Brownian functionals, we adopt the space of CKS entire functionals as test functions. As applications, the stochastic integral with respect to a complex Brownian motion are defined and studied. The Itô formula for complex Brownian functionals is obtained and it is shown that the evaluation of stochastic integral with respect to a complex Brownian motion follows the rule of Stratonovich integral.

1. Introduction

In this paper we are devoted to a systematic study of the complex Brownian functionals, by which we mean functions of complex Brownian motion given by

$$Z(t, \omega) = \frac{1}{\sqrt{2}} [B_1(t, \omega) + iB_2(t, \omega)],$$

where B_1 and B_2 are independent real-valued standard Brownian motions. Clearly $Z(t)$ is normally distributed with mean zero and variance parameter $|t|$.

As the calculus of the complex Brownian motion will play the main role, we need to represent $Z(t)$ as a function on a certain probability space. In this paper, we choose $(\mathcal{S}'_c, \mathcal{B}(\mathcal{S}'_c), \nu(dz))$ as the underlying probability space , where $\mathcal{S}$ is the Schwartz space with dual space $\mathcal{S}'$, $\mathcal{S}'_c$ is the complexification of $\mathcal{S}'$ which is identified as the product space $\mathcal{S}' \times \mathcal{S}'$, $\mathcal{B}(\mathcal{S}'_c)$ the Borel field of $\mathcal{S}' \times \mathcal{S}'$ and $\nu(dz)$ denotes

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the product measure $\mu_1(dx)\mu_1(dy)$, where μ_t represents the Gaussian measure defined on $\mathcal{S}'$ with characteristic function given by

$$C(\xi) = \int_{\mathcal{S}'} e^{i(x,\xi)} \mu_t(dx) = e^{-t|\xi|^2/2}.$$

One sees easily that the complex Brownian motion on $(\mathcal{S}'_c, \mathcal{B}(\mathcal{S}'_c), \nu(dz))$ may be represented by

$$Z_t(x,y) = \frac{1}{\sqrt{2}} \left(\langle x, h_t \rangle + i \langle y, h_t \rangle \right),$$

where

$$h_t = \begin{cases} 1_{(0,t]} & , t > 0, \\ -1_{[t,0]} & , t < 0. \end{cases}$$

The calculus of complex Brownian functionals is then performed with respect to the measure $\mu(dz)$. For example, let $f : \mathbb{C} \to \mathbb{C}$ be an entire function of exponential growth and $Z(t) = \frac{1}{\sqrt{2}}\langle x + iy, h_t \rangle$ the complex Brownian motion. Then we have

$$E[|f(Z(t))|^2] = \int_{\mathcal{S}'} \int_{\mathcal{S}'} |f\left(\langle x + iy, h_t \rangle\right)|^2 \mu_{1/2}(dx)\mu_{1/2}(dy).$$

The above identity gives a connection between the calculus of complex Brownian motion and the Segal–Bargmann entire functionals (see §3 for definition).

Being motivated by the (real) white noise analysis initiated by Hida (see, e.g. [4] and [2]), it is desirable to develop a theory of generalized complex functionals by white noise calculus approach. The main results are given as follows:

(1) A theory of generalized Segal–Bargmann functionals is established using CKS entire functionals [1] as test functionals.
(2) A Stochastic integral with respect to complex Brownian motion is defined and studied (for related work, we referred the reader to [3]).
(3) An Itô formula for complex Brownian functionals is derived.

As an example, it is shown that, for any Segal–Bargmann entire function F, the Itô formula is given by

$$F(Z(b)) - F(Z(a)) = \int_a^b F'(Z(t)) dZ(t).$$

The formula is then extended to a generalized CKS entire function F by using Hitsuda–Skorokhod integral given below:

$$\frac{d}{dt} \langle\!\langle F(Z(t)), \phi \rangle\!\rangle_c = \langle\!\langle \partial_t^* F'(Z(t)), \phi \rangle\!\rangle_c,$$

where $\langle\!\langle \cdot, \cdot \rangle\!\rangle_c$ denotes the pairing of generalized functional and test functional, $\partial_t = D_{\delta_t}$ and ∂_t^* is its adjoint, and ϕ is any CKS entire functional.

2. White Noise Calculus

The (Gaussian) white noise is generally understood in the engineering literature as a stationary stochastic process with constant spectral density. In mathematics, it can be shown that the white noise is the time derivative $\{\dot{B}(t) : t > 0\}$ of a Brownian motion . Since almost all sample paths of a Brownian motion are differentiable nowhere, the (Gaussian) white noise is realized in mathematics as a generalized process. In this paper, the Brownian motion denoted by $\{B(t) : t > 0\}$ will be regarded as a regular generalized functional on $\mathcal{S}'$. Hence for almost all x (μ_t) the white noise $t \mapsto \dot{B}_t(x)$ is a generalized function in $\mathcal{S}'$. Thus $\mathcal{S}'$ is regarded as the state space of white noise.

