SCS 28: The Lattice of Open Subsets of a Topological Space

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The lattice of open subsets of a topological space


REFERENCE

If $X$ is a topological space, then the space of open subsets of $X$, $O(X)$, is a complete lattice. This memo is intended to give some results about when $O(X)$ is a continuous lattice or a compact semilattice. These results are not all new, and they are not exhaustive; however, we hope they will shed some light on the problem, and eventually lead to a solution of it.

If we denote by $2^X$ the complete algebraic lattice of all subsets of $X$, then there is a natural kernel operator $k : 2^X \rightarrow 2^X$ with image $O(X)$, namely, $k(A) = \text{int } A$, the interior of the set $A$. The following lemma shows that the now well-known lemma in reference 1 is of virtually no use in determining when $O(X)$ is a continuous lattice.

Lemma 1. Let $X$ be a $T_1$ space, and define $k : 2^X \rightarrow 2^X$ by $k(A) = \text{int } A$. If $k$ preserves sups of up-directed sets, then $X$ is discrete.

Proof. $X = \sup \{ F : F \subseteq X \text{ is finite} \}$, and this is an up-directed sup. Hence, if $k$ preserves up-directed sups, we have $X = \sup \{ k(F) : F \subseteq X \text{ is finite} \}$. Thus, if $x \in X$, then there is some $F \subseteq X \text{ finite with } x \in k(F)$, and $k(F)$ is a finite open set. Since $X$ is $T_1$, points are closed, and so it follows that each point of $k(F)$ is open. Therefore $(x)$ is open, and so $X$ is discrete.

As a result of this lemma, we see that whether or not $O(X)$ is a continuous lattice must be determined independently of the lattice $2^X$; thus the way-below relation on $O(X)$ must be determined.

Definition 2. Let $L$ be a complete lattice. For $x, y \in L$, we write $x \ll y$ if and only if for each up-directed set $A \subseteq L$ with $y \leq \sup A$, there is some $a \in A$ with $x \leq a$.

We write $x \ll y$ if and only if, for each up-directed subset $A$ of $L$ with $y \leq \sup A$, there is some $a \in A$ with $x \ll a$.

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**Definition 3.** Let $X$ be a topological space, $U \subseteq V$ open subsets of $X$. We say $U$ is relatively compact in $V$ if each open cover of $V$ admits a finite subcover of $U$. Clearly $U$ is relatively compact in $V$ if and only if $U \ll V$ in $O(X)$.

**Proposition 4.** Let $X$ be a Hausdorff space, and let $A$, $B$ be open subsets of $X$. The following are equivalent:

1. $A \ll B$ in $O(X)$.
2. $\overline{A} \subseteq B$ and $\overline{A}$ is compact.

**Proof.** Suppose that $A \ll B$ in $O(X)$, and let $x \in X \setminus B$. Then $X$ Hausdorff implies that the family of closed neighborhoods of $x$ is downwards directed and has intersection $\{x\}$, and so the family $\{B \setminus N : N$ is a closed neighborhood of $x\}$ is an up-directed family in $O(X)$ whose sup is $B$. $A \ll B$ then implies that there is some closed neighborhood $N$ with $A \subseteq B \setminus N$. It follows that $\overline{A} \subseteq \overline{B}$. Since $A \ll B$ implies $A \ll B$, we have $A \ll B$ implies that $\overline{A} \subseteq B$. Second, assume that $A \ll B$, and let $\{O_i\}$ be an open cover of $\overline{A}$. Then, the family $\{O_i \cup B \setminus \overline{A}\}$ is an open cover of $B$, and since $A \ll B$, it follows that $A \ll O_i \cup B \setminus \overline{A}$ for some $i$, if we assume that the $O_i$ are up-directed, which is possible by taking finite unions of the $O_i$'s if necessary. Then, the first part of the proof shows that $\overline{A} \subseteq O_i$; this then demonstrates the compactness of $\overline{A}$. Hence we have shown that 1 implies 2.

