Seminar on Continuity in Semilattices

Volume 1 | Issue 1 Article 36

4-18-1977

SCS 35: Dedekind Complete Posets and Scott Topologies

Oswald Wyler
Carnegie Mellon University, Pittsburgh, PA USA

SEMINAR ON CONTINUITY IN SEMILATTICES (SCS)

NAMES: Oswald Wyler

DATE: April 18, 1977

TOPIC: Dedekind complete posets and Scott topologies

REFERENCES: [1] Gierz, Hofmann et al., On complete lattices L for which O(L) is continuous. SCS memo, 4/8/77.
[2] B.J. Day and G.M. Kelly, On topological quotient maps. Proc. Camb. Phil. Soc. 67 (1970), 553 - 558.

For T_0 spaces X, Y, Z, Proposition 1.1 of [1] says in effect: if a map $f : X \times Y \longrightarrow Z$ is separately continuous and Y quasilocally compact, then f is jointly continuous. This is well known to be false if $X = Y = Z = [0,1]$, the unit interval. Prop. 1.1 was used in [1] to prove Corollaries 1.2 and 1.3. Both corollaries are valid. Corollary 1.2 says that if the functors $- \times Y_1$ and $- \times Y_2$ on T_0 spaces have right adjoints, then their composite (up to natural equivalence) $- \times (Y_1 \times Y_2)$ has a right adjoint. This is of course true. Corollary 1.3 in [1] is important for [1] and probably also otherwise; thus a rescue effort is in order. This effort led to a study of the basic properties of Scott topologies which I present here.

It seemed reasonable to study Scott topologies in the maximal feasible generality; this led to the consideration of a category of upper Dedekind complete posets, with maps preserving suprema of updirected sets as morphisms. We denote this category

Seminar on Continuity in Semilattices, Vol. 1, Iss. 1 [2023], Art. 36

by $\underline{D}$. Scott topologies define a full and faithful functor from $\underline{D}$ to T_0 spaces. We denote this functor, with an appropriate codomain restriction, by S .

As pointed out by the Grothendieck school in SGA 4-1 (Lecture Notes in Math. 269), and used by D. Scott in LNM 274 for the creation of Scott topologies, every T_0 space X has an induced poset structure, with $x \leqslant y$ in X iff, equivalently, every open neighborhood of x in X is also a neighborhood of y , or $x \in \overline{\{y\}}$, the closure of $\{y\}$. We shall call X a d-space if X with this order is upper Dedekind complete, and the topology of X is coarser than the induced Scott topology.

Every sober space turns out to be a d-space. Not every d-space is sober, for every T_1 space is a d-space. If X is a T_1 space, then the induced order on X is discrete, hence Dedekind complete, with the discrete topology as Scott topology. d-spaces define an epireflective full subcategory of T_0 spaces; we skip the proof of this result as irrelevant for our present purpose. Warning: reflections for d-spaces are not surjective; they are strict embeddings.

We do not know whether an object of $\underline{D}$ with the Scott topology is always a sober space. It is always a d-space; the order induced by the Scott topology is the given order. Aus der Not eine Tugend machend, we substitute d-spaces for sober spaces in the present study. The category $\underline{D}$ is cartesian closed; it is also embedded into d-spaces as a full coreflective subcategory. Combining the last two statements, we re-establish Cor. 1.3 of [1].

1. Dedekind complete posets

1.1. We call a poset S (upper) Dedekind complete if every updirected subset of S has a supremum in S . A morphism of Dedekind complete posets S , T is a mapping $f : S \longrightarrow T$ which preserves order, and suprema of updirected subsets. With composition of mappings as composition of morphisms, Dedekind complete posets and their morphisms form a category which we denote by $\underline{D}$. By a mathematician's habitual lazyness, an object of $\underline{D}$ will be called a d-set in this memo.

1.2. A sup semilattice S is a d-set iff S is complete, but there are many d-sets which are not complete lattices. In fact, every discrete poset is a d-set; the only updirected subsets of a discrete poset are singletons. This includes the empty poset: an updirected subset, having upper bounds of all finite subsets, cannot be empty. One verifies easily that discrete posets are free d-sets; they define a left adjoint of the forgetful functor from $\underline{D}$ to sets.

The category $\underline{D}$ has products (and in fact limits of all small diagrams), and the forgetful functor from $\underline{D}$ to posets preserves these products. Thus the product poset of a family of d-sets is the product of this family in $\underline{D}$.

