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SCS 34: On Complete Lattices L for which O(L) is Continuous - A Lattice Theoretical Characterization of CS.

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TOPIC On complete lattices L for which O(L) is continuous- A lattice theoretical characterisation of <u>CS</u>

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For a complete lattice L, O(L) will denote the Scott topology. The property " $O(L) \in$ <u>CL</u>" is an apparently important lattice theoretical property for which we don't have a name yet. By Theorem 1.6 below it means that L with its Scott topology is a locally quasi-compact (sober) space (in view of what was done in 2-8-77). Theorem 1.6 also says that O(L) will always satisfy condition ((O)) and thus have a <u>CL</u>-closed, hence compact T_2 spectrum Spec O(L) which, moreover, is order anti-isomorphic to L itself. This will be utilized in order to show that for meet continuous complete lattices L we have $L \in$ <u>CS</u> iff $O(L) \in$ <u>CL</u>, and $L \in$ <u>CL</u> iff both $O(L) \in$ <u>CL</u> and ~~&&~~ O(L) has enough coprimes.

1. Some facts in general topology

1.1 PROPOSITION. Let X and Y be (T_o) spaces and T a topology on $X \times Y$ such that

$$pr_1: X \times_T Y \longrightarrow X \quad \text{and} \quad s_{1x'}: Y \longrightarrow X \times_T Y,\ s_{1x}(y) = (x',y),$$

$$pr_2: X \times_T Y \longrightarrow Y \quad \text{and} \quad s_{2y'}: X \longrightarrow X \times_T Y,\ s_{2y}(x) = (x,y')$$

are continuous for all $(x',y') \in X \times Y$.

If $O(Y)$ is a continuous lattice (i.e. Y is a CL-space [quasi locally compact]) then T is the product topology.

Proof. i) For each $f \in Top(X, O(Y))$, where $O(Y)$ carries the Scott topology, we define a function $a(f): X \times Y \longrightarrow 2$ by

$$a(f) = \begin{cases} 1 & \text{for } y \in f(x) \\ 0 & \text{for } y \notin f(x). \end{cases}$$

We claim that $a(f)$ is continuous relative to the product topology. Suppose now that $a(f)(x,y) \in 1$. Since $O(Y) \in \underline{CL}$, there is a $V \in O(Y)$ with $y \in V << f(x)$. Since now $f(x) \in \Uparrow V$, and since f is continuous and $\Uparrow V$ open in $O(Y)$, there is an open neighborhood U of x such that $f(U) \subseteq \Uparrow V$. If we now take $(u,v) \in U \times V$, then $v \in V \subseteq f(u)$ whence $a(f)(u,v) = 1$. Thus

(1) $a(f): X \times Y \longrightarrow 2$ is continuous when $X \times Y$ has the product topology, i.e. $a(f) \in Top(X \times Y, 2)$.

Thus $a: Top(X, O(Y)) \longrightarrow Top(X \times Y, 2)$ is a well-defined function.

ii) Let us take $F \in Top(X \times_T Y, 2)$; define $b(F): X \to 2^Y$ by $b(F)(x)(y) = F(x,y)$. Now $F(x,y) = (F \circ s_{1x})(y)$. Since s_{1x} is continuous, $b(F)(x) \in Top(Y,2)$. Since the function $x \longmapsto b(F)(x)(y)$ equals $F \circ s_{2y}$ and s_{2y} is continuous, $b(F): X \longrightarrow Top(Y,2)$ is continuous if we consider on $Top(Y,2)$ the topology of pointwise convergence (2 having the Scott topology). Thus b (2) $b(F) \in \mathrm{Top}(X,\mathrm{Top}(Y,2))$ and

$b\colon \mathrm{Top}(X \times_T Y,2) \longrightarrow \mathrm{Top}(X,\mathrm{Top}(Y,2))$ is a well-defined function.

