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# New symplectic 4-manifolds with $b_+=1$

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## Abstract

Symplectic 4-manifolds  $(X, \omega)$  with  $b_+=1$  are roughly classified by the canonical class  $K$  and the symplectic form  $\omega$  depending upon the sign of  $K^2$  and  $K \cdot \omega$ . Examples are known for each category except for the case when the manifold satisfies  $K^2 = 0$ ,  $K \cdot \omega > 0$ ,  $b_1 = 2$ , and fails to be of Lefschetz type. The purpose of this paper is to construct an infinite number of examples of such manifolds. Furthermore, we will show that these manifolds have very special properties — they are not complex manifolds, their Seiberg-Witten invariants are independent of the chamber structure, and they do not have metrics of positive scalar curvature.

*Key words:* Symplectic 4-manifolds, symplectic topology

*1991 MSC:* 53D05, 57R17, 57R57, 57M60

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## 1 Introduction

Symplectic manifolds with  $b_+ = 1$  are useful because they often form the basic building blocks for symplectic manifolds with  $b_+ > 1$ . Furthermore, the Seiberg-Witten invariants have a richer structure when  $b_+ = 1$  allowing one to make and test hypotheses about symplectic manifolds that might be true

generally. In this paper we describe new examples of symplectic manifolds with  $b_+ = 1$ .

We begin the discussion with a quick treatment of the classification of symplectic 4-manifolds and mention the history behind the examples described in this paper. Symplectic 4-manifolds have a natural, well-defined extension of the Kodaira dimension used to classify compact complex surfaces defined by the sign of two numbers:

$$K^2 \quad \text{and} \quad K \cdot [\omega],$$

where  $K$  is the canonical class of a symplectic 4-manifold  $(X, \omega)$  [6,10]. In particular, for a minimal symplectic 4-manifold,

$$\kappa(X, \omega) = \begin{cases} -\infty & \text{if } K^2 < 0 \text{ or } K \cdot [\omega] < 0 \\ 0 & \text{if } K^2 = 0, \quad K \cdot [\omega] = 0 \\ 1 & \text{if } K^2 = 0, \quad K \cdot [\omega] > 0 \\ 2 & \text{if } K^2 > 0 \text{ and } K \cdot [\omega] > 0. \end{cases}$$

For a general symplectic 4-manifold, the Kodaira dimension is defined to be the Kodaira dimension of a minimal model of  $(X, \omega)$ . (The minimal model may not be unique, but  $\kappa(X, \omega)$  is still well-defined.) Note that examples with  $K^2 > 0, K \cdot \omega = 0$  do not exist [6,12].

Kähler surfaces provide many of the known examples of minimal symplectic 4-manifolds. For instance,  $\kappa(X, \omega) = -\infty$  if and only if  $X$  is diffeomorphic to a rational or ruled surface [8]. Minimal symplectic 4-manifolds with  $\kappa = 0$

include  $K3$ , Enriques surface, and hyperelliptic surfaces. Incidentally, the first example of a non-Kähler symplectic 4-manifold satisfied  $\kappa = 0$  [13]. Products  $T^2 \times \Sigma_g$  where  $g > 1$  satisfy  $\kappa(T^2 \times \Sigma_g, \omega) = 1$  and Kähler surfaces of general type are examples of symplectic 4-manifolds with  $\kappa = 2$ .

In this paper we are interested in symplectic 4-manifolds that satisfy  $\kappa = 1$  and  $b_+ = 1$ . For these manifolds the first Betti number is always zero or two: Noether's formula requires that  $b_1$  is even and the restriction on the size of  $b_1$  follows from the Hirzebruch signature theorem,

$$0 = K^2 = 2\chi + 3\sigma = 9 - 4b_1 - b_-,$$

where  $\chi$  is the Euler class of  $X$  and  $\sigma$  is its signature.