$\mathcal{S}'$ also has a nuclear space structure described as follows:

Let A denote the operator $A = 1+t^2-(d/dt)^2$ with domain $\mathcal{D}(A) \subset L^2 = L^2(\mathbb{R})$ and $\mathcal{D}(A)$ contains a CONS $\{e_n : n \in \mathbb{N}\}$ of L^2 consisting of eigenfunctions of A with corresponding eigenvalues $\{2n+2 : n = 1, 2, ...\}$. $\{e_n\}$ are known as Hermite functions.

For $p \geq 0$, let $\mathcal{S}_p = \mathcal{D}(A^p)$. Its dual space is given by $\mathcal{S}_p^* = \mathcal{S}_{-p}$. Then we have $\mathcal{S} = \cap_p \mathcal{S}_p$ which is provided with the projective limit topology and the dual space of $\mathcal{S}(\mathbb{R}^1)$ is given by $\mathcal{S}' = \cup_p \mathcal{S}_p$ which is equipped with the inductive topology. Moreover, the spaces

$$\mathcal{S} \subset L^2(\mathbb{R}^1) \subset \mathcal{S}'$$

form a Gel'fand triple.

It is well-known that $\mathcal{S}'$ carries a standard Gaussian measure μ_t with variance parameter t. Then the calculus on $\mathcal{S}'$ is performed with respect to μ_t. Let $\mu = \mu_1$. Then $(\mathcal{S}', \mathcal{B}, \mu)$ is called the white noise space which serves as the underlying probability space for white noise functionals.

3. Test and Generalized Functions

Definition 3.1. (Segal–Bargmann space[6]) Let H be a real separable Hilbert space and H_c the complexification of H. A single-valued function f defined on H_c is called a Segal–Bargmann entire function if it satisfies the following conditions:

(i) f is analytic in H_c.
(ii) The number

$$M_f := \sup_P \int_H \int_H |f(Px + iPy)|^2 n_t(dx) n_t(dy)$$

is finite, where n_t denotes as the Gaussian cylinder measure on H with variance parameter $t > 0$ (for details, we refer the reader to [5]) and P's run through all finite dimensional orthogonal projections on H.

Denote the class of Segal–Bargmann entire functions on H by $\mathcal{SB}_t[H]$ and define $\|f\|_{\mathcal{SB}_t[H]} = \sqrt{M_f}$. Then $(\mathcal{SB}_t[H], \|\cdot\|_{\mathcal{SB}_t[H]})$ is a Hilbert space.

It follows immediately from [6] that we have

$$\begin{aligned} \|f\|^2_{\mathcal{SB}_t[H]} &= \sum_{k=0}^{\infty} \frac{(2t)^k}{k!} \left(\sum_{i_1,\ldots,i_k=1}^{N} \left| D^k f(0) e_{i_1} \cdots e_{i_k} \right|^2 \right) \\ &= \sum_{k=0}^{\infty} \frac{(2t)^k}{k!} \|D^n f(0)\|^2_{HS^2[H]}, \end{aligned} \tag{3.1}$$

where $\|S\|_{HS^n[H]}$ denotes the Hilbert–Schmidt norm of an n-linear operator $S \in L^n(H)$ defined by

$$\|S\|_{HS^n(H)} := \left(\sum_{i_1,\ldots,i_k=1}^{\infty} |Se_{i_1} \cdots e_{i_k}|^2 \right)^{1/2}$$

which is independent of the choice of CONS $\{e_i\}$ of H.

Next, we introduce the CKS entire functionals [1].

Let $\alpha(n)$, $n \geq 0$, be a sequence of real numbers satisfying the following conditions:

(a) $\alpha(0) = 1$ and $\inf_{n \geq 0} \alpha(n) > 0$,
(b) $\lim_{n\to\infty} n^{-1}\alpha(n)^{1/n} = 0$,
(c) $\gamma_{n+2}/\gamma_{n+1} \leq \gamma_{n+1}/\gamma_n$, for all $n \geq 0$, where $\gamma_n = \alpha(n)/n!$.