iii) Since pr_1 and pr_2 are continuous, the identity $X \times_T Y \to X \times Y$ is continuous. Thus $\mathrm{Top}(X \times Y, 2)$ is a subset of $\mathrm{Top}(X \times_T T,2)$. The function $f \longmapsto f^{-1}(1)\colon \mathrm{Top}(Y,2) \longrightarrow O(Y)$ is a homeomorphism (relative to the Scott topologies), inducing and isomorphism $\mathrm{Top}(X, \mathrm{Top}(Y,2)) \longrightarrow \mathrm{Top}(X,O(Y))$. One verifies straightforwardly that the following diagram commutes

$$\begin{tikzcd}
\mathrm{Top}(X \times Y,2) \arrow[rr, "\subseteq"] \arrow[dd, "\mathrm{id}"'] & & \mathrm{Top}(X \times_T Y,2) \arrow[d, "b"] \\
& & \mathrm{Top}(X,\mathrm{Top}(Y,2)) \arrow[d, "\cong"] \arrow[dll, "a'"'] \\
\mathrm{Top}(X \times Y,2) & & \mathrm{Top}(X,O(Y)) \arrow[ll, "a"]
\end{tikzcd}$$

$$a'(f)(x,y) = f(x)(y).$$

This shows that $\mathrm{Top}(X \times Y,2) = \mathrm{Top}(X \times_T Y,2)$, i.e. $O(X \times Y) = O(X \times_T Y) = T$. $\square$

1.2.COROLLARY. The product of two CL -spaces is a CL-space.

Proof. We proved in 1.1 that $O(X \times Y) \cong \mathrm{Top}(X,O(Y))$ if $O(Y) \in \underline{CL}$. By Isbell's Theorem on function spaces ,$O(X) \in \underline{CL}$ implies that $\mathrm{Top}(X,O(Y)) \in \underline{CL}$ if $O(L) \in \underline{CL}$. $\square$

1.3 COROLLARY. If K, L are complete lattices and $O(L) \in \underline{CL}$ (i.e. L is a $\underline{CL}$-space in the Scott topology) then the Scott topology on $L \times K$ is the product of the Scott topologies.

Proof. Apply the proposition 1.1 with ~~the~~ T being the Scott topology on $K \times L$; the hypotheses of 1.1 are fulfilled. $\square$

1.4. COROLLARY. If L is a complete lattice with $O(L) \in \underline{CL}$, then $\vee : L \times L \longrightarrow L$ is jointly continuous.

Proof. The binary sup operation clearly preserves arbitrary sups, hence is Scott continuous. The assertion then follows from 1.3. $\square$

1.5. ~~COROLLARY~~ LEMMA (MISLOVE). Let L be a complete lattice such that $\vee : L \times L \longrightarrow L$ is continuous. Then

(1) For two quasicompact Scott saturated sets Q_1 and Q_2 the intersection $Q_1 \cap Q_2$ is quasicompact .

(2) $U \in O(L)$ is prime iff $U = L$ or $U = L \setminus \downarrow x$ for $x = \max (L \setminus U)$.

Proof. (1) Saturation relative to the Scott topology means being upwards closed. Then $Q_1 \cap Q_2 = Q_1 \vee Q_2$; thus the assertion follows fro, the continuity of $\vee$.

(2) Clearly all $L \setminus \downarrow x$ are prime. Now suppose that $U \neq L$ is prime. We must show that $x = \max (L \setminus U)$ exists. Since $L \setminus U$ is Scott closed this is the case if $L \quad U$ is a lattice ideal, i.e. is up-directed. If that were not the case, then we would find $a, b \notin U$ with $a \vee b \in U$. By the continuity of $\vee$ we then had open neighborhoods A and B of a, b respectively such that $A \cap B = A \vee B \subseteq U$, and since $a \in A, b \in B$ and thus $A, B \not\subseteq U$, this would contradict~~s~~ the primeness of U $\square$

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We have proved the following Theorem

1.6. THEOREM. Let L be a complete lattice such that $O(L)$ is a continuous lattice. Then Spec $O(L)$ is closed in $O(L)$ is the CL -topology (hence is compact Hausdorff in this topology) and the function $x \mapsto L \setminus \downarrow x : L \longrightarrow \text{Spec } O(L)$ is an order-anti-isomorphism.

Proof. The first assertion follows from Misloves Lemma 1.5,part 1, (which applies because of 1.4) and from Theorem 1.25 of Hofmann and Lawson SCS 2-8-77.