The  $b_1 = 0$  case is covered by Dolgachev surfaces. Recently, using a construction of Fintushel and Stern, Park produced similar non-complex symplectic 4-manifolds (see [11]). The  $b_1 = 2$  case is interesting because there are no examples from Kähler surfaces. McDuff and Salamon mentioned this fact in a 1995 survey paper and went on to discuss how one might construct symplectic manifolds for this case [10]. They broke the search for such manifolds into two cases based upon cup product structure of  $H^1(X; \mathbb{Z})$ . These two cases can be reformulated in terms of Kähler-like condition called Lefschetz type.

Symplectic 4-manifolds  $(X, \omega)$  are said to be of *Lefschetz type* if  $[\omega] \in H^2(X; \mathbb{R})$  satisfies the conclusion of the Hard Lefschetz Theorem, namely, that  $\cup[\omega] : H^1(X; \mathbb{R}) \rightarrow H^3(X; \mathbb{R})$  is an isomorphism. Essentially, McDuff and Salamon broke the search into 4-manifolds which either did or did not have this condition.

Examples of  $\kappa = 1$ ,  $b_+ = 1$ ,  $b_1 = 2$  manifolds of Lefschetz type can be

constructed by first doing zero-surgery on a fibered knot in  $S^3$  to get a closed 3-manifold, then taking the product with  $S^1$ . The resulting 4-manifold then has the desired properties. However, McDuff and Salamon pointed out in their survey paper that the discovery of manifolds not of Lefschetz type remained open.

At about the same time, Li and Liu discovered a general wall-crossing formula for the Seiberg-Witten invariant when  $b_+=1$  [7]. (Recall that the Seiberg-Witten invariant generally depends upon the chamber in which it is calculated when  $b_+=1$ .) In their paper Li and Liu questioned whether there existed symplectic 4-manifolds where the wall-crossing number was always equal to zero. They were interested in such examples because the Seiberg-Witten invariant would depend only on the smooth structure, implying for instance that these manifolds did not have any metrics with positive scalar curvature. The only known examples were limited to special  $T^2$ -bundles over  $T^2$ . These manifolds have Kodaira dimension  $\kappa = 0$  (c.f. [3]) and one basic class (given by the canonical  $\text{Spin}^c$  structure).

There is in fact a robust number of examples which satisfy the conditions prescribed by both McDuff-Salamon and Li-Liu. This paper describes the conditions needed to produce them and provides an infinite family of such examples. In particular, we prove:

**Theorem 1** *For each integer  $g > 1$ , there exists a smooth, closed,  $\kappa = 1$ ,  $b_1=2$ ,  $b_+=1$  symplectic 4-manifold  $(X_g, \omega)$  not of Lefschetz type constructed in §2 below with the following properties:*

- (1)  $X_g$  does not admit a complex structure,
- (2)  $X_g$  does not carry a metric with positive scalar curvature, and

(3) The Seiberg-Witten invariant is a smooth invariant of  $X_g$  given by

$$SW_{X_g}(\mathfrak{s}) = (\mathfrak{s}^{-2} - 3 + \mathfrak{s}^2)^{g-1}.$$

In particular,  $X_g$  is SW-simple type.

When  $g = 1$ , the manifold  $X_g$  satisfies all of the conditions in Theorem 1 except that  $\kappa = 0$ .

## 2 Construction

In this section we construct a manifold  $X_g$  with the properties above for each integer  $g > 1$  using a technique similar to those found in [13,5]. Let  $\Sigma$  be a surface of genus  $g > 0$ . Let  $\langle a_1, b_1, \dots, a_g, b_g \rangle$  be smooth loops which represent a symplectic basis of  $H_1(\Sigma; \mathbb{Z})$ , and let  $\langle \alpha_1, \beta_1, \dots, \alpha_g, \beta_g \rangle$  be closed 1-forms