Conversely, it is clear from the definitions that $\overline{A} \subseteq B$ and $\overline{A}$ compact imply $A$ is relatively compact in $B$, and so $A \ll B$. Hence, if $\{O_i\}$ is any up-directed family of open subsets of $X$ with $B \subseteq \sup O_i$, then $\overline{A} \subseteq \bigcup O_i$, and so there is some $i$ with $\overline{A} \subseteq O_i$. But, the comment just made then implies that $A \ll O_i$, and so $A \ll B$.

**Corollary 5.** (Isbell) For a Hausdorff space $X$, the following are equivalent:

1. $O(X)$ is a continuous lattice.
2. $X$ is locally compact.

**Proof.** Suppose that $O(X)$ is a continuous lattice, and let $x \in X$. Since $X = \sup \{A : A \ll X\}$, there is some $A \in O(X)$ with $x \in A \ll X$. Then, there is some $B \in O(X)$ with $A \ll B \ll X$, and it follows that $A \ll X$. This shows that $\overline{A}$ is compact by the Proposition, and so we have the desired compact neighborhood of $x$.

Conversely, suppose that $X$ is locally compact, and let $A$ be an open subset of $X$. Then, the local compactness and Hausdorff properties imply that $A$ is the union of compact neighborhoods of each of the points in $A$, and the interior of such a neighborhood is then way-below $A$ by the Proposition. Hence each open set of $X$ is the sup of the open subsets way-below it, and so $O(X)$ is a continuous lattice.
Example 6. Let $D$ be the closed unit disk in the plane, and let $D$ have the usual topology. Let $D'$ be the open unit disk. We define a new topology, $k$, on $D$ as follows: A subset $U$ of $D$ is $k$-open if and only if, for each $x \in U$, there is an open subset $V$ of $D$ in the usual topology on $D$ such that $x \in V$ and $V \cap D' \subseteq U \cap D'$. The effect of this is to give $D'$ the usual topology, but the boundary of $D$ is now discrete in the $k$-topology. We claim that $D'$ is relatively compact in $D$ in the $k$-topology: Indeed, let $\{0_i\}$ be a family of $k$-open sets which covers $D$. For each $i$, if $x \in 0_i \cap D'$, we let $0_{i,x} = 0_i$; if $x \in 0_i \setminus D'$, then we let $0_{i,x} = (0_i \cap D') \cup V_x$, where $x \in V_x$ is an open subset of $D$ in the usual topology such that $V_x \cap D' \subseteq 0_i \cap D'$ (such a $V_x$ exists by the definition of the $k$-topology). Now, since the family $\{0_i\}$ covers, it follows that the family $\{0_{i,x}\}$ covers $D$, and it is clear that each set $0_{i,x}$ is open in $D$ in the usual topology. Hence, since $D$ is compact in the usual topology, there is a finite subfamily $\{0_{j,x_j} : j = 1, \ldots, n\}$ which also covers $D$

Now, $D' = D' \cap (\bigcup\{0_{j,x_j}\}) = \bigcup(\{D' \cap 0_{j,x_j}\}) \subseteq \bigcup 0_{j,x_j}$, since $0_{j,x_j} = (0_j \cap D') \cup V_{x_j}$ and $V_{x_j} \cap D' \subseteq 0_j$ for each $j$. This shows that the family $0_1, \ldots, 0_n$ forms a finite cover of $D'$, and so we have our claim. It then follows that $D' \ll D$ in the $k$-topology.

The point of the example is to show that $U \ll V$ does not imply $\overline{U}$ compact even for Hausdorff spaces. The following result gives a characterization of $U \ll V$ for regular $T_1$ spaces.

Proposition 7. Let $X$ be a regular $T_1$ space, and let $Y$ be an open dense subset of $X$. The following are then equivalent:

1. $Y \ll X$ in $O(X)$.
2. Let $O'(X)$ be the family of all open sets $U$ of $X$ which satisfy: For each $x \in X$, if there is some $V \in O(X)$ with $x \in V$ and $V \cap Y \subseteq U \cap Y$, then $x \in U$. Then $O'(X)$ is a basis for a compact Hausdorff topology on $X$.