The fact that the function $L \longrightarrow \text{Spec } O(L)$ given by $x \mapsto L \setminus \downarrow x$ is well defined and bijective again follows from Mislove's Lemma (part (2)) in view of 1.4.

Remark. Under the hypotheses of 1.6 we have induced a compact Hausdorff topology on L which has a closed graph.

Warning: One should not mix up the map in Theorem 1.6 with the lattice isomorphism $x \mapsto L \setminus \uparrow x : L \longrightarrow O(\text{Spec } L)$ introduced and discussed for continuous L in SCS-2-8-77 ,see loc.cit 1.4.

Proposition 1.1 appears to be similar if not equivalent to theorem 2.10 in Isbell's "Meet continuous lattices" and some of the developments in his "Function spaces and adjoints".

We would like to see examples satisfying the hypotheses of 1.6 such that Spec O(L) is not sup-closed in O(L).

The following proposition gives additional information on the links between L and O(L).

1.7. PROPOSITION. Let L be a complete lattice. Then the following statements are equivalent:

(1) L is meet continuous.

(2) O(L) is join continuous.

(3) O(L) is join Brouwerien.

(4) The lattice $C(L)$ of Scott closed sets is meet continuous.

Proof. Since O(L) is distributive and complete (2)<=>(3), and (2)<=>(4) is clear. (4)=>(1): The function $x \longmapsto \downarrow x : L \longrightarrow C(L)$ is an embedding preserving infs and up directed sups. (1)=>(4): We observe that for any pair of lower sets A and B with $\bar{A} \subseteq \bar{B}$ we have $\bar{A} = \bar{A}\bar{B} \subseteq \overline{AB}$ (since inf is a Scott continuous operation in meet continuous lattices) $= (A \cap B)^-$. If we now have any family B_j of closed lower sets, then $B = \bigcup_j B_j$ is a lower set, and if $A = \bar{A}$ is a closed lower set with $A \subseteq \bar{B}$, then we have from the preceding remark $A \subseteq (A \cap B)^- =$ $(A \cap \bigcup_j B_j)^- = (\bigcup_j (A \cap B_j))^- = \sup_j (A \cap B_j)$, which shows in fact that C(L) is (meet) Brouwerien.$\square$

2. More on the spectrum of distributive continuous lattices.

2.1. PROPOSITION. Let L be a complete lattice. Then the following statements are equivalent:

(1) Spec L is closed under arbitrary sups and down directed infs.

(2) Spec L is closed under arbitrary sups.

(3) The inclusion map $i\colon \mathrm{Spec}\ L \longrightarrow L$ has a left adjoint $\pi\colon L \longrightarrow \mathrm{Spec}\ L$, $\pi(x) = \sup_{\mathrm{Spec}\ L} (\downarrow x \cap \mathrm{Spec}\ L)$.

(4) For each $x \in L$ there is a unique largest prime $p \neq 1$ such that $p \leq x$.

Remark. Note that (2) implies that $0 = \sup \emptyset \in \mathrm{Spec}\ L$. Note further ~~that~~ in case every element is inf of primes that (4) implies that $L \setminus \{1\}$ has a maximum, i.e. that 1 is "attached".

Proof. Since down directed infs of primes are primes, (1) <=> (2). (2) <=> (3) is a consequence of the theory of Galois connections, see e.g. ATLAS 1.7-1.8.

Clearly (3)=>(4) with $p = \pi(x)$. Conversely, (4) shows that $\max_{\mathrm{Spec}\ L} (\downarrow x \cap \mathrm{Spec}\ L)$ exists yielding the desired left adjoint for the inclusion map. ▯

One could call the function π the <u>prime picker</u>. When followed by the inclusion, the prime picker is a kernel operator on L with image Spec L.

2.2. LEMMA. (LAWSON). Let $L \in$ <u>CL</u>. ~~The following~~ If L is join Brouwerien, then L is a topological lattice (relative to the Lawson topology).