Definition 3.2. (Infinite-dimensional CKS entire functionals) For each $p \in \mathbb{R}$, define

$$|\!|\!|\phi|\!|\!|_{\alpha,p} = \left(\sum_{n=0}^{\infty} \alpha(n) \frac{\|D^n\phi(0)\|^2_{HS^n[\mathcal{S}_{-p}]}}{n!} \right)^{1/2}$$

and set

$$\mathcal{SB}_{p,\alpha} = \{\phi \in \mathcal{SB}_{1/2}[H] : |\!|\!|\phi|\!|\!|_{\alpha,p} < \infty\}.$$

Let $\mathcal{SB}_\alpha$ be the projective limit of $\mathcal{SB}_{p,\alpha}$ for $p \geq 0$ and let $\mathcal{SB}'_\alpha$ be the dual space of $\mathcal{SB}_\alpha$. Then $\mathcal{SB}_\alpha$ is a nuclear space and we have the following continuous inclusions:

$$\mathcal{SB}_\alpha \subset \mathcal{SB}_{p,\alpha} \subset \mathcal{SB}[L^2] \subset \mathcal{SB}'_{p,\alpha} \subset \mathcal{SB}'_\alpha,$$

where $\mathcal{SB}[L^2] := \mathcal{SB}_{1/2}[L^2]$.

The space $\mathcal{SB}_\alpha$, which is referred to as the CKS entire functionals on $\mathcal{S}'{}_c$, will serve as test functionals and $\mathcal{SB}'_\alpha$ is referred to as the generalized complex Brownian functionals.

The space $\mathcal{SB}'_{p,\alpha}$ may be identified as the space of entire functions defined on $\mathcal{S}_{p,c}$ such that : $|\!|\!|\phi|\!|\!|_{\alpha^{-1},-p} < \infty$ and the pairing of $\mathcal{SB}'_\alpha$ and $\mathcal{SB}_\alpha$ is defined by

$$\langle\!\langle \Phi, \varphi \rangle\!\rangle = \sum_{n=0}^{\infty} \frac{1}{n!} \langle\!\langle D^n\overline{\Phi}(0), D^n\varphi(0) \rangle\!\rangle_{HS^n},$$

where

$$\langle\!\langle D^n\overline{\Phi}(0), D^n\varphi(0) \rangle\!\rangle_{HS^n} := \sum_{i_1,\ldots,i_n=1}^{\infty} \left[\overline{D^n\Phi(0)e_{i_1} \cdots e_{i_n}} D^n\varphi(0)e_{i_1} \cdots e_{i_n} \right].$$

Definition 3.3. (One-dimensional CKS entire functions) If $\phi(z)$ can be represented by a formal power series $\sum_{n=0}^{\infty} a_n z^n$, we define

$$|\!|\!|\phi|\!|\!|_\alpha = \left(\sum_{n=0}^{\infty} \alpha(n) n! |a_n|^2 \right)^{1/2}$$

and let

$$\mathcal{SB}_\alpha(\mathbb{R}) = \{\phi : |\!|\!|\phi|\!|\!|_{\alpha,p} < \infty\}.$$

If $\phi(z)$ is a formal power series represented by $\sum_{n=0}^{\infty} b_n z^n$, we define

$$|||\phi|||_{\alpha^{-1}} = \left(\sum_{n=0}^{\infty} \frac{n!|b_n|^2}{\alpha(n)} \right)^{1/2}.$$

Then the dual space of $\mathcal{SB}_\alpha$ is characterized by

$$\mathcal{SB}'_\alpha(\mathbb{R}) = \{\phi : |||\phi|||_{\alpha^{-1}} < \infty\}.$$

Remark 3.4. Let k be any positive integer and let $f(w) = w^k$ $(w \in \mathbb{C})$. Then, for any $h \in H$, the functional $\Phi = f(\langle \cdot, h\rangle)$ is clearly a Segal–Bargmann entire function defined on H. Φ is also defined on $\mathcal{S}_c$ a.e. (ν) and we have, for $\varphi \in \mathcal{SB}_\alpha$,

$$\langle\langle \Phi, \varphi \rangle\rangle = \int_{\mathcal{S}'} \int_{\mathcal{S}'} \overline{f(\langle x + iy, h\rangle)} \varphi(x + iy) \mu_{\frac{1}{2}}(dx) \mu_{\frac{1}{2}}(dy).$$

The above identity will be used in the proof of Theorem 5.7.

4. Examples of Complex Brownian Functionals

The following lemma will be used frequently in computation.