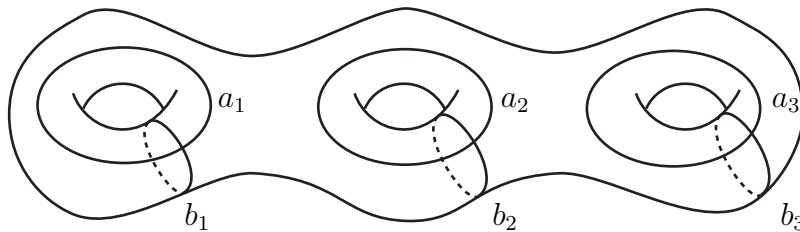


Fig. 1. Surface of genus 3.

which represent a dual basis for  $H^1(\Sigma; \mathbb{Z})$  with respect to the  $a_i$ 's and  $b_i$ 's, i.e.,

$$\alpha_i(a_j) = \delta_{ij}, \quad \beta_i(b_j) = \delta_{ij}, \quad \text{and zero otherwise.}$$

Consider an orientation-preserving diffeomorphism  $\varphi : \Sigma \rightarrow \Sigma$  with the following matrix with respect to the basis above:

$$\varphi^* = \begin{pmatrix} 1 & 0 & \\ & & 0 \\ 1 & 1 & \\ & & \\ 0 & A & \end{pmatrix}$$

where  $\ker(A - Id) = 0$ . Note that this means  $\varphi^*[\beta_1] = [\beta_1]$  and  $\varphi^*[\alpha_1] = [\beta_1 + \alpha_1]$ , and that  $\dim \ker(\varphi^* - Id) = 1$ .

Such diffeomorphisms exist for any genus. For the purposes of this paper, we will consider diffeomorphisms given by the following sequence of Dehn twists acting on the left,

$$\varphi = (T_{b_g} T_{a_g}^{-1}) \cdots (T_{b_2} T_{a_2}^{-1}) \cdot T_{a_1}$$

for each genus  $g$ .

Next, create the mapping torus  $Y$  obtained by crossing  $\Sigma$  with the interval  $[0, 1]$  and identifying the ends by  $\varphi$ . Then

$$Y = (\Sigma \times [0, 1]) / ((x, 1) \sim (\varphi(x), 0))$$

is a smooth, closed, 3-dimensional manifold.

The 4-manifold that we will construct is a nontrivial circle bundle over  $Y$ . This circle bundle has to be chosen carefully because 4-manifolds which are circle bundles over a 3-manifold which fibers over  $S^1$  are not necessarily symplectic (cf. [1]).

In order to show that the 4-manifold has the desired properties we need to describe its cohomology and the cup products explicitly. We begin by first writing down a basis for the cohomology of  $Y$ .

The mapping torus  $Y$  comes with a projection map,  $p : Y \rightarrow S^1$ , with fiber  $\Sigma$ . Let  $dt$  be the volume form on  $S^1$ . Then  $\theta = p^*(dt)$  is a nowhere-zero closed 1-form in  $\Omega^1(Y; \mathbb{R})$ . The form  $\theta$  is one of the components needed to construct the symplectic form on the 4-manifold.

There is another important 1-form on  $Y$  to consider. The cohomology class  $[\beta_1]$  is invariant under  $\varphi^*$ , so there exists a function  $f \in \Omega^0(\Sigma)$  such that  $\varphi^*(\beta_1) = \beta_1 + df$  point-wise. We can use this 1-form to construct a closed 1-form on  $Y$  which represents a nontrivial integral cohomology class. Let  $\rho : [0, 1] \rightarrow [0, 1]$  be a smooth function which is identically 0 near 0 and identically 1 near 1. Extend  $\beta_1$  to  $\Sigma \times [0, 1]$  by writing

$$\beta(x, t) = \rho(t)\beta_1(x) + (1 - \rho(t))(\varphi^*(\beta_1(x))) - \left(\frac{d}{dt}\rho(t)\right)f(x)dt.$$

This glues up to give a smooth, closed, 1-form on  $Y$  since

$$\frac{d^n}{dt^n}\varphi^*(\beta)|_{\Sigma \times \{0\}} = \frac{d^n}{dt^n}\beta|_{\Sigma \times \{1\}}$$

for all nonnegative integers  $n$  (we will also call the wrapped up 1-form  $\beta$ ).