1.3. THEOREM. *The category $\underline{D}$ is cartesian closed.*

Proof. We denote by $D[S,T]$, for d-sets S and T, the set of all morphism $f : S \longrightarrow T$ in $\underline{D}$, ordered point-wise. If $F \subset D[S,T]$ is updirected, then every set $F(x)$ of points $f(x)$ of T, with $f \in F$, is updirected, for $x \in X$. We put $(\sup F)(x) = \sup F(x)$; this clearly is the desired supremum of F if it is a morphism of $\underline{D}$. For $\varphi \subset S$ updirected, we have

$$(\sup F)(\sup \varphi) = \sup_{f\in F} f(\sup \varphi) = \sup_{f\in F} \sup_{x\in\varphi} f(x)$$
$$= \sup_{x\in\varphi} \sup_{f\in F} f(x) = \sup_{x\in\varphi} (\sup F)(x) .$$

Thus $D[S,T]$ is indeed a d-set, with pointwise suprema.

Now let $f^* : R \longrightarrow D[S,T]$ correspond to $f : R \times S \longrightarrow T$ by $f^*(x)(y) = f(x,y)$, for d-sets R, S, T and $(x,y) \in R \times S$. If $f \in \underline{D}$ and $\psi \subset S$ is updirected, then $\{x\} \times \psi$ is updirected in $R \times S$ for $x \in R$, with supremum $(x, \sup \psi)$, and $f^*(x)(\sup \psi) = f(x, \sup \psi) = \sup f(x,\psi) = \sup f^*(x)(\psi)$. Thus each $f^*(x)$ is in $\underline{D}$. A similar computation shows that f^* preserves suprema of updirected sets.

Conversely, let $f^* : R \longrightarrow D[S,T]$ in $\underline{D}$, and let Φ be updirected in $R \times S$, with projections $\varphi \subset R$ and $\psi \subset S$. Then $f : R \times S \longrightarrow T$ clearly preserves order, and $\sup f(\Phi) \leq f(\sup \varphi, \sup \psi) = f(\sup \Phi)$ follows. We have

$$f(\sup \Phi) = f(\sup \varphi, \sup \psi) = f^*(\sup \varphi)(\sup \psi)$$
$$= (\sup_{x\in\varphi} f^*(x))(\sup \psi) = \sup_{y\in\psi} \sup_{x\in\varphi} f^*(x)(y) .$$

If $x \in \varphi$ and $y \in \psi$, then $(x,y') \in \Phi$ and $(x',y) \in \Phi$, for suitable $x' \in R$ and $y' \in S$. These points have a common upper bound (x'',y'') in Φ, with $x \leq x''$ and $y \leq y''$, and with $f^*(x)(y) \leq f(x'',y'')$. Now $f(\sup \Phi) \leq \sup f(\Phi)$ follows; thus f is a morphism of $\underline{D}$ and 1.3 is proved.

2. d-spaces

2.1. We recall that every T_0 space X has an induced order, with the following statements equivalent for x, y in X. (a) $x \leq y$. (b) $x \in \overline{\{y\}}$, the closure of $\{y\}$. (c) Every neighborhood of x in X is also a neighborhood of y. This order is discrete iff X is a T_1 space.

From now on, every T_0 space will be provided with the induced order. We note that every open set is increasing, and that $\overline{\{x\}} = \downarrow x$ for $x \in X$.

2.2. We recall that a closed set F in a T_0 space X is called irreducible in X if F is not empty, and not the set union of two proper closed subsets. One sees easily that a closed set F in X is irreducible in X iff the open sets $V \subset X$ such that $V \cap F \neq \emptyset$ form a filter in the lattice of open sets of X. *)

If $x \in X$, then $\overline{\{x\}} = \downarrow x$ is irreducible. We say that x is a generic point of an irreducible closed set F if $F = \downarrow x$.

If $\varphi \subset X$ with irreducible closure $\overline{\varphi}$, then $x \in X$ is a generic point of $\overline{\varphi}$ iff $x \in V \iff \varphi \cap V \neq \emptyset$ for every open set $V \subset X$. It follows that $x = \sup \varphi$ in the induced order of X.

2.3. LEMMA. If $\varphi \subset X$ is updirected for the induced order of a T_0 space X, then $\overline{\varphi}$ is irreducible in X.

Proof. If V is open in X, then $V \cap \overline{\varphi} \neq \emptyset \iff V \cap \varphi \neq \emptyset$. If V meets φ in y, and an open set U meets φ in x, then x and y have a common upper bound z in φ, with $z \in U \cap V$. Thus the open sets meeting $\overline{\varphi}$ form a filter, and $\overline{\varphi}$ is irreducible.

2.4. DEFINITION. We say that a T_0 space X is a d-space if $\overline{\varphi}$ has a generic point for every $\varphi \subset X$ which is updirected in the induced order of X.

We recall that a T_0 space X is called sober (primal has also been used) if every irreducible closed set in X has a generic point. Thus every sober space is a d-space.

If X is a T_1 space, then an updirected subset is a singleton $\{x\}$, with generic point x. Thus every T_1-space is a d-space.