Lemma 4.1. *For the functional representation of a complex Brownian motion $Z(t)$ given by*

$$Z(t, x, y) = \frac{\langle x + iy, h_t\rangle}{\sqrt{2}},$$

we have

(i)

$$\langle\langle Z(t), 1\rangle\rangle_c = \int_{\mathcal{S}'} \int_{\mathcal{S}'} \langle x + iy, h_t\rangle \mu_{1/2}(dx) \mu_{1/2}(dy) = 0.$$

(ii)

$$\langle\langle Z(t)^m, Z(t)^n \rangle\rangle_c = n! \delta_{m,n} t^n.$$

Proof. (i) is clear and the proof of (ii) follows immediately from integration by parts formula (see, for example [6]). □

Example 4.2. If $\xi \in \mathcal{S}$, then clearly $(\cdot, \xi) \in \mathcal{SB}_\alpha$. For any $\varphi \in \mathcal{SB}_\alpha$, it follows immediately from the definition of the $\langle\langle \cdot, \cdot \rangle\rangle_c$ pairing that we have

$$\langle\langle (\cdot, \xi), \varphi) \rangle\rangle_c = D_\xi \varphi(0) = D\varphi(0)\xi, \tag{4.1}$$

where $D\varphi(0)$ denotes the Fréchet derivative of φ at 0. Note that the right hand side of the the identity in Equation (4.1) remains meaningful even for $\xi \in \mathcal{S}'$. $(\cdot, \xi)$ defined a continuous linear functional on $\mathcal{SB}_\alpha$ so that $(\cdot, \xi) \in \mathcal{SB}'_\alpha$.

It is worthwhile to note that, for $\varphi \in \mathcal{SB}[L^2]$, φ is $L^2[\mathbb{R}]-$differentiable i.e. φ is Fréchet differentiable in the directions of $L^2[\mathbb{R}]$. If, for $\xi \in L^2[\mathbb{R}]$, we interpret $D_\xi \varphi(0)$ as the $L^2[\mathbb{R}]-$derivative of φ at 0 in the direction of ξ, then the identity in Equation (4.1) remains valid. It follows that one can apply this identity with $\xi = (h_{t+\epsilon} - h_t)/\epsilon$ and $\varphi \in \mathcal{SB}_\alpha$ to get

$$\langle\langle \langle \cdot, (h_{t+\epsilon} - h_t)/\epsilon\rangle, \varphi \rangle\rangle_c = D\varphi(0)[(h_{t+\epsilon} - h_t)/\epsilon].$$

Let $\epsilon \to 0$ in the above identity. We obtain the definition of the complex white noise $\dot{Z}(t)$ as follows:

$$\langle\langle \dot{Z}(t), \varphi \rangle\rangle_c = D\varphi(0)\delta_t.$$

Clearly $\dot{Z}(t) \in \mathcal{SB}'_\alpha$.

Example 4.3. Given $h \in L^2[\mathbb{R}]$, we have

$$\begin{aligned} & \langle\langle e^{\langle \cdot, h\rangle}, \phi \rangle\rangle_c \\ = & \sum_{n=0}^{\infty} \frac{1}{n!} \langle\langle h^{\otimes n}, D^n\phi(0) \rangle\rangle \\ = & \phi(h). \end{aligned}$$

If $h = h_t$ then we obtain

$$\langle\langle e^{Z(t)}, \phi \rangle\rangle_c = \phi(h_t). \tag{4.2}$$

It is easy to see that $e^{Z(t)} \in \mathcal{SB}_\alpha$.

Example 4.4. Let $h, k \in L^2[\mathbb{R}]$ such that $\langle h, k\rangle = 0$. Then it follows from Equations (4.1) and (4.2) that

$$\begin{aligned} \langle\langle \langle \cdot, h\rangle e^{\langle \cdot, k\rangle}, \varphi \rangle\rangle_c &= \langle h, k\rangle \langle\langle 1, \varphi \rangle\rangle_c + \langle\langle e^{\langle \cdot, k\rangle}, \varphi \rangle\rangle_c \\ &= \langle\langle e^{\langle \cdot, k\rangle}, \varphi \rangle\rangle_c. \end{aligned} \tag{4.3}$$

Taking $h = (h_{t+\epsilon} - h_t)/\epsilon$ and $k = h_t$ in Equation (4.3) and then letting $\epsilon \to 0$, we are led to the definition of $\dot{Z}(t)\exp(Z(t)$ given in the following

Definition 4.5. For $\varphi \in \mathcal{SB}_\alpha$, we define

$$\langle\langle \dot{Z}(t)e^{Z(t)}, \varphi \rangle\rangle := D\varphi(h_t)\delta_t. \tag{4.4}$$

Clearly $\dot{Z}(t)e^{Z(t)} \in \mathcal{SB}'_\alpha$.