Next we show that the cohomology classes  $[\theta]$  and  $[\beta]$  form a basis for  $H^1(Y; \mathbb{Z})$ .

It is clear that they are integral and independent. Therefore we need only show that the dimension of  $H^1(Y)$  is 2, which follows from the Wang sequence and the fact that  $\dim \ker(\varphi^* - 1) = 1$ .

$$\begin{array}{ccccccccc} H^0(\Sigma) & \longrightarrow & H^1(Y) & \xrightarrow{|\Sigma} & H^1(\Sigma) & \xrightarrow{\varphi^*-1} & H^1(\Sigma) & \longrightarrow & H^2(Y) \\ \parallel & & \parallel & & \parallel & & \parallel & & \parallel \\ \mathbb{Z} & \longrightarrow & \mathbb{Z} \oplus \mathbb{Z} & \longrightarrow & \mathbb{Z} \oplus \mathbb{Z}^{2g-1} & \longrightarrow & \mathbb{Z}^{2g-1} \oplus \mathbb{Z} & \longrightarrow & \mathbb{Z} \oplus \mathbb{Z} \end{array}$$

By Poincaré duality,  $H^2(Y; \mathbb{Z})$  is also two dimensional; we will need an explicit



basis for this space as well. Let  $\Omega_\Sigma$  be the volume form on  $\Sigma$ . There exists a 2-form  $\Omega$  on  $Y$  which restricts to the volume form  $\Omega_\Sigma$  on each fiber  $\Sigma$  of  $p : Y \rightarrow S^1$ . One way to get this 2-form is to take a metric  $g_Y$  on  $Y$  such that  $\theta$  is harmonic, then the Hodge dual  $\Omega = \star\theta$  is the desired form.

To get the second basis element, glue up  $\alpha_1$  to get a 1-form on  $Y$  by setting

$$\alpha(x, t) = \rho(t)\alpha_1(x) + (1 - \rho(t))(\varphi^*(\alpha_1(x))) - \left(\frac{d}{dt}\rho(t)\right)g(x)dt,$$

where  $g \in \Omega^0(Y)$  is the function given by  $\varphi^*(\alpha_1) = \alpha_1 + \beta_1 + dg$ . This 1-form is not closed, in fact

$$d\alpha = \beta \wedge \left(\frac{d}{dt}\rho\right)\theta,$$

but  $\alpha \wedge \theta \in \Omega^2(Y; \mathbb{R})$  represents a closed, nontrivial, integral cohomology class.

The desired 4-manifold  $X_g$  is a circle bundle over  $Y$  with Euler class  $\chi = [\alpha \wedge \theta]$  in  $H^2(Y; \mathbb{Z})$ . There is a connection 1-form  $\eta$  for this bundle whose curvature form is  $\alpha \wedge \theta$ , i.e.,  $d\eta = \pi^*(\alpha \wedge \theta)$ . Set

$$\omega = \pi^*(\Omega) + \pi^*(\theta) \wedge \eta,$$

then  $\omega$  is non-degenerate because  $\omega^2$  is the pullback of the volume form on  $Y$  wedge a nowhere-zero 1-form  $\eta$ . It is also closed since

$$d\omega = -\pi^*(\theta) \wedge d\eta = 0.$$

Thus  $(X_g, \omega)$  is a symplectic manifold.

### 3 Properties of $X_g$

Next we show that  $b_+(X_g) = 1$ , and that  $X_g$  satisfies the conditions  $K \cdot [\omega] > 0$  and  $K^2 = 0$ . A calculation using the Gysin sequence and the fact that  $\chi \neq 0$  shows that the cohomology  $H^2(X_g; \mathbb{Z})$  is 2-dimensional.