We denote by TOP_d the category of d-spaces and their continuous maps.

---

*) The filters thus obtained are the completely prime filters of open sets.

2.5. PROPOSITION. Induced orders define a functor $I : TOP_d \longrightarrow \underline{D}$ which preserves underlying sets and mappings.

Proof. If X is a d-space and $\varphi \subset X$ updirected, then φ has a supremum in the induced order, by Def. 2.4 and 2.2. Thus X with the induced order is a d-set which we denote by $I\,X$.

If $f : X \longrightarrow Y$ is a map of d-spaces and $\varphi \subset X$ is updirected with supremum x, then we show that $f(x)$ is a generic point of $\overline{f(\varphi)}$. It follows that $f(x) = \sup f(\varphi)$; thus $f : I\,X \longrightarrow I\,Y$ in $\underline{D}$. Indeed, if $V \subset X$ is open, then

$$f(x) \in V \iff x \in f^{-1}(V) \iff f^{-1}(V) \cap \varphi \neq \emptyset \iff V \cap f(\varphi) \neq \emptyset ;$$

this verifies our claim, by 2.2.

2.6. As noted in the introduction, d-spaces form a reflective full subcategory of T_0 spaces, with strict embeddings as reflections. We shall not prove this here; all we need is a much more modest result.

PROPOSITION. The product of two d-spaces is a d-space.

Proof. Let X and Y be d-spaces, and let $\Phi \subset X \times Y$ be updirected, with projections φ and ψ. Then φ and ψ have generic points $u = \sup \varphi$ and $v = \sup \psi$ in X and Y. Let W be a neighborhood of $(u,v) = \sup \Phi$, with $U \times V \subset W$ for open neighborhoods U of u and V of v. Then U meets φ, and V meets ψ, in points x and y. As in the proof of 1.3, there is (x'',y'') in Φ with $(x,y) \leq (x'',y'')$, and hence $(x'',y'') \in U \times V \subset W$. Thus every neighborhood of (u,v) meets Φ, and (u,v) is a generic point of $\overline{\Phi}$.

3. Scott topologies

3.1. We define the Scott topology of a d-set L in the usual way. $U \subset L$ is Scott-open iff U is increasing, and $\sup \varphi \in U \implies U \cap \varphi \neq \emptyset$ for every updirected subset φ of L. In particular, a discrete ordered set has a discrete Scott topology. If $2 = \{0,1\}$ with $0 \leq 1$, then 2 with the Scott topology is the Sierpiński space. If L is a d-set, then we denote by $S\,L$ the topological space obtained by providing L with the Scott topology.

3.2. PROPOSITION. If L is a d-set, then SL is a d-space, with induced order $ISL = L$.

Proof. Sets $L \setminus \downarrow x$ are Scott-open; it follows that SL is a T_0 space, with $x \not\leq y$ in the induced order if $x \not\leq y$ in L. On the other hand, $x \leq y$ for the induced order of SL if $x \leq y$ in L; thus $ISL = L$. By the definition of the Scott topology and 2.2, $\sup \varphi$ is a generic point of φ for $\varphi \subset L$ updirected; thus SL is a d-space.

Our next result is thoroughly predictable.

3.3. PROPOSITION. Scott topologies define a functor $S : D \longrightarrow TOP_d$ which preserves underlying sets and mappings.

Proof. We must only show that $f : SL \longrightarrow SM$ in TOP for $f : L \longrightarrow M$ in $\underline{D}$. If V is Scott-open in M, then $f^{-1}(V)$ obviously is increasing in L. If $\varphi \subset L$ is updirected and $\sup \varphi \in f^{-1}(V)$, then $f(\sup \varphi) = \sup f(\varphi)$ is in V. But then $f(\varphi) \cap V \neq \emptyset$, and this says $\varphi \cap f^{-1}(V) \neq \emptyset$. Thus $f^{-1}(V)$ is Scott-open in L, and $f : SL \longrightarrow SM$ is continuous.

3.4. THEOREM. The Scott topology functor $S : \underline{D} \longrightarrow TOP_d$ is a left adjoint right inverse of the functor $I : TOP_d \longrightarrow \underline{D}$.

Proof. If L is a d-set and X a d-space, then we show that $f : SL \longrightarrow X$ in TOP_d, for a mapping f of the underlying sets, iff $f : L \longrightarrow IX$ in $\underline{D}$. As both functors preserve underlying mappings, this provides a natural bijection.

If $f : L \longrightarrow IX$ and V is open in X, then V is increasing in IX, and thus $f^{-1}(V)$ increasing in L. If $\varphi \subset L$ is updirected and $f(\sup \varphi) = \sup f(\varphi)$ in V, then $f(\varphi) \cap V \neq \emptyset$ in the d-space X; thus $\varphi \cap f^{-1}(V) \neq \emptyset$. Now $f^{-1}(V)$ is Scott-open, and $f : SL \longrightarrow X$ follows.