Example 4.6. If $Z(t)$ is a complex Brownian motion, then it follows from Example 4.2 and Definition 4.5 that we have

$$\frac{d}{dt} e^{Z(t)} = e^{Z(t)}\dot{Z}(t).$$

Example 4.7. (Composition of generalized function with a complex Brownian motion) Let $f \in \mathcal{SB}'_\alpha(\mathbb{R})$ be an one dimensional generalized CKS entire function represented by $f(z) = \sum_{n=0}^{\infty} b_n z^n$. Assume that $\sum_{n=0}^{\infty} \alpha(n)^{-1} n! |b_n|^2 < \infty$. Then, for $\psi \in \mathcal{SB}_\alpha$, we define

$$\langle\langle f(Z(t)), \psi \rangle\rangle_c = \sum_{n=0}^{\infty} b_n D^n\psi(0)h_t^n. \tag{4.5}$$

It will be proved in the next section that $f(Z(t)) \in \mathcal{SB}'_\alpha$.

5. Itô Formula

We first prove that the composition $f(Z(t))$ defined in Example 4.7 is in fact a generalized Segal–Bargmann functional.

Theorem 5.1. *Let $f \in \mathcal{SB}'_\alpha(\mathbb{R})$ be an one dimensional generalized CKS entire function represented by $f(z) = \sum_{n=0}^\infty b_n z^n$. Assume that that $\{b_n\}$ satisfies $\sum_{n=0}^\infty \alpha(n)^{-1} n! |b_n|^2 < \infty$. Then $f(Z(t))$, defined by Equation (4.5), is a member of $\mathcal{SB}'_\alpha$.*

Proof. Recall that

$$\langle\langle f(Z(t)), \psi \rangle\rangle_c = \sum_{n=0}^\infty b_n D^n \psi(0) h_t^n.$$

For each $t > 0$, there exists p such that $|h_t|_{-p} \leq 1$. Thus

$$\begin{aligned}
|\langle\langle f(Z(t)), \psi \rangle\rangle_c| &= \left| \sum_{n=0}^\infty \left(b_n \left(\frac{n!}{\alpha(n)} \right)^{1/2} \right) \left(\frac{D^n \psi(0) h_t^n}{\sqrt{n!}} \right) (\alpha(n))^{1/2} \right| \\
&\leq \|\!| f \|\!|_{\alpha^{-1}} \sqrt{\sum_{n=0}^\infty \frac{|D^n \psi(0)|^2}{n!} h_t^n \alpha(n)} \\
&\leq \|\!| f \|\!|_{\alpha^{-1}} \sqrt{\sum_{n=0}^\infty \left(\frac{\|D^n \psi(0)\|^2_{HS[\mathcal{S}_{-p}]}}{n!} |h_t|^2 n_{-p} \right) \alpha(n)} \\
&\leq \|\!| f \|\!|_{\alpha^{-1}} \sqrt{\sum_{n=0}^\infty \left(\frac{\|D^n \psi(0)\|^2_{HS[\mathcal{S}_{-p}]}}{n!} \right) \alpha(n) |h_t|^{2n}_{-p}} \\
&\leq \|\!| f \|\!|_{\alpha^{-1}} \|\!| \psi \|\!|_{\alpha,p}.
\end{aligned}$$

This proves that $f(Z(t)) \in \mathcal{SB}'_\alpha$. □

Now we are ready to derive the Itô formula. Let $f \in \mathcal{SB}'_\alpha(\mathbb{R})$. Then we have

$$\begin{aligned}
\frac{d}{dt} \langle\langle f(Z(t)), \phi \rangle\rangle_c &= \sum_{n=0}^\infty b_n D^n \phi(0) h_t^{n-1} \delta_t \\
&= \sum_{n=0}^\infty b_n n D^{n-1} (D\phi(0) \delta_t) h_t^{n-1} \\
&= \sum_{n=0}^\infty b_n n D^{n-1} (\partial_t \phi)(0) h_t^{n-1} \\
&= \langle\langle \partial_t^* f'(Z(t)), \phi \rangle\rangle_c,
\end{aligned}$$

where $\partial_t = \partial_{\delta_t}$ and ∂_t^* is the adjoint operator of ∂_t. It follows that

$$\frac{d}{dt} f(Z(t) = \partial_t^* f'(Z(t)).$$

This proves the Itô formula for a complex Brownian motion. As a summary, we state the above result as a theorem.