$$\begin{array}{ccccccccc}
 H^0(Y) & \xrightarrow{\cdot \cup \chi} & H^2(Y) & \xrightarrow{\pi^*} & H^2(X_g) & \longrightarrow & H^1(Y) & \xrightarrow{\cdot \cup \chi} & H^3(Y) & \longrightarrow & 0 \\
 \parallel & & \parallel & & \parallel & & \parallel & & \parallel & & \\
 \mathbb{Z} & \longrightarrow & \mathbb{Z} \oplus \mathbb{Z} & \longrightarrow & \mathbb{Z} \oplus \mathbb{Z} & \longrightarrow & \mathbb{Z} \oplus \mathbb{Z} & \longrightarrow & \mathbb{Z} & \longrightarrow & 0
 \end{array}$$

Hence  $\langle [\pi^*(\Omega)], [\pi^*(\theta) \wedge \eta] \rangle$  is a basis for  $H^2(X; \mathbb{Z})$  with intersection matrix

$$Q_X = \begin{pmatrix} 0 & 1 \\ 1 & d \end{pmatrix},$$

for some  $d \in \mathbb{Z}$ . ( $[\pi^*(\Omega)]^2 = 0$  by naturality of  $\pi^*$ .) This quadratic form clearly has signature zero, implying that  $b_+(X) = 1$ .

Next we define a compatible complex structure  $J$  and use it to calculate  $K$ . Because  $\Sigma$  intersects trivially with the Poincaré dual of  $\chi$ , the tangent space  $T\Sigma$  lifts to a subspace of  $TX$  at each point in  $X$ . Fix a complex structure on the  $\Sigma$ . On the subspace  $T\Sigma \subset TX$ , define  $J$  to be the complex structure given by  $\Sigma$ . Also at each point in  $X$ , there are two natural vectors,  $\frac{\partial}{\partial t}$  and  $T$ , given by  $\theta(\frac{\partial}{\partial t}) = 1$  and  $\eta(T) = 1$ , which are linearly independent to each other and the subspace  $T\Sigma$ . Define  $J\frac{\partial}{\partial t} = T$  and  $JT = -\frac{\partial}{\partial t}$  on the span of these vectors. It is easy to see that  $J$  is compatible with  $\omega$ . With this complex structure the tangent space of  $TX$  splits into a direct sum and

$$K = -c_1(TX, J) = -c_1(T\Sigma \oplus \underline{\mathbb{C}}) = (2g - 2)[\pi^*\Omega].$$

(In Section 6 we will see another proof of this equality.) Hence  $K^2 = 0$  and  $K \cdot [\omega] = 2g - 2 \geq 0$ .

We are interested in  $g > 1$ , but the discussion also holds for the case when  $g = 1$ . In that case the construction yields a symplectic version of a Calabi-Yau 4-manifold, i.e., a symplectic manifold with  $K = 0$ .

#### 4 $X_g$ does not admit a complex structure

Recall that a closed 4-manifold with even  $b_1$  is complex if and only if it is Kähler (cf. [4]). By the Gysin sequence,  $\pi^* : H^1(Y; \mathbb{Z}) \rightarrow H^1(X_g; \mathbb{Z})$  is an isomorphism, and therefore  $b_1(X_g) = 2$ . Thus to show that  $X_g$  is not complex it is enough to show that it is not of Lefschetz type.

Suppose that  $X_g$  is Kähler. Then for some  $L \in H^2(X_g; \mathbb{Z})$ ,

$$\cdot \cup L : H^1(X_g; \mathbb{Z}) \rightarrow H^3(X_g; \mathbb{Z})$$

is an isomorphism by the Hard Lefschetz Theorem. This in turn implies that there is a non-degenerate quadratic form,

$$q : H^1(X_g; \mathbb{Z}) \otimes H^1(X_g; \mathbb{Z}) \rightarrow \mathbb{Z},$$

given by  $q(a, b) = a \cup b \cup L$ .

We will show that the class  $[\pi^*(\theta)]$  annihilates  $H^1(X_g; \mathbb{Z})$  by cup product, contradicting that  $q$  is non-degenerate. Notice that this contradiction also means that  $X_g$  is not of Lefschetz type.