Proof. Suppose $x = \lim x_j$ and $y = \lim y_k$. Then

$$\sup_{j',k'} \inf_{(j,k)\geq(j',k')} x_j \vee y_k = \sup_{j',k'} (\inf_{j\geq j'} x_j) \vee (\inf_{k\geq k'} y_k)$$

[since L is join Brouwerien] $= (\sup_{j'} \inf_{j \geq j'} x_j) \vee (\sup_{k'} \inf_{k \geq k'} y_k)$

$= (\lim x_j) \vee (\lim y_k)$ [since $L \in \underline{CL}$] $= x \vee y$.

Since this argument applies to every subnet of x_j ,resp. y_k, we have shown $x \vee y = \lim x_j \vee y_k$ since we operate in a $\underline{CL}$-object.

2.3. LEMMA (LAWSON). Let L be a compact semilattice. For a subset X let $I(X) = \{y \mid y = \inf X' \text{ for some } X' \subseteq X\}$ and $D(X) = \{y \mid y = \sup X' \text{ for some up-directed } X' \subseteq X\}$. Then DIDI(X) is the smallest closed subsemilattice containing X.

For a proof one has to sharpen the argument given by Lawson in "Intrinsic Topologies... " for the plus-plus business in Theorem 13 and Corollary 14.

2.4. PROPOSITION. Let L be a continuous join Brouwerien lattice satisfying the equivalent conditions of 2.1. Then in the induced Lawson topology, Spec L is a compact topological sup-semilattice.

Proof. By 2.1 , Spec L is closed under arbitrary sups and down-directed infs. By 2.2, L is a compact topological sup-semilattice and so by 2.3 , Spec L is a closed subsemilattice of the sup-semilattice L. □

2.5. NOTATION. Under the hypotheses of 2.4. we denote the compact topological sup-semilattice on Spec L with the compact Hausdorff topology induced by the Lawson topology Spec^{op} L.

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2.6. REMARK. Let L, L' be lattices $F: L \longrightarrow L'$ left adjoint to $R: L \longrightarrow L'$. Consider

(1) F preserves primes.

(2) R is a lattice morphism.

Then (2) => (1), and if every element is an inf of primes in L, then both conditions are equivalent.

Proof. Let $a, b \in L'$ and $p \in$ PRIME L. Then $F(p) \geq ab$ is equivalent to $p \geq R(ab)$.

If (2) then $p \geq R(ab) = R(a)R(b)$ implies $p \geq R(a)$ or $p \geq R(b)$, and thus $F(p) \geq a$ or $F(p) \geq b$. Thus $F(b)$ is prime, i.e. (1) holds. If (1), then $F(p)$ is a prime and $F(p) \geq ab$ implies $F(p) \geq a$ or $F(p) \geq b$ i.e. $p \in \uparrow R(a) \cup \uparrow R(b)$, and since p is prime this is equivalent to $p \in \uparrow R(a)R(b)$. Thus $p \in \uparrow R(ab)$ and $p \in \uparrow R(a)R(b)$ are equivalent properties, and if every element in L is the inf of primes, $R(ab) = R(a)R(b)$ follows.□

2.7. DEFINITION. Let $\underline{H}$ be the category whose objects are complete lattices L satisfying the following conditions:

(i) $L \in \underline{CL}$. (ii) L is join Brouwerien (i.e. distributive and join continuous).

(iii) L satisfies the equivalent conditions of 2.1.

The morphisms of $\underline{H}$ are functions $f: L \longrightarrow L'$ satisfying the following conditions:

(1) $f \in \underline{CL}$. (2) f is a lattice morphism.

(3) f preserves primes.

By Lemma 2.6, condition (3) is equivalent to the following

(3') The right adjoint $r: L' \longrightarrow L$ is a lattice morphism, and by ATLAS (1) can be rephrased as follows:

(1') f has a right adjoint r which respects the $\ll$ relation.

2.8. PROPOSITION. There is a well defined functor $\mathrm{Spec}^{op}: \underline{H} \longrightarrow \underline{CS}$ which associates with an $\underline{H}$-morphism $f: L \longrightarrow L'$ the restriction and corestriction $f|\mathrm{Spec}\ L: \mathrm{Spec}^{op}\ L \longrightarrow \mathrm{Spec}^{op}\ L'$.

Proof: Clear.