Note: The motivation for the topology $O'(X)$ given in part 2 stems from the idea of recovering the original topology on the unit disk $D$ from the topology described in Example 5.

Proof. Suppose that 1 holds. It is routine to show that $O'(X)$ is a basis for a topology on $X$; moreover, if $x, y \in X$ with $x \neq y$, then there are disjoint open sets $U$ and $V$ containing $x$ and $y$, respectively. Now, let $U' = \{z \in X : W \subseteq U \cap Y \subseteq U \cap Y \}$ for some open set $W$ in $X$ with $z \in W$, and let $V' = \{z \in X : W \subseteq V \cap Y \subseteq V \cap Y \}$ for some open set $W \subseteq X$ with $z \in W$. Then $U'$ and $V'$ are open in the new topology on $X$ (they are in fact members of $O'(X)$), and since $U$ and $V$ are disjoint, we have that $U'$ and $V'$ are
disjoint. Moreover, clearly $U \subseteq U'$ and $V \subseteq V'$, so $U'$ and $V'$ are the disjoint open subsets in the new topology which we seek. Notice that a variant of this argument also shows that the new topology is regular, since the original topology is regular.

We now show that the new topology is compact. Let $\{A_i\}$ be a descending family of closed sets in the new topology, which, for brevity sake, we shall call the $k$-topology. Fix an index $i$, and let $x \in X$. If $x \notin A_i$, then since the $k$-topology is regular (as we noted above), there is a closed neighborhood $C(i,x)$ of $A_i$ which doesn't contain $x$. If $x \in A_i$, then we let $C(i,x) = X$. For a finite subset $F$ of $X$, we let $C(i,F) = \bigcap_{x \in F} C(i,x)$, which we note is a closed neighborhood of $A_i$. We claim the family $\{Y \cap C(i,F) : i \in I, F \subseteq X$ finite$\}$ has the finite intersection property. Indeed, suppose that $C(i_1,F_1), \ldots, C(i_n,F_n)$ are given. Then, $F = F_1 \cup \ldots \cup F_n$ is a finite subset of $X$, and since the family $\{A_i\}$ is descending, there is some $A_j$ with $A_j \subseteq A_i$ for $k = 1, \ldots, n$. Then $C(j,F)$ is a closed neighborhood of $A_j$, as is $C(i_k,F_k)$ for each $k = 1, \ldots, n$, since $A_j \subseteq A_i$ for each $k = 1, \ldots, n$. Hence, since $Y$ is dense in $X$, it follows that $Y \cap C(j,F) \cap \bigcap_{k \leq n} C(i_k,F_k) \neq \emptyset$, and this establishes the claim. Since $Y$ is relatively compact in $X$ in the original topology, it follows that $\bigcap (C(i,F) : i \in I, F \subseteq X$ finite$) \neq \emptyset$ since each of these sets has non-empty interior. Now, it is clear that $\bigcap A_i \subseteq \bigcap C(i,F)$; conversely, if $x \notin A_i$ for some $i$, then since $C(i_1,\{x\})$ is a closed neighborhood of $A_i$ not containing $x$, it follows that $x \notin \bigcap C(i,F)$. Thus $\bigcap A_i = \bigcap C(i,F)$, and since the right side is non-empty, so also is the left. We have therefore shown that each descending family of closed subsets of $X$ in the $k$-topology has a non-empty intersection, and so we conclude that $X$ is compact in the $k$-topology. This finishes the proof that 1 implies 2.

