Seminar on Continuity in Semilattices

11-10-1976

SCS 21: ≤ (n)

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SEMINAR ON CONTINUITY IN SEMILATTICES (SCS)

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		Nov 11	10	76

TOPIC $\leq$(n)

REFERENCE   Lawson SCS Memo dated 7/12/76   Other Memos quoted herein
Mislove SCS memo dated 8/13/76

Now that everyone seems to agree that "way below" should have

as its definition:

(a) $x \ll y$ iff $y \leq \sup A$ implies $x \leq \sup F$ for some finite $F \subseteq A$.

rather than the original:

(b) x is way below y iff $y = \sup A$ implies $x \leq \sup F$ for some

finite $F \subseteq A$.

our seminar would suggest yet a new notation and term for this concept.
Before everyone curses us out loud, please hear us out.

In his memo of 7/12/76 , Lawson defines $\lll$ by

(c) $x \lll y$ iff $y \leq \sup A$ implies $x \ll \sup F$ for some finite $F \subseteq A$.

and proves that, for compact S, $x \lll y$ iff $y \in (\uparrow x)^\circ$. He also

asserts (Corollary 5) that $x \in \Lambda(S) = \left\{ z \in S \mid S \text{ has small semi-lattices at } z \right\}$ iff $x = \sup \left\{ y \in S \mid y \ll x \right\}$. No proof is given

and, in fact, the assertion remains an open question. In a personal

correspondence received here on 10/5/76 Lawson did establish the

assertion for compact metric S.

West Germany: TH Darmstadt (Gierz, Keimel)
U. Tübingen (Mislove, Visit.)

England: U. Oxford (Scott)

USA: U. California, Riverside (Stralka)
LSU Baton Rouge (Lawson)
Tulane U., New Orleans (Hofmann, Mislove)
U. Tennessee, Knoxville (Carruth, Crawley)

Next, Mislove (8/18/76) used this open assertion to construct an example of a compact S such that $L(S) = \Lambda(S)$ is not a CL - object. First of all, there is a typographical error in the definition of T in step 1. The set Q should have been defined as $P \cup P^{-1} \cup \Delta$, where

$$P = \left\{ ((n,1),(n+1,0)) \mid n \in \mathbb{N} \right\} .$$

(Note: $((n,1),(n+1,0))$, not $((n,0),(n+1,1))$.) This example can also be viewed as a generalized hormos with

$X = \left\{ 1 - 1/n \mid n \in \mathbb{N} \right\} \cup \left\{ 1 \right\}$ in an obvious fashion. Next, he correctly observes that $1 \in L(T)$, as $\{1\} = \bigcap_{n \in \mathbb{N}} (\uparrow[n,0])^{\circ}$ and $\uparrow[n,0]$ is a subsemilattice. He then incorrectly "shows" that each semilattice neighborhood of $[n,r]$ must contain $[0,0]$. Both of his assertions "each semilattice neighborhood of $[n,r]$ must contain $[n,0]$ " and "each semilattice neighborhood of $[n,0]$ must contain $[n-1,0]$ " are correct, however, it does not follow that "each semilattice neighborhood of $[n,r]$ must contain $[n-1,0]$ ". In fact, if this were true, each semilattice neighborhood of 1 would necessarily contain $[0,0]$, yielding $1 \notin L(T)$. The problem, of course, is that each semilattice neighborhood of $[n,r]$ must contain $[n,0]$ but it need not be a neighborhood of $[n,0]$. In fact, if

$r \neq 0$, $\uparrow[n,0]$ is a semilattice neighborhood of $[n,r]$ (which is not a neighborhood of $[n,0]$) (unless $n=0$).

It can, however, be easily argued that $L(T) = \left\{ [0,0], 1 \right\}$ anyway.

Now, in step 2 , Mislove concludes that $(1,1) \in L(S)$ by using the fact that $(1,1) = \sup_{S} \left\{ z \in S \mid z \ll (1,1) \right\}$. As we have

observed, the sufficiency of this condition remains in question.

However, one sees that $(1,1) \in \uparrow([n,0],[n,0])^o$ as before and so

$(1,1) \in L(S)$ (or a glutton for punishment might use Lawson's

observation that the condition is sufficient for compact metric S).

Similar remarks apply to the "proof" that $([n,0],1) \in L(S)$.

We hasten to say that the example is quite nice. Along these

lines, is it obvious that L(S) is always a closed subspace of S ?

Is it even true ?

Now to the topic of this memo. Throughout, S will be a complete

lattice. We also adopt the convention that D will always be an

arbitrary up-directed subset of S and d will always denote an

element of D which, when written, is asserted to exist. Hence,

we know that:

(d) $x \ll y$ iff $y \leq \sup D$ implies $x \leq d$; and

(e) $x \lll y$ iff $y \leq \sup D$ implies $x \ll d$.

