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SCS 22: Representations of Colimits in CL, Part I

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TOPIC Representation of Colimites in CL, Part I

REFERENCE (1) Gierz, Hofmann, Keimel, Mislove: Relations with the interpolation property, extended version by Keimel, SCS-memo from 8/1/76

(2) Hofmann: The space of lower semicontinuous functions into a CL-object, SCS-Memo from 9/10/76

In (2) K.H.Hofmann gave a representation of copowers in CL. But this representation is false, because the following (wrong) assertion is used in the proof of lemma 2.4:
For every compact space S and every set J, ß(JxS) = ßJxS.
So we have to start again.

Let us recall some facts from (1):

I) Let CSRIP be the following category:

Objects: $(S, \sqsubset)$ where
(1) S is a complete lattice with smallest element 0.
(2) $\sqsubset$ is a binary relation on S such that
(2.1) (Interpolation property) $a \sqsubset b \Rightarrow \exists c \in S: a \sqsubset c \sqsubset b$
(2,2) $a \sqsubset b \Rightarrow a \leq b$
(2.3) $0 \sqsubset 0$
(2.4) $a \leq b \sqsubset c \Rightarrow a \sqsubset c$
(2.5) $a \sqsubset b \leq c \Rightarrow a \sqsubset c$
(2.6) $a \sqsubset c,\ b \sqsubset c \Rightarrow a \vee b \sqsubset c$

Morphisms: Let $(S, \sqsubset)$ and $(T, \sqsubset)$ be objects. A morphism between $(S, \sqsubset)$ and $(T, \sqsubset)$ is a mapping φ between S and

West Germany: TH Darmstadt (Gierz, Keimel)
U. Tübingen (Mislove, Visit.)
England: U. Oxford (Scott)
USA: U. California, Riverside (Stralka)
LSU Baton Rouge (Lawson)
Tulane U., New Orleans (Hofmann, Mislove)
U. Tennessee, Knoxville (Carruth, Crawley)

T which preserves arbitrary suprema and the relation $\sqsubset$, i.e. $a \sqsubset b$ implies $\varphi(a) \sqsubset \varphi(b)$.

Let $(S, \sqsubset)$ be an <u>CSRIP</u>-object. A subset $I \subseteq S$ is called an $\sqsubset$-ideal if $0 \in I$, if $a,b \in I$ implies $a \vee b \in I$, if $a \leq b \in I$ implies $a \in I$ and if for every $a \in I$ there exists an $b \in I$ such that $a \sqsubset b$. Every lattice ideal $I \subseteq S$ contains a largest $\sqsubset$-ideal denoted by $c(I)$. Denote by $P_{\sqsubset}(S)$ the set of all $\sqsubset$-ideals and by $P(S)$ the set of all lattice ideals. Then $c: P(S) \to P_{\sqsubset}(S)$ is a kernel operator and $P_{\sqsubset}(S)$ is a continuous lattice.

Furthermore, let $\varphi:(S, \sqsubset) \to (T, \sqsubset)$ be a <u>CSRIP</u>-morphism. Then $P_{\sqsubset}(\varphi): P_{\sqsubset}(T) \to P_{\sqsubset}(S)$; $I \mapsto c(\varphi^{-1}(I))$ is a <u>CL</u>-morphism, i.e. preserves arbitrary infima and updirected suprema. Therefore, $P_{\sqsubset}$: <u>CSRIP</u> $\to$ <u>CL</u> is a contravariant functor.

II) Let L, L' be continuous lattices and let $g: L \to L'$ be a map preserving arbitrary infima. Then its right adjoint $D(g): L' \to L$ preserves arbitrary suprema. Moreover, g is a <u>CL</u>-morphism iff for all $x,y \in L$, $x \ll y$ implies $D(g)(x) \ll D(g)(y)$ (see ATLAS, 1.19). Therefore D:<u>CL</u> $\to$ <u>CSRIP</u>, $L \mapsto (L, \ll)$; $g \mapsto D(g)$ is a contravariant functor onto a full subcategory of <u>CSRIP</u>.

III) <u>Theorem</u> (see (1), 4.2): The functors P :<u>CSRIP</u> $\to$ <u>CL</u> and D:<u>CL</u> $\to$ <u>CSRIP</u> are adjoint on the left and $P_{\sqsubset} \circ D = 1_{\underline{CL}}$. Especially, P and D both transfer limits to colimits.

The last theorem says that for the calculation of comlimites in <u>CL</u> it is very usefull to know the limits in <u>CSRIP</u>.