Conversely, suppose that 2 holds, and let $\{0_i\}$ be an open cover of $X$ in the original topology. For each index $i$, let $0_i' = \{z \in X : z \in V \land V \cap Y \subseteq 0_i \cap Y\}$, and note that $0_i' \subseteq O'(X)$ and $0_i \subseteq 0_i'$ for each $i$. Hence, the family $0_i'$ covers $X$, and since these sets are in $O'(X)$ which generates a compact topology, it follows that there is a finite subfamily $0_1', \ldots, 0_n'$ which cover $X$. Now, $Y \cap (\bigcup_{j \leq n} 0_j') = \bigcup_{j \leq n} (Y \cap 0_j') \subseteq \bigcup_{j \leq n} 0_j'$ by the definition of $0_j'$. Hence, for the cover $0_i$ of $X$, we have found a finite subcover $\{0_j : j \leq n\}$ which covers $Y$, and this shows that 1 holds.

This completes the proof of the Proposition.
The reason that this characterizes the way-below relation in \( O(X) \) for regular \( T_1 \) spaces \( X \) is as follows: If \( U \subset V \) are open sets, and if \( U \preceq V \), then we claim \( U \preceq \overline{U} \) in \( O(U) \): Indeed, if \( \{0_i\} \) is an open cover of \( \overline{U} \), then each \( 0_i \) can be written as \( 0_i \cap \overline{U} \), where \( 0_i \) is open in \( X \). Hence the family \( \{0_i\} \cup (V \setminus \overline{U}) \) is an open cover of \( V \), and since \( U \preceq V \), there is a finite subcover of \( U \). Clearly this gives rise to a finite subcover of \( U \) from the \( \{0_i\} \). Conversely, suppose that \( U \preceq \overline{U} \) in \( O(U) \). Then, for any open subset \( V \) of \( X \) with \( U \subset V \), it is easily seen that \( U \preceq V \) in \( O(X) \). Thus, our Proposition does indeed characterize the way-below relation on \( O(X) \) for \( X \) regular and \( T_1 \).

We now consider the question of whether \( O(D) \) is a compact semilattice, where \( D \) is the unit disk with the topology described in Example 5. Since \( D \) is not locally compact in this topology, it is clear from Corollary 4 that \( O(D) \) is not a continuous lattice. The following definition and lemma are taken from reference 2:

**Definition 8.** Let \( L \) be a complete lattice, and \( A \subset L \) any subset. We define \( A^+ = \{ \sup B : B \subset A \text{ and } B \text{ is up-directed} \} \).

**Lemma 9.** (Lawson). Let \( S \) be a compact semilattice and \( I \) a semilattice ideal of \( S \). Then, the closure of \( I \) satisfies \( \overline{I} = I^{++} \).

**Example 10.** We show in this example that \( O(D) \) is not a compact semilattice, where \( D \) is the unit disk with the topology described in Example 5. Let \( E = D' \cup \{(1,0)\} \), and give \( E \) the relative topology from \( D \). Then \( E \) is open in \( D \), and so \( O(E) \) is an ideal of \( O(D) \) which is closed under up-directed sups. The lemma then implies that \( O(E) \) is a compact semilattice if \( O(D) \) is, and we show that this is not the case.