2.9. ~~PROPOSITION~~ NOTATION. If L is a complete lattice then $O(L)$ and $\hat{O}(L)$ will both denote the lattice of Scott opne sets, and if $f: L \longrightarrow L'$ is a Scott continuous function, then $O(f): O(L') \to O(L)$ is given by $O(f)(U) = f^{-1}(U)$, and $\hat{O}(f): \hat{O}(L) \longrightarrow \hat{O}(L')$ is its left adjoint.$\square$

2.10 LEMMA. Let ~~L~~ $S \in \underline{CS}$. Then $O(S) \in \underline{H}$, and $U \ll V$ in $O(S)$ iff $\bar{U} \subseteq V$.

Proof. If $S \in \underline{CS}$, then S is meet continuous, and so $O(S)$ is join Brouwerien by 1.7. Since S is compact Hausdorff, the lattice of all open sets of S is continuous, and $O(S)$ is a complete sublattice thereof, hence is continuous, and $U \ll V$ is tantamount to $\bar{U} \subseteq V$ in $O(S)$ since this equivalence holds in the lattice of all open sets of S. Now Theorem 1.6 applies and shows that condition 2.1.1 is satisfied by $O(L)$.

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2.11. LEMMA. Let $f: S \longrightarrow S'$ be in <u>CS</u>. Then $O(f)$ is a lattice morphism respecting the relation $\ll$, and its ~~x~~ left adjoint $\hat{O}(f)$ is given by $\hat{O}(f)(U) = S' \setminus \downarrow f(S \setminus U)$ and preserves finite sups.

Proof. Clearly $O(f)$ is always a lattice morphism preserving arbitrary sups. If $U \ll V$ in $O(S')$, then $\bar{U} \subseteq V$ by 2.10 and so $O(f)(U)^- = f^{-1}(U)^- \subseteq f^{-1}(\bar{U}) \subseteq f^{-1}(V) = O(f)(V)$, thus $O(f)$ respects the $\ll$ relation. In order to identify the left adjoint of $O(f)$ we take an arbitrary $V \in O(S')$ and $U \in O(S)$ and note that $O(f)(V) = f^{-1}(V) \subseteq U$ iff $f^{-1}(V) \cap (S \setminus U) = \emptyset$ iff $V \cap f(S \setminus U) = \emptyset$ iff $V \cap \downarrow f(S \setminus U) = \emptyset$ (since V is an upper set) iff $V \subseteq S' \setminus \downarrow f(S \setminus U)$, and ~~Thus~~ indeed $\hat{O}(f)(U) = S' \setminus \downarrow f(S \setminus U)$. If A,B are closed semilattice ideals, then $\downarrow f(AB) = SF(AB) = SF(A)f(B) = Sf(A)Sf(B) = \downarrow f(A) \downarrow f(B)$, and thus $\hat{O}(f)$ will preserve finite sups.[]

Now 2.10 and 2.11 yield

2.12. PROPOSITION. There is a well-defined functor $\hat{O}: \underline{CS} \longrightarrow \underline{H}$ with $\hat{O}(f)(U) = S' \setminus \downarrow f(S \setminus U)$ for $f: S \longrightarrow S'$ in <u>CS</u>. []

2.13. LEMMA. If $S \in \underline{CS}$ and $L \in \underline{H}$, then we have two isomorphisms $x \longmapsto S \setminus \downarrow x: S \longrightarrow \mathrm{Spec}^{op}\, O(S)$ in <u>CS</u>

and $x \longmapsto \mathrm{Spec}^{op} L \setminus \uparrow x : L \longrightarrow O(\mathrm{Spec}^{op} L)$. in <u>H</u>.

Proof. The first assertion follows from Theorem 1.6, and the second from Hofmann-Lawson SCS 2-8-77, 14 and the following Lemma

2.14. LEMMA. For $L \in \underline{H}$, the hull kernel topology on Spec L is the Scott topology of $\mathrm{Spec}^{op} L$.

