Seminar on Continuity in Semilattices

Volume 1 | Issue 1 Article 30

12-29-1976

SCS 29: On the Closedness of the Set of Primes in Continuous Lattices

Karl Heinrich Hofmann
Technische Universitat Darmstadt, Germany, hofmann@mathematik.tu-darmstadt.de

Oswald Wyler
Carnegie Mellon University, Pittsburgh, PA USA

SEMINAR ON CONTINUITY IN SEMILATTICES (SCS)

NAME(S) K.H.Hofmann and O.Wyler	DATE M	D	Y
	12	28	76

TOPIC On the closedness of the set of primes in continuous lattices

REFERENCE

[1] K.Keimel and M.Mislove SCS-Memo 9-30-76, notably part 2

[2] K.H.Hofmann ,SCS-Memo 11-23-76

[3] O.Wyler, Algebraic theories of continuous lattices, preprint (old title Compact complete lattices).

[4] K.H.Hofmann and J.D. Lawson, Irreducibility...,preprint.

The following question is fairly pressing:

(P) Let S be a <u>CL</u>-object. When is $\overline{\text{PRIME } S}$ = PRIME S?

In [1] ,Keimel and Mislove have given a conclusive answer for distributive S, and in [4] the question of distributivity in <u>CL</u> was discussed at some length (Chapter 3). The condition Keimel and Mislove found to be necessary and sufficient in the distributive case was the following:

((0)) For all $a,x,y \in S$, the relations $a \ll x$ and $a \ll y$ imply $a \ll xy$.

This condition is evidently equivalent to any of the following

((0')) For any $s \in S$ the set int $\uparrow s$ is a filter. [2]

((0")) For all $a,b,x,y \in S$, the relations $a \ll x$ and $b \ll y$ imply $ab \ll xy$.

((0"')) Graph $\ll$ is a subsemilattice of $S \times S$.

Keimel and Mislove give an example of a sublattice of the square violating this condition.

O.Wyler proves the following fact in [4] 12.5, old version:

If L is a lattice with 0 and 1 , then PRIME $\hat{L}$ is closed , where $\hat{L}$ is the <u>Z</u> -dual of L (HMS -DUALITY).

The purpose of the Memo is to observe that this fact together with the observations by Keimel and Mislove in [1] suffice to show that ((0)) is sufficient for the closedness of PRIME S regardless of distributivity. In order to make this a bit more selfcontained, we present a proof of Wyler's proposition.

Recall that the CL-topology of a CL-object S has a basis of the open sets

$$U(u;v_1,\ldots,v_n) = \operatorname{int}\uparrow u \setminus (\uparrow v_1 \cup \ldots \cup \uparrow v_n),$$

$$\operatorname{int}\uparrow u = \{x \in S:\ u \ll x\}.$$

LEMMA 1 . If $v_1 \ldots v_n \leq u$, then $U(u;v_1,\ldots,v_n) \cap \text{PRIME } S = \emptyset$.
Clear from the definitions.□

PROPOSITION 2. In a CL-object S, condition ((0)) implies $\overline{\text{PRIME}}\ S = \text{PRIME } S$.

Proof. Let $s \notin$ PRIME S. Then there exist elements $x,y \notin \downarrow s$ with $xy \leq s$. Since S is a continuous lattice, we find elements $a \ll x$ and $b \ll x$ with $a,b \notin \downarrow s$. By condition ((0")) we know $ab \ll xy$. Thus $U(ab;\ a,b)$ is an open neighborhood of s which according to Lemma 1 does not contain primes.□

RECALL 3 . An algebraic lattice $S \in Z$ is arithemetic if K(S) is a sublattice (i.e. is closed under finite infs).

REMARK 4. Any arithmetic lattice $S \in Z \subseteq CL$ satisfies condition ((0)).
Proof. Let $a \ll x,y$, i.e. $x,y \in \operatorname{int}\uparrow a$. Since $s = \sup(\downarrow s \cap K(S))$ for all $s \in S$, there are compact elements $c,k \in K(S)$ such that $a \ll c,k$ and $c \leq x$ and $k \leq y$. Since S is arithmetic, $ck \in K(S)$. Thus $a \leq ck \ll ck$,whence $a \ll ck$.□

Steht auch in [9]

COROLLARY 5.(Wyler) The set of primes is closed in any arithmetic lattice.
This follows now from Proposition 2 and Remark 4.

COROLLARY 6. If $S \in CL$ and PS is as in ATLAS (the Z-object of all lattice ideals), then PRIME PS is closed.□

For distributive S this occurs in the proof of 2.1 in Keimel-Mislove [1]. From ATLAS we recall the morphism $r_S: PS \longrightarrow\!\!> S$, $r_L(J) = \sup J$. By the preceding Corollary, r(PRIME PS) is a closed subspace of S which contains PRIME S (since $p \in$ PRIME S implies $\downarrow p \in$ PRIME PS). Keimel and

Mislove demonstrate that condition ((0)) implies that $J \in$ PRIME PS always gives sup $J \in$ PRIME S. We have

((n)) => ((n+1)):

((0))

((1)) If J is a prime ideal of S , then sup J is a prime.

((2)) $\overline{\text{PRIME}}$ S = PRIME S.

If S is distributive, then these conditions are equivalent. Example 4.2 in ATLAS satisfies ((2)) , but not ((0)).

<u>Question</u>. Does ((2)) imply ((1))?

If S satisfies ((2)) ,then the distributive <u>CL</u>-subobject $S' = \{x \in S: \ x = \inf(\uparrow x \cap \text{PRIME } S)\}$ appears to play ~~an~~ role. It is not clear which. The closure operator $f: S \longrightarrow S'$, $f(s) = \inf(\uparrow s \cap \text{PRIME } S)$ is a lattice homomorphism. So what?