Instead of working with the cup product structure of  $X_g$ , we can work with  $\theta$  and the cup product structure on  $Y$  instead. This is because of the naturality

of the cup product and the isomorphism  $H^1(Y; \mathbb{Z}) \cong H^1(X_g; \mathbb{Z})$  given by the Gysin sequence.

Apply  $[\theta]$  to the basis elements  $\{[\theta], [\beta]\}$  described above. Clearly

$$[\theta] \cup [\theta] = [\theta \wedge \theta] = 0$$

since  $\theta$  is a closed 1-form. To show that  $[\theta] \cup [\beta] = 0$ , evaluate the 2-form  $\theta \wedge \beta$  on a basis of  $H_2(Y; \mathbb{Z})$ . Consider  $\Sigma$  as one of the fibers of the fibration  $p : Y \rightarrow S^1$ , then

$$\langle [\theta] \cup [\beta], [\Sigma] \rangle = \int_{\Sigma} \theta \wedge \beta = 0.$$

The other basis element of  $H_2(Y; \mathbb{Z})$  is represented by  $a_1 \times S^1$  — the torus that is transverse to the fiber  $\Sigma$  in  $Y$ . (Recall that  $a_1$  is the loop such that  $\varphi_*(a_1) = a_1$ .) Then

$$\langle [\theta] \cup [\beta], [a_1 \times S^1] \rangle = \int_{a_1 \times S^1} \theta \wedge \beta = \beta_1(a_1) = 0.$$

Thus  $X_g$  is not of Lefschetz type and does not admit a complex structure.

## 5 Seiberg-Witten invariants and the wall crossing formula

The Seiberg-Witten invariants of a 4-manifold  $X$  for a fixed  $\text{Spin}^c$  structure is the count of signed solutions to a partial differential equation on the spinor bundle. In this section we give a brief introduction of the relevant topics needed to understand why the Seiberg-Witten invariants are diffeomorphism invariants for  $X_g$  and to understand why they do not carry metrics of positive scalar curvature.

Fix a metric  $h$  on  $X$ . A  $\text{Spin}^c$  structure on the 4-manifold  $X$  is a pair  $\xi = (W, \sigma)$

consisting of a rank 4 complex bundle  $W$  with a hermitian metric (the spinor bundle) and an action  $\sigma$  of 1-forms on spinors,

$$\sigma : T^*X \rightarrow \text{End}(X),$$

which satisfies the property that, if  $e^1, e^2, e^3, e^4$  are an orthonormal coframe at a point in  $X$ , then the endomorphisms  $\sigma(e^i)$  are skew-adjoint and satisfy the Clifford relations

$$\sigma(e^i)\sigma(e^j) + \sigma(e^j)\sigma(e^i) = -2\delta_{ij}.$$

The bundle  $W$  decomposes into two bundles of rank 2,  $W = W^+ \oplus W^-$ , with  $\det W^+ = \det W^-$ . The bundle  $W^-$  is the subspace annihilated by the action of self-dual 2-forms ( $\sigma$  can be extended to an action of 2-forms).

By coupling it with a  $U(1)$ -connection  $A$  on  $\det W^+$  with the Levi-Civita connection we can define a connection on  $W^+$ . Use this connection to define a Dirac operator  $\mathcal{D}_A^+ : \Gamma_X(W^+) \rightarrow \Gamma_X(W^-)$  from the space of smooth sections of  $W^+$  to  $W^-$ . The 4-dimensional perturbed Seiberg-Witten equations for a section  $\Psi \in \Gamma_X(W^+)$  and a  $U(1)$ -connection  $A$  on  $\det W^+$  are:

$$F_A^+ + \delta - q(\Psi) = 0,$$

$$\mathcal{D}_A^+(\Psi) = 0.$$

Here  $F_A^+$  is the projection of the curvature onto the self-dual two forms,  $\delta \in \Omega^+(X; \mathbf{i}\mathbb{R})$  is self-dual 2-form used to perturb the equations, and  $q : \Gamma_X(W^+) \rightarrow \Omega^+(X, \mathbf{i}\mathbb{R})$  defined by  $q(\Psi) = \Psi \otimes \Psi^* - \frac{1}{2}|\Psi|^2$  is the adjoint of Clifford multiplication by self-dual 2-forms, i.e.,