Proof. The hull kernel open sets $(\mathrm{Spec}\ L) \setminus \uparrow x$ are clearly Scott open. Now let U be Scott open in $\mathrm{Spec}^{op}\ L$. Then U is an open upper set in $\mathrm{Spec}^{op}\ L$ and $A = (\mathrm{Spec}\ L) \setminus U$ is a closed lower set in $\mathrm{Spec}^{op}\ L$. Thus A is compact in the CL-topology of L. Let $x = \inf A$. We claim that $A = \uparrow x \cap \mathrm{Spec}\ L$: Let $p \in \uparrow x \cap \mathrm{Spec}\ L$; then THE LEMMA implies $p \in \uparrow A \cap \mathrm{Spec}\ L = A$. So $\uparrow x \cap \mathrm{Spec}\, A \subseteq A$. The other inclusion is trivial. Now $U = (\mathrm{Spec}\ L) \setminus \uparrow x$ and thus U is a hull-kernel open set. □

We are now ready for the principal theorem.

2.15 . MAIN THEOREM. The categories CS and H are equivalent under the pair of inverse functors

$$\mathrm{Spec}^{op}: \underline{H} \longrightarrow \underline{CS} \quad \text{and} \quad O: \underline{CS} \longrightarrow \underline{H}.$$

The proof follows from the previous discussion. It certainly serves a useful purpose to isolate what this means for the objects:

2.16. THEOREM. Let L be a complete lattice and $\mathcal{L} = O(L)$. Then

(I) L is meet continuous iff $\mathcal{L}$ is join Brouwerien, and

(II) L carries a (unique) compact Hausdorff topology making it into a compact topological semilattice iff $\mathcal{L}$ is join Brouwerien and continuous.

3 . Characterisation of continuous lattices through O(L).

3.1. PROPOSITION. Let X be a topological space and write $x \leq y$ in X iff $x \in U$ implies $y \in Y$ for all $U \in O(X)$ (i.e. if $x \in \{y\}^-$). Then $U \in O(X)$ is coprime (i.e. join irreducible) iff U is down directed relative to $\leq$.

Proof. If U is not down directed, then for some $u, v \in U$ one has $\downarrow u \cap \downarrow v \cap U = \emptyset$ i.e. $(U \setminus \downarrow u) \cup (U \setminus \downarrow v) = U$; thus U is not join irreducible. If U is not join irreducible, then there are two proper open subsets V and W of U with $U = V \cup W$. We pick $v \in V \setminus W$ and $w \in W \setminus V$ and notice $\downarrow v \cap W = \{v\}^- \cap W = \emptyset$ and $\downarrow w \cap V = \{w\}^- \cap V = \emptyset$ so $(U \setminus \downarrow v) \cup (U \setminus \downarrow w) = U$,i.e. $\downarrow v \cap \downarrow w \cap U = \emptyset$ and U is not down directed.

We say that a lattice has enough coprimes iff every element is a sup of coprimes.

3.2.COROLLARY. Let L be a complete lattice and O(L) the Scott topology. Then O(L) has enough coprimes iff the Scott topology has a basis of open filters.

3.3. LEMMA (MISLOVE). Let L be a complete lattice. Then we have (1)=>(2)=>(3) , where

(1) O(L) has enough coprimes.

(2) $\bigcap \{U: x \in U \in O(L)\} = \uparrow x$.

(3) L is meet continuous.

Remark.
We will see that (3) does not imply (1). We do not know whether (1) and (2) are equivalent.

Proof. (1) means that O(L) has a basis of open filters by 3.2. Since $L \setminus \downarrow x \in O(L)$ for all $x \in L$ we have (1)=>(2). We now assume that (3) fails assume ~~and show not~~ (2) and derive a contradiction: If not (3) then there must be an up-directe set D and an element x with $x \leq \sup D$ bu $x \not\leq \sup xD$. Thus ~~xx∈xLxxx↓xxx~~ $x \in L \setminus \downarrow \sup xD$. By (2) there is an open filter U with $x \in U$ and $\sup xD \notin U$, whence $xd \notin U$ for ~~xxxx~~ all $d \in D$. On the other hand $\sup D \in U$ since $x \in U$ and $s \leq \sup D$. Since U is Scott open, there is a $d \in D$ with $d \in U$.As U is a filter, $xd \in U$ and this is the desired contradiction.[]

3.~~3~~4. THEOREM. For any complete lattice the following conditions are equivalent:

(I) L is continuous.

(II) (i) O(L) has enough coprimes and (ii) O(L) is continuous.