IV) Let <u>Compl</u> be the category of complete lattices with arbitrary sup-preserving maps as morphisms. It is well known that the forgetfull functor <u>Compl</u> $\to$ <u>Set</u> preserves limits. Let U:<u>CSRIP</u> $\to$ <u>Compl</u> be the forgetfull functor, <u>X</u> be a small category and F:<u>X</u> $\to$ <u>CSRIP</u> be a diagramm in <u>CSRIP</u>. Then the limit of F in <u>CSRIP</u> may be calculated as follows:
Let $\varprojlim U \circ F$ be the limit of $U \circ F$ in <u>Compl</u> and for every $x \in |\underline{X}|$ let $pr_x: \varprojlim U \circ F \to U \circ F(x)$ be the canonical projection. Define a relation $\sqsubset$ on $\varprojlim U \circ F$ by $a \sqsubset b$ iff $pr_x(a) \sqsubset pr_x(b)$ in $(F(x), \sqsubset)$ for all $x \in |\underline{X}|$. It is easily checked that $\sqsubset$ satisfies (2.2)-(2.6). In (1) we constructed a largest relation $\boxtimes$ contained in $\sqsubset$ which satisfies (2.1)-(2.6) in the following way: Let B denote the set of all dyadic rationals between 0 and 1 i.e. the set of all rational numbers $r = n/2^m$, $n \in \mathbb{N}$, $m \in \mathbb{N}$, $n \leq 2^m$. Let $a,b \in \varprojlim U \circ F$. A dyadic chain from a to b is a map $\varphi: B \to \varprojlim U \circ F$ such that $r < s$ implies $\varphi(r) \sqsubset \varphi(s)$, $\varphi(0) = a$ and $\varphi(1) = b$. Define $a \boxtimes b$ iff there is a dyadic chain from a to b.

V) Theorem: Let $F:\underline{X} \to \underline{CSRIP}$ be a diagram in CSRIP. Then $(\varprojlim U\circ F, ⊑⊑)$ is the limit of F in CSRIP; the projections are the same as in Compl.

Proof: As ⊑⊑ is coarser than ⊏ the projection $pr_x: \varprojlim U\circ F \to U\circ F(x)$ is a CSRIP-morphism. Let $(L, \sqsubset)$ be a CSRIP-object and let $(\varphi_x : L \to F(x))_{x\in|\underline{X}|}$ be a projective cone of CSRIP-morphisms. Then there exists a unique supremum-preserving $\varphi : L \to \varprojlim U\circ F$ such that $\varphi_x = pr_x \circ \varphi$ for all $x \in |\underline{X}|$. We have to show that φ preserves the additional relation ⊏. Firstly, note that for all $a,b \in L$, $a \sqsubset b$ implies $\varphi_x(a) = pr_x\circ\varphi(a) \sqsubset \varphi_x(b) = pr_x\circ\varphi(b)$ for all $x \in |\underline{X}|$ and hence $\varphi(a) \sqsubset \varphi(b)$ by the definition of ⊏ on $\varprojlim U\circ F$. But $a \sqsubset b$ in L implies that there exists a dyadic chain from a to b in L as L has the interpolation property. Hence there exists a dyadic chain from $\varphi(a)$ to $\varphi(b)$ in $\varprojlim U\circ F$. Thus, $a \sqsubset b$ in L implies $\varphi(a)$ ⊑⊑ $\varphi(b)$.

VI) Theorem: Let $F:\underline{X} \to \underline{CL}$ be a diagram in CL. Then the colimit $\varinjlim F$ is given by $P_{\sqsubset}(\varprojlim U\circ D\circ F, ⊑⊑)$, the canonical injection $i_x : F(x) \to P_{\sqsubset}(\varprojlim U\circ D\circ F, ⊑⊑)$ sends every $a \in F(x)$ to $c(\{f \in \varprojlim U\circ D\circ F ;\ pr_x(f) \ll a\})$.

VII) Corollary: Let L_j, $j \in J$, be continuous lattices. Then $\coprod L_j \cong P_{\sqsubset}(\prod L_j)$, where for $f,g \in \prod L_j$ we have $f \sqsubset g$ iff $f(j) \ll g(j)$ for all $j \in J$. The canonical injection $i_j : L_j \to \coprod L_j$ is given $i_j(a) = \{ f \in \prod L_j ;\ f(j) \ll a \text{ and } f(k) \ll 1 \text{ for } j \neq k\}$.

In part II I will prove a representation of coproducts by upper semi-continuous sections in a "bundle" of continuous lattices and discuss some model theoretical properties of the "stalks" of that "bundle".