$$\langle \sigma(\beta)\Psi, \Psi \rangle_{W^+} = 4\langle \beta, q(\Psi) \rangle_{\mathbf{i}\Lambda^+}$$

for all self-dual 2-forms  $\beta \in \Omega^+(X; \mathbf{i}\mathbb{R})$  and all sections  $\Psi$ .

For a fixed metric  $h$  and perturbation term  $\delta$ , the moduli space  $\mathcal{M}(X, \xi, h, \delta)$  is the space of solutions to the Seiberg-Witten equations modulo the action of the gauge group  $\mathcal{G} = \text{Map}(X, S^1)$ . For a generic choice of  $(h, \delta)$  the moduli space is a compact, oriented, smooth manifold of dimension

$$d(\xi) = \frac{1}{4} (c_1(\xi)^2 - 2\chi(X) - 3\sigma(X))$$

which is independent of metric and perturbation when  $b_+(X) > 1$ .

The Seiberg-Witten invariant  $SW_X(\xi)$  is a suitable count of solutions. Fix a base point in  $M$  and let  $\mathcal{G}^0 \subset \text{Map}(X, S^1)$  denote the group of maps which map that point to 1. The based moduli space, denoted by  $\mathcal{M}^0$ , is the quotient of the space of solutions by  $\mathcal{G}^0$ . When the moduli space  $\mathcal{M}(X, \xi, h, \delta)$  is smooth,  $\mathcal{M}^0$  is a principal  $S^1$ -bundle over  $\mathcal{M}(X, \xi, h, \delta)$ . For a given  $\text{Spin}^c$  structure  $\xi$ , the 4-dimensional Seiberg-Witten invariant  $SW_X(\xi)$  is defined to be 0 when  $d(\xi) < 0$ , the sum of signed points when  $d(\xi) = 0$ , or if  $d(\xi) > 0$ , it is the pairing of the fundamental class of  $\mathcal{M}(X, \xi, h, \delta)$  with the maximal cup product of the Euler class of the  $S^1$ -bundle  $\mathcal{M}^0$ .

Let  $G(X)$  be the product space of metrics and  $\Gamma(\Lambda_+)$ . A pair  $(h, \delta) \in G(X)$  is called a *good pair* if the moduli space  $\mathcal{M}(X, \xi, h, \delta)$  is a smooth manifold without reducible solutions (ie., solutions  $(A, \Psi)$  where  $\Psi \equiv 0$ ). The bad pairs form a ‘wall’ inside  $G(X)$  of dimension  $b_+(X)$ . When  $b_+ > 1$  a cobordism can be constructed between any two moduli spaces of good pairs, making the Seiberg-Witten invariants independent of metric and perturbation. However, when  $b_+(X)=1$  it is possible that two good pairs cannot be connected through a generic smooth path in  $G(X)$  without crossing a wall of bad pairs where

reducible solutions occur. Passing through a bad pair could cause a singularity to occur in the cobordism. For a general  $b_+ = 1$  manifold, this will often break the smooth invariance of the Seiberg-Witten invariant.

In 1995 Li and Liu proved a general wall crossing formula which describes how the Seiberg-Witten invariants change when crossing a wall [7]. The wall crossing formula for  $X_g$  can be stated as follows:

**Theorem 2 (Li-Liu)** *Let  $\xi \in H^2(X_g; \mathbb{Z})$  a  $\text{Spin}^c$  structure with  $d(\xi) \geq 0$ . There exists a basis  $\{y_1, y_2\}$  of  $H^1(X_g; \mathbb{Z})$  depending on  $\xi$  such that*

$$\langle y_i \cup y_i \cup \frac{\xi}{2}, [X_g] \rangle = 0$$

for  $i = 1, 2$ . Then, after crossing a wall, the  $SW_X(\xi)$  changes by  $\pm y_1 y_2 \frac{\xi}{2} [X_g]$ .