(III) (L,O(L)) is a locally quasicompact space with a basis of open filters.

Proof. (I) =>(II) : If L is continuous, then every Scott open set is an opne upper sets in the Lawson topology, hence ~~x~~ is a union of open ~~fixxxxx~~ subsemilattices and thus of open filters. Further $O(L) \cong$ Top(L,2) with the Scott topology on 2,but Top(L,2) is a continuous lattice according to Scott.

(II) =>(III): Since O(L) is continuous, (L,O(L)) is a primal (sober) space by Theorem 1.3, since $\downarrow x = \{x\}^-$ relative to the Scott topology. Then (ii) implies that (L,O(L)) is locally quasicompact by Hofmann and Lawson SCS 2-8-77 , 1.16. The rest follows from 3.2.

(III)=>(I): Let $x \in L$ and suppose that $y \not\geq x$. Find an open filter U with $x \in U \not\ni y$. Since (L,O(L)) is locally quasicompact, there is a quasicompact set Q and an open neighborhood of x (which we may and will assume to be a filter) such that $V \subseteq Q \subseteq U$. The identity function of V is a down directed net with a cluster point $q \in Q$ in the quasicompact space Q. We claim $V \subseteq \uparrow q$; for if not, then there would be an open filter W and a $v \in V$ with $q \in W \not\ni v$; since q is a cluster point of V, then V is cofinally in W which cannot be the case if $v \notin W$ for some $v \in V$. Thus $q \leq \inf V$, whence $\inf V \in \uparrow q \subseteq \uparrow Q \subseteq U$. We now have shown that $y \notin \downarrow \sup\{\inf V: V$ an open filter with $x \in V\}$ and this is certainly $\leq \sup \Downarrow x$. Thus $x = \sup \Downarrow x$.□

3.5. COROLLARY. For any ~~compa~~complete lattice L, the following conditions are equivalent:

(I) L is algebraic

(II) (i) $O(L)$ has enough coprimes

(ii) $O(L)$ is algebraic.

Proof. We observe the following easily proved facts:

Fact 1: If a union $U_1 \cup \ldots \cup U_n$ of open filters is quasicompact and U_1 is not contained in the union $U_2 \cup \ldots \cup U_n$, then $U_1 = \uparrow k_1$ for some $k_1 \in K(L)$.

Proof. $U = U_1 \setminus (U_2 \cup \ldots \cup U_n)$ is down directed and quasicompact. If U has no minimal element, then it is covered by the sets $L \setminus \Downarrow u$, $u \in U$. Hence $U \subseteq L \setminus \Downarrow u$ for some $u \in U$, which is patently false. Thus $U \subseteq \uparrow k_1$ with $k_1 = \min U$. Then $U_1 = \uparrow k_1$ since U_1 is a filter. Clearly $k_1 \in K(L)$.

Fact 2. If $k_1, \ldots, k_n \in K(L)$ and $U \in O(L)$ is maximal w.r.t. not containing $\uparrow k_1 \cup \ldots \cup \uparrow k_n$, then $U = L \setminus \uparrow k_m$ for some $m \in \{1, \ldots, n\}$.

The proof of (I)=>(II) is clear. Conversely, if (II) holds, then by (i) and Fact 1 , every quasicompact open set is of the form $\uparrow k_1 \cup \ldots \cup \uparrow k_n$ with $k_m \in K(L)$. Then by Fact 2 , the complete irreducibles of $O(L)$ are precisely the $L \setminus \downarrow k$ with $k \in K(L)$. Since Irr $O(L)$ is order generating, every $L \setminus \downarrow x$ is an inf of sets $L \setminus \downarrow k$, whence $x = \sup(\Downarrow x \setminus K(L))$.□

Let us summarize the results of 1.6, 2.16 and 3.4 in the following statement:

3.6. THEOREM. Let L be a complete lattice such that O(L) is a continuous lattice. Then

(A) Spec O(L) is closed and ~~anti~~ order anti-isomorphic to L.

(B) $L \in \underline{CS}$ iff O(L) is join continuous.

(C) $L \in \underline{CL}$ iff O(L) has enough coprimes.

Note that this shows that meet continuity of L is weaker than the existence of enough coprimes in O(L) (see 3.3.).