Let \( x_0 = (1,0) \), and let \( \{x_n\}_{n \geq 1} \) be a sequence in \( D' \) with no convergent subsequence in \( E \) (e.g., take \( \{x_n\}_{n \geq 1} \) to be a sequence which converges to \( (0,1) \) in \( D \) in the usual topology). Define \( A_n = E \setminus \{x_m \mid m > n\} \), and note that the family \( \{A_n\} \) is an increasing set of open subsets of \( E \) whose union is all of \( E \). Thus \( O(A_n) \), the family of open subsets of \( A_n \), is an ideal of \( O(E) \), and the family \( \{O(A_n)\} \) is an increasing family of ideals in \( O(E) \). Now, let \( \{y_n\}_n \) be a sequence in \( D' \setminus E \) which converges to \( x_0 \), and, for each \( n \), choose a sequence \( \{y_{n,m} \} \) in \( E \) which converges to \( y_n \) in the usual topology on \( D \). We define \( B_{n,m} = A_n \setminus \{y_{n,p} : p > m\} \), and note that, for each \( n \), the family \( \{B_{n,m}\} \) is an increasing family of open subsets in \( E \) whose union is \( A_n \); thus \( O(B_{n,m}) \) is an ideal of \( O(E) \), and the family of these is up-directed. We let \( I = \bigcup O(B_{n,m}) \), and note that \( E = \bigcup A_n = \bigcup B_{n,m} \in I^{++} \).
We claim that \( E \notin I^+ \): Indeed, suppose that \( \{O_i\} \) is an up-directed subset of \( I \) and that \( x_0 \in \bigcup_0 \). Then, we can assume that \( x_0 \in O_i \) for each \( i \). Fix \( i \); since \( \{y_n\} \) converges to \( x_0 \), and since \( \{y_{n,m}\} \) converges to \( y_n \) for each \( n \), it follows that there are \( r, s \) with \( y_{p,q} \in O_i \) for \( p \geq r \) and \( q \geq s \). It then follows that \( 0_i \subset B_{n,m} \) implies \( n \leq r \) and \( m \leq s \), and so this also holds for each \( j \geq i \) since the \( O_j \) are up-directed. Hence, \( \bigcup_0 \bigcup \{B_{n,m} : n \leq r \text{ and } m \leq s\} \subset A_{r}' \), and so \( \bigcup_0 \neq E \).

This establishes our claim.

Now, we repeat the above construction as follows: We choose yet another sequence \( \{z_n\} \subset \bigcap E \) which converges to \( x_0 \), and which is disjoint from \( \{y_n\} \), and for each \( n \), we choose a sequence \( \{z_{n,m}\} \) in \( E \) which converges to \( z_n \) in the usual topology of \( D \). Now, for \( n,m \) in \( IN \), we define \( C_{n,m,p} = B_{n,m} \setminus \{z_{n,m,r} : r \geq p\} \), and note that \( C_{n,m,p} \) is open on \( E \) for each triple \( n,m,p \), and that the union of the \( C_{n,m,p} \), for fixed \( n,m \) is \( B_{n,m} \). This time we let \( J = \bigcup_{n,m,p} O(C_{n,m,p}) \), and note that \( J \) is an ideal of \( O(E) \). Now, \( E = \bigcup_n A_n = \bigcup_{n,m} B_{n,m} = \bigcup_{n,m,p} C_{n,m,p} \in J^+ \).

We show that \( E \notin J^{++} \): Suppose that \( \{O_i\} \) is an up-directed subset of \( J^+ \) and that \( x_0 \in \bigcup_0 \); then, as before we can assume \( x_0 \) is in each \( O_i \). We show that \( O_i \subset B_{n,m} \) for some \( n,m \) depending on \( i \); our previous remarks will then show that \( E \neq \bigcup_0 \) since this is an up-directed family. Now, for a fixed \( i \), \( O_i \in J^+ \), and so there is an up-directed family \( \{O_{i,j}\} \) in \( J \) with \( O_i = \bigcup O_{i,j} \). Again, we can assume that \( x_0 \in O_{i,j} \) for each \( j \); fixing one \( j \), since \( O_{i,j} \) is open, there are \( r, s \) in \( IN \) with \( z_{p,q} \in O_{i,j} \) for \( p \geq r \) and \( q \geq s \). It follows that \( O_{i,j} \subset C_{n,m,p} \) for some \( n,m,p \) implies \( n+m \leq r \) and \( p \leq s \), so that \( O_i \subset B_{n,m} \) and \( n+m \leq r \). Since the \( O_{i,j} \) are up-directed, it follows that \( O_{i,k} \subset C_{n,m,p} \) implies \( n+m \leq r \) and \( p \leq s \) for \( k \geq j \), and so \( O_{i,k} \subset B_{n,m} \) with \( n+m \leq r \) for \( k \geq j \). We conclude that \( O_i = \bigcup_{i,j} \bigcup_{n+m \leq r} B_{n,m} \subset B_{r,r}' \) as is clear from the definitions. Thus each \( O_i \) is a subset of some \( B_{n,m} \), and so they form an up-directed subset of \( I \). Since \( E \notin I^+ \), it follows that \( \bigcup_0 \neq E \). Therefore \( E \notin J^{++} \).