If there exists an element in  $H^1(X; \mathbb{R})$  which annihilates  $H^1(X; \mathbb{R})$  by cup product, the Seiberg-Witten invariant of  $X$  does not change after crossing any wall in  $G(X)$ . In that case the Seiberg-Witten invariants of  $X$  are independent of the metric and perturbation, and are smooth invariants of the manifold.

We finish this section by showing that  $X_g$  does not carry a metric of positive scalar curvature. Since  $X_g$  is symplectic, the Seiberg-Witten invariants are nontrivial by a theorem of Taubes [12]. It was shown in Section 4 that  $[\pi^*(\theta)]$  annihilates  $H^1(X_g; \mathbb{R})$ , thus the Seiberg-Witten invariants for a given  $\text{Spin}^c$  structure are the same in both chambers by the wall crossing formula above. The existence of an irreducible solution for all good pairs then implies the standard fact in Seiberg-Witten theory that  $X_g$  can not have a metric of positive scalar curvature (cf. [4]).

## 6 Seiberg-Witten invariants of $X_g$

In this section we explicitly calculate the Seiberg-Witten invariants of the manifold  $X_g$  for any genus  $g > 0$  using the formula described in [2].

First we calculate the Milnor torsion of the three manifold  $Y$  directly from the fundamental group of  $Y$ ,

$$\pi_1(Y) = \langle t, a_1, b_1, \dots, a_g, b_g \mid \prod [a_i, b_i], ta_i t^{-1} \varphi_*^{-1}(a_i), tb_i t^{-1} \varphi_*^{-1}(b_i) \rangle.$$

Here the loop  $t$  is  $pt \times [0, 1]$  in  $Y = \Sigma \times [0, 1] / \sim$ . The Alexander matrix, calculated by Fox calculus from the group presentation above, is equal to:

$$\begin{pmatrix} 0 & 1 - b_1 & 0 \\ 0 & t - 1 & 0 \\ 1 - b_1 & -b_1 & t - 1 \end{pmatrix} \bigoplus \underbrace{K_8 \oplus \dots \oplus K_8}_{g-1},$$

where  $K_8 = \begin{pmatrix} t - 1 & -1 \\ -1 & t - 2 \end{pmatrix}$ . The Alexander polynomial is the polynomial

which generates the first elementary ideal of this matrix. Symmetrizing this polynomial gives the Milnor torsion of  $Y$ ,

$$\Delta_Y(t) = (t^{-1} - 3 + t)^{g-1}.$$

By a theorem of Meng-Taubes [9], the Seiberg-Witten invariant of  $Y$  is

$$\mathcal{SW}_Y(\mathfrak{t}) = \Delta_Y(\mathfrak{t}^2) = (\mathfrak{t}^{-2} - 3 + \mathfrak{t}^2)^{g-1}$$

where  $\mathfrak{t} = \exp(PD[t]) = \exp(\Omega_\Sigma)$ .



By the formula in [2], the Seiberg-Witten invariants of  $X_g$  are found by adding the coefficients of  $\mathcal{SW}_Y$  for all  $\text{Spin}^c$  structures which differ by a multiple of  $\chi = [\alpha \wedge \theta]$ . Since  $\text{Spin}^c$  structures in  $\mathcal{SW}_Y$  are multiples of  $\Omega_\Sigma$ , the polynomial does not change, except that we need to replace  $\mathfrak{t}$  with  $\mathfrak{s} = \exp(\pi^*(\Omega_\Sigma))$ .

Note that this calculation confirms that  $K = (2g - 2)\pi^*[\Omega_\Sigma]$ . (Taubes showed that the canonical class in this case is the top power of  $\mathcal{SW}_{X_g}$ .) It also shows that  $X_g$  is not diffeomorphic to  $X_h$  for  $g \neq h$  although this already follows from the calculation of the fundamental group.

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