Theorem 5.2. (Itô formula) *Let $f \in \mathcal{SB}'_\alpha(\mathbb{R})$. Then we have*

$$\frac{d}{dt} f(Z(t)) = \partial_t^* f'(Z(t).$$

or in the integral form,

$$f(Z(b)) - f(Z(a)) = \int_a^b \partial_t^* f'(Z(t)dt.$$

As in the case of a real Brownian motion, the term on the right hand side may be interpreted as a stochastic integral as shown below.

Definition 5.3. Suppose that $f \in \mathcal{SB}'_\alpha$. Define the stochastic integral $f(Z(t))$ as follows:

$$\langle\!\langle \int_a^b f(Z(t))dZ(t), \phi \rangle\!\rangle_c := \lim_{\|\triangle_n\| \to 0} \langle\!\langle \sum_{i=1}^n f(Z(t_{i-1}))(Z(t_i) - Z(t_{i-1})), \phi \rangle\!\rangle_c,$$

where $a = t_0 < t_1 < t_2 < \cdots < t_n = b$ and $\|\triangle_n\| = max_j |t_j - t_{j-1}|$.

Lemma 5.4. *Let $\widetilde{h} = \langle x + iy, h \rangle$ and $\widetilde{k} = \langle x + iy, k \rangle$, then*

$$\langle\!\langle \widetilde{h}\widetilde{k}, \phi \rangle\!\rangle_c = \langle D^2\phi(0), h \otimes k \rangle.$$

Proof. The proof is straightforward. □

Corollary 5.5. *Let $\widetilde{h}_t = \langle x + iy, h_t \rangle$, then $\langle\!\langle \widetilde{h}_t^{\,n}, \phi \rangle\!\rangle_c = \langle D^n\phi(0), h_t^{\otimes n} \rangle$.*

Example 5.6. To compare the stochastic integrals for complex Brownian motions with the stochastic integral for real Brownian motions, we first show that $\int_a^b Z(t)dZ(t)dt = \frac{1}{2}(Z^2(b) - Z^2(a))$. In fact, it follows from Definition 5.3 that, for any test functional φ, we have

$$\begin{aligned}
&\langle\!\langle \int_a^b Z(t)dZ(t), \varphi \rangle\!\rangle_c \\
&= \lim_{\|\triangle_n\| \to 0} \sum_{i=1}^n \langle\!\langle Z(t_{i-1})(Z(t_i) - Z(t_{i-1})), \varphi \rangle\!\rangle_c \\
&= \lim_{\|\triangle_n\| \to 0} \sum_{i=1}^n D^2\varphi(0) h_{t_{i-1}} h_{t_i - t_{i-1}} \\
&= \lim_{\|\triangle_n\| \to 0} \sum_{i=1}^n \frac{1}{2} \left\{ \left[D^2\varphi(0)(h_{t_i}^{\otimes^2} - h_{t_{i-1}}^{\otimes^2}) \right] + \left[D^2\varphi(0)(h_{t_i} - h_{t_{i-1}})^{\otimes^2} \right] \right\} \\
&= \lim_{\|\triangle_n\| \to 0} \{ (\mathrm{I}) + (\mathrm{II}) \}.
\end{aligned}$$

It is easy to see that

$$(\mathrm{I}) = \langle\!\langle \frac{1}{2}(Z^2(b) - Z^2(a)), \phi \rangle\!\rangle_c$$

and (II) $\to 0$ as $\| \triangle_n \| \to 0$. Therefore we have

$$\int_a^b Z(t)dZ(t) = \frac{1}{2}(Z^2(b) - Z^2(a)). \tag{5.1}$$

On the other hand,

$$\begin{aligned}\langle\langle\int_a^b \partial_t^* Z(t)dt, \phi\rangle\rangle_c &= \int_a^b \langle\langle\partial_t^* Z(t), \phi\rangle\rangle_c dt \\ &= \int_a^b \langle D^2\phi(0), h_t \otimes \delta_t\rangle dt \\ &= \frac{1}{2}\int_a^b \frac{d}{dt}\langle D^2\phi(x+iy), h_t \otimes h_t\rangle dt \\ &= \frac{1}{2}\langle\langle(Z^2(b) - Z^2(a)), \phi\rangle\rangle_c \qquad (5.2)\end{aligned}$$

It follows from Equations (5.1) and (5.2) that

$$\int_a^b Z(t)dZ(t) = \int_a^b \partial_t^* Z(t)dt.$$

Theorem 5.7. *Let $f \in \mathcal{SB}_\alpha(\mathbb{R})$ and $\phi \in \mathcal{SB}_\alpha$ then*

$$\langle\langle\int_a^b f(Z(t))dZ(t), \phi\rangle\rangle_c = \langle\langle\int_a^b \partial_t^* f(Z(t))dt, \phi\rangle\rangle_c.$$