If $f : SL \longrightarrow X$ and $x \leq y$ in L, then $x \in f^{-1}(V) \implies y \in f^{-1}(V)$ for V open in X, and $f(x) \leq f(y)$ in IX follows. If now $\varphi \subset L$ is updirected, and $f(\sup \varphi) \in V$ for V open in X, then $\sup \varphi \in f^{-1}(V)$, and $\varphi \cap f^{-1}(V) \neq \emptyset$ follows for the Scott-open set $f^{-1}(V)$. Now $f(\varphi) \cap V \neq \emptyset$. Thus $f(\sup \varphi)$ is the generic point $\sup f(\varphi)$ of $\overline{f(\varphi)}$, and $f : L \longrightarrow IX$ in $\underline{D}$.

The unit $L \longrightarrow ISL$ of the adjunction corresponds to $\mathrm{id}\, SL$ by the adjunction described above. By 3.2, this is the identity map $\mathrm{id}\, L$ in $\underline{D}$; thus S is a right inverse, as well as a left adjoint, of I.

3.5. REMARKS. Since the unit of the adjunction $S \dashv I$ is an isomorphism, the functor S is full and faithful. This is well known for the restrictions of S which have been studied by D. Scott and other authors.

The functor S embeds $\underline{D}$ into TOP_d as a full subcategory of d-spaces with Scott topologies. Since S has a right adjoint I, this is a coreflective subcategory of TOP_d, with coreflections $id_X : S\,I\,X \longrightarrow X$. We note in particular that the topology of a d-space X is coarser than the Scott topology of the d-set I X.

It is of course possible to define a "well below" relation for d-sets, and continuous d-sets. I have not studied these concepts.

4. An application

4.1. If Y is a topological space and O(Y) the complete lattice of open sets of Y, then the Scott topology of O(Y) is the Ω-topology of Day and Kelly [2]. As shown in [2], this is the finest topology of O(Y) with the property that for every topological space X and every open set $U \subset X \times Y$, putting $y \in f_U(x) \Longleftrightarrow (x,y) \in U$, for $(x,y) \in X \times Y$, defines a continuous map $f_U : X \longrightarrow O(Y)$. Conversely, every continuous map $f : X \longrightarrow O(Y)$ is of the form $f = f_U$, for an open set $U \subset X \times Y$, if the map id O(Y) is of this form, i.e. if the set

$$E_Y = \{(V,y) \in O(Y) \times Y \mid y \in V\}$$

is open in $S O(Y) \times Y$. In this situation, $f = f_U$ for $U = (f \times id_Y)^{-1}(E_Y)$. Spaces with this property have been called Ω-compact in [2], and quasilocally compact by A.S. Ward and other authors.

It is well known that a topological space Y is quasilocally compact if and only if O(Y) is a continuous lattice.

For a d-set L, the characteristic functions of the Scott-open subsets of L are the maps $f : S\,L \longrightarrow S\,2$, and hence the elements $f : L \longrightarrow 2$ of the d-set $D[L,2]$. This bijection between open sets and their characteristic functions clearly is an isomorphism of the complete lattices $D[L,2]$ and $O(S\,L)$.

We now obtain a slight generalization of Corollary 1.3 in [1].

<u>4.2</u>. PROPOSITION. <u>If</u> L <u>is a</u> d-<u>set for which</u> $O(S\,L)$ <u>is a continuous lattice</u>, <u>then</u> $S\,K \times S\,L = S\,(K \times L)$ <u>for every</u> d-<u>set</u> K .

<u>Proof</u>. We note that $S(K \times L) = SI(SK \times SL)$ is the coreflection of $SK \times SL$; thus $S(K \times L)$ has a finer topology than $SK \times SL$, and more open sets.

On the other hand, a Scott-open subset U of $K \times L$ is given by its characteristic function $h_U : K \times L \longrightarrow 2$ which is a morphism of d-sets, by the remark made above. By our Thm. 1.3, this corresponds to a morphism $g_U : K \longrightarrow D[L,2]$ in <u>D</u> , and composing this with the isomorphism between $D[L,2]$ and $O(S\,L)$, we obtain a morphism $f_U : K \longrightarrow O(S\,L)$ of d-sets. It is easily verified that f_U is given by $y \in f_U(x) \iff (x,y) \in U$, for $(x,y) \in K \times L$.

Now if $O(S\,L)$ is a continuous lattice, then the map $f_U : SK \longrightarrow SO(SL)$ corresponds to an open subset U of $SK \times SL$. Thus $SK \times SL$ and $S(K \times L)$ have the same open sets; this proves 4.2.