Now, \( J \) is an ideal in the semilattice \( O(E) \) with \( J^{++} = J^{+++} \). But, Lemma 8 shows that the closure of each ideal \( I \) is \( I^{++} \), and clearly closed ideals are closed under the formation of up-directed supers (since these are then limits in the topology). Hence \( O(E) \) cannot be a compact semilattice, since \( J \) is an ideal with \( J^{++} \) not closed under up-directed supers, and so also not closed in any semilattice topology.
We close this memo with an observation on a result of Lawson's which appears in the proof of Theorem 13 of reference 2.

**Definition 11.** Let \( L \) be a complete lattice. For \( x, y \in L \) we define \( x \ll_c y \) if and only if for each subset \( A \) of \( L \) with \( y \leq \text{sup} A \), there is some countable subset \( \{a_n\} \) of \( A \) with \( x \leq \text{sup} a_n \).

**Lemma 12.** For a compact semilattice \( S \) and an \( x \in S \), we have \( x = \text{sup} \{ y \in S : y \ll_c x \} \).

**Proof.** (Lawson(2)). For any compact neighborhood \( W_0 \) of \( x \) in \( S \), choose recursively a family \( W_n \) of compact neighborhoods of \( x \) with \( W_n^2 \subseteq W_{n-1} \) for each \( n \). Let \( U \) be the intersection of the family \( W_n \). Then, it is readily seen that \( U \) is a compact subsemilattice of \( S \) containing \( x \). We let \( u = \text{inf} U \), and claim that \( u \ll_c x \).

Indeed, suppose that \( A \) is a subset of \( S \) with \( x \leq \text{sup} A \). Then, \( S \) is a compact semilattice, and so \( x = \text{sup} xA \). Hence, since \( \text{sup} A = \text{sup} \{ \text{sup} F : F \subseteq A \text{ finite} \} \), and the right side is an up-directed sup, the right side is also a limit. Hence, for each \( n \in \mathbb{N} \), there is some finite subset \( F_n \subseteq A \) with \( \text{sup} xF_n \leq W_n \). Now, the set \( C = \bigcup_{n} F_n \) is a countable subset of \( A \), and \( \text{sup} xF_n \leq W_n \) for each \( n \) implies that \( \text{sup} \{ \text{sup} xF_n : n \in \mathbb{N} \} = \lim_{n} \text{sup} xF_n \leq U \), and so \( u = \text{inf} U \leq \text{sup} xC \).

Since \( \text{sup} xC \leq \text{sup} C \), it follows that \( u \leq \text{sup} C \), and we have established our claim.

Now, we have shown that, for each compact neighborhood \( W_0 \) of \( x \), there is some \( u \in W_0 \) with \( u \ll_c x \) (that \( u \in W_0 \) follows from the fact that \( V \subseteq V^2 \) for each subset \( V \) of \( S \), so that the family \( W_n \) is towered). It then follows that \( x = \text{sup} \{ y \in S : y \ll_c x \} \), and so the lemma is proved.

We note in closing the following properties of the Example 9: \( 0(E) \) is a lower continuous complete lattice (since sups are just unions and finite infima just intersections), and \( 0(E) \) satisfies the property that each \( U \in 0(E) \) is the sup of the \( V \in 0(E) \) with \( V \ll_c U \) in \( 0(E) \); this follows from the fact that \( E \) is a countable union of compact subsets, as is also each open subset of \( E \).

Finally, note that the proof that \( 0(E) \) is not a compact semilattice can be used to show the following:

**Proposition 13.** Let \( X \) be a Hausdorff space which is embeddable in compact first countable (or, in particular, compact metrisable) space. Then the following are equivalent:

1. \( 0(X) \) is a compact semilattice
2. \( 0(X) \) is a continuous lattice.
3. \( X \) is locally compact.