Proof. Since f can be represented by a formal power series, it is sufficient to prove the theorem for $f(z) = z^k$ for arbitrary non-negative integer k. By definition,

$$\langle\langle\int_a^b f(Z(t))dZ(t), \varphi\rangle\rangle_c = \lim_{\|\triangle_n\|\to 0}\sum_{i=1}^n \langle\langle f(Z(t_{i-1}))(Z(t_i) - Z(t_{i-1})), \varphi\rangle\rangle_c$$

for any $\varphi \in \mathcal{SB}_\alpha$. According to Remark 3.4, for each i, we have

$$\begin{aligned}&\langle\langle f(Z(t_{i-1}))(Z(t_i) - Z(t_{i-1})), \varphi\rangle\rangle_c \\ &= \int_{\mathcal{S}'} f(\langle x - iy, h_{t_i}\rangle)\langle x - iy, h_{t_j} - h_{t_{j-1}}\rangle\varphi(x+iy)\mu_{\frac{1}{2}}(dx)\mu_{\frac{1}{2}}(dy) \\ &= \int_{\mathcal{S}'} f(\langle x - iy, Ph_{t_i}\rangle)\langle x - iy, P[h_{t_j} - h_{t_{j-1}}]\rangle\varphi(x+iy)\mu_{\frac{1}{2}}(dx)\mu_{\frac{1}{2}}(dy). \qquad (5.3)\end{aligned}$$

Next, apply integration by parts formula, Equation (5.3) becomes

$$\begin{aligned}&\int_{\mathcal{S}'}\int_{\mathcal{S}'} f(\langle x - iy, h_{t_{i-1}}\rangle)\langle x - iy, h_{t_i} - h_{t_{i-1}}\rangle\varphi(x+iy)\mu_{\frac{1}{2}}(dx)\mu_{\frac{1}{2}}(dy) \\ &= \int_{\mathcal{S}'}\int_{\mathcal{S}'} f(\langle x - iy, h_{t_{i-1}}\rangle)\langle x, h_{t_i} - h_{t_{i-1}}\rangle\varphi(x+iy)\mu_{\frac{1}{2}}(dx)\mu_{\frac{1}{2}}(dy) \\ &\quad - i\int_{\mathcal{S}'}\int_{\mathcal{S}'} f(\langle x - iy, h_{t_{i-1}}\rangle)\langle y, h_{t_i} - h_{t_{i-1}}\rangle\phi(x+iy)\mu_{\frac{1}{2}}(dx)\mu_{1/\frac{1}{2}}(dy) \\ &= \frac{1}{2}\int_{\mathcal{S}'}\int_{\mathcal{S}'}\langle f(\langle x - iy, h_{t_i}\rangle)\langle D\varphi(x+iy), h_{t_i} - h_{t_{i-1}}\rangle\mu_{\frac{1}{2}}(dx)\mu_{\frac{1}{2}}(dy) \\ &\quad - \frac{i}{2}\langle if(\langle x - iy, h_{t_{i-1}}\rangle)\langle D\varphi(x+iy), P[h_{t_i} - h_{t_{i-1}}]\rangle\mu_{\frac{1}{2}}(dx)\mu_{\frac{1}{2}}(dy) \\ &= \int_{\mathcal{S}'}\int_{\mathcal{S}'} f(\langle x - iy, h_{t_{i-1}}\rangle)\langle D\varphi(x+iy), h_{t_i} - h_{t_{i-1}}\rangle\mu_{\frac{1}{2}}(dx)\mu_{\frac{1}{2}}(dy).\end{aligned}$$

Then we have

$$\begin{aligned}
&\langle\!\langle f(Z(t_{i-1}))(Z(t_i)-Z(t_{i-1})),\varphi\rangle\!\rangle_c \\
&=\int_{\mathcal{S}'}\int_{\mathcal{S}'} f(\langle x-iy,h_{t_{i-1}}\rangle)\langle D\varphi(x+iy),h_{t_i}-h_{t_{i-1}}\rangle\mu_{\frac{1}{2}}(dx)\mu_{\frac{1}{2}}(dy) \\
&=\int_{\mathcal{S}'}\int_{\mathcal{S}'} f(\langle x-iy,h_{t_{i-1}}\rangle)\langle D\phi(x+iy),\frac{h_{t_i}-h_{t_{i-1}}}{t_i-t_{i-1}}\rangle(t_i-t_{i-1}).
\end{aligned} \tag{5.4}$$

Finally, by Equation (5.4), we have

$$\begin{aligned}
&\sum_{i=1}^{n}\langle\!\langle f(Z(t_{i-1}))(Z(t_i)-Z(t_{i-1})),\varphi\rangle\!\rangle_c \\
&=\sum_{i=1}^{n}\langle f(Z(t_{i-1}))\langle D\phi(x+iy),\frac{h_{t_i}-h_{t_{i-1}}}{t_i-t_{i-1}}\rangle(t_i-t_{i-1}) \\
&\to\langle\!\langle\int_a^b \partial_t^* f(Z(t))dt,\phi\rangle\!\rangle_c \qquad \text{as } \|\triangle_n\|\to 0.
\end{aligned}$$

The last step follows from the fact that $\dfrac{h_{t_i}-h_{t_{i-1}}}{t_i-t_{i-1}}$ converges to δ_{t_i} uniformly with respect to the norm of $\mathcal{S}_p$ for $p>\frac{1}{2}$. We complete the proof. □

Remark 5.8. The reason why the second derivative term, from the classic Itô formula, disappears may be explained as follows:

Let $F(z)=P(x,y)+iQ(x,y)$, where $z=x+iy$. Since F is analytic, from the Cauchy-Riemann equations $\dfrac{\partial P}{\partial x}=\dfrac{\partial Q}{\partial y}$ and $\dfrac{\partial P}{\partial y}=-\dfrac{\partial Q}{\partial x}$, it follows that both P and Q are harmonic functions. Thus $\Delta P=\Delta Q=0$. Because of this fact, the second derivative term in the Itô formula for $F(Z(b))-F(Z(a))$ becomes:

$$\begin{aligned}
&\frac{1}{2}\int_a^b\left[\Delta(B_1(t),B_2(t))+i\Delta Q(B_1(t),B_2(t))\right]dt \\
&+\int_a^b\left[\frac{\partial^2 P}{\partial x\partial y}(B_1(t),B_2(t))+i\frac{\partial^2 Q}{\partial x\partial y}(B_1(t),B_2(t))\right]d\langle B_1,B_2\rangle_t.
\end{aligned}$$

Since B_1 and B_2 are independent processes, we get $\langle B_1,B_2\rangle_t=0$. Thus the second derivative term in Itô formula, for holomorphic functions, is vanishing.

Remark 5.9. Conclusion: We have shown that the stochastic integral with respect to a complex Brownian motion behaves exactly in the same manner as Stratonovich integral. Since the Segal–Bargmann entire functions play the same role as the U-functionals in white noise analysis, one can take the inverse Segal–Bargmann transform S^{-1} to obtain the corresponding white noise functional, for example,

$$S^{-1}\left\{\int_a^b Z(t)^n dZ(t)\right\}=\int_a^b B(t)^n dB(t),$$

where

$$S^{-1}\varphi(x)=\int_{\mathcal{S}'}\varphi(\sqrt{2}(x+iy))\mu(dy).$$

It is worthwhile to mention here that $S^{-1}\varphi(x)$ is nothing but the conditional expectation of $\varphi(\sqrt{2}z)$ with respect to the real Brownian motion.

As an application, we consider the linear Itô stochastic differential equation

$$X_t = X_0 + c\int_0^t X_s ds + \sigma \int_0^t X_s dB_s,\ t \in [0,T], \tag{5.5}$$

for given constants c and $\sigma > 0$. Let us first consider the corresponding stochastic differential equation with B_t being replaced by the complex Brownian motion $\sqrt{2}Z(t)$, i.e.

$$\frac{dX_t^c}{dt} = cX_t^c + \sqrt{2}\sigma X_t^c \dot{Z}(t), \quad X_0^c = X_0. \tag{5.6}$$

We can solve Equation (5.6) by employing the method of usual ordinary differential equation,

$$X_t^c = X_0 e^{ct+\sqrt{2}\sigma Z_t}.$$

Finally, taking conditional expectation for X_t^c with respect to the real Brownian motion, we obtain

$$\begin{aligned} X_t &= X_0 e^{ct}\int_{\mathcal{S}'} e^{\sigma\langle x+iy,h_t\rangle}\mu(dy) \\ &= X_0 e^{(c-\frac{1}{2}\sigma^2)t+\sigma\langle x,h_t\rangle} \\ &= X_0 e^{(ct-\frac{1}{2}\sigma^2)t+\sigma B_t}\ t \in [0,T]. \end{aligned}$$

Note that X_t is a geometric Brownian motion which solves the stochastic differential equation in Equation (5.5).

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