Let us define:

(1) $x \leq(1) y$ iff $x \leq y$;

(2) $x \leq(2) y$ iff $y \leq \sup D$ implies $x \leq(1) d$;

(3) $x \leq(3) y$ iff $y \leq \sup D$ implies $x \leq(2) d$;

and recursively define:

(n) $x \leq(n) y$ iff $y \leq \sup D$ implies $x \leq(n-1) d$.

One might read $x \leq(2) y$ as "x is two-below y", and, in general,

$x \leq(n) y$ as "x is n-below y". Recall that $x \prec y$ means $y \in (\uparrow x)^o$.

<u>Observation 1.1.</u>　(a)　For a compact　S,　$\leq(3) = \prec$　according to

Lawson (7/12/76) ;

(b)　For a compact Lawson　S,　$\leq(2) = \prec$　and so

$\leq(2) = \leq(3)$　by (a) .

<u>Problem.</u>　Is　$\leq(2) = \leq(3)$　for　all compact　S ?　Equivalently,

is　$\leq(2) = \prec$　for all compact　S ?

Surely almost everyone has tried to prove this in one form or another

and it appears to be quite difficult.

We offer some observations about　$\leq(n)$.　Perhaps these

relations have been studied for complete lattices in the literature,

but we are not aware of any such studies.

<u>Proposition 1.2.</u>　$P = \left\{ n \in \mathbb{N} \mid \leq(n+1) \subseteq \leq(n) \right\} = \mathbb{N}$.

<u>Proof.</u>　First we show　$1 \in P$.　Fix　$(x,y) \in \leq(2)$.　Let　$D = \left\{ y \right\}$.

Then　$y \leq(1) \sup D = y$　implies　$x \leq(1) d = y$.　Hence,　$x \leq(1) y$,

$\leq(2) \subseteq \leq(1)$, and　$1 \in P$.

Assume that　$k \in P$.　That is, assume that　$\leq(k+1) \subseteq \leq(k)$.

Fix　$(x,y) \in \leq(k+2)$　and fix　D　such that　$y \leq(1) \sup D$.　Then,

since　$x \leq(k+2) y$, we have　$x \leq(k+1) d$.　Now, by the induction

hypothesis,　$x \leq(k) d$.　Hence, by definition,　$x \leq(k+1) y$　so

that　$\leq(k+2) \subseteq \leq(k+1)$　and　$k+1 \in P$.　∎

<u>Corollary 1.3.</u>　$\leq(n)$　is antisymmetric for all　$n \in \mathbb{N}$.　∎

<u>Example 1.4.</u>　Let　$D = \left\{ 1 - 1/n \mid n \in \mathbb{N} \right\}$　and let　$S = D \cup \left\{ 1 \right\}$

with the usual ordering.　Then,　$1 \leq(1) \sup D$, but　$1 \not\leq(1) d$　for

any　$d \in D$.　Hence,　$1 \not\leq(2) 1$, and so　$\leq(n)$　is not reflexive for

any　$n \geq 2$　(in view of　1.2) .

The next proposition proves to be indispensible in the study of $\leq(n$

Proposition 1.5. $\leq(n) \circ \leq(m) \subseteq \leq(n+m-1)$.

Proof. The proof is by induction on n .

n = 1 : For m = 1 the assertion is clear since $\leq(1)$ is transitive. Assume for m = k and fix x $\leq(1)$ y $\leq(k+1)$ z . Now, fix D such that z $\leq(1)$ supD . Then y $\leq(k)$ d and by the I. H. , x $\leq(k)$ d . Hence, x $\leq(k+1)$ z..

Assume for n = k: i.e. $\leq(k) \circ \leq(m) \subseteq \leq(k+m-1)$ for all m $\in$ I' We proceed to show that $\leq(k+1) \circ \leq(m) \subseteq \leq(k+m)$ for all m $\in$ N . m = 1 :   Fix x $\leq(k+1)$ y $\leq(1)$ z and D such that z $\leq(1)$ supD . Then y $\leq(1)$ supD implies x $\leq(k)$ d . Hence, x $\leq(k+1)$ z as was to be shown .

Assume for m = j : i.e. $\leq(k+1) \circ \leq(j) \subseteq \leq(k+j)$. Fix x $\leq(k+1)$ y $\leq(j+1)$ z and D such that z $\leq(1)$ supD . Then, y $\leq(j)$ d and by the I. H. x $\leq(k+j)$ d . Hence, x $\leq(k+j+1)$ z . ◼

Corollary 1.6. (a) $\leq(n)$ is transitive for all n $\in$ N . *Menge der kompk Elemente*

(b) $\Delta \cap \leq(n) = \Delta \cap \leq(m)$ for all n,m $\geq$ 2 .

(c) If $1 \leq n < m$ and $\leq(n) = \leq(m)$, then $\leq(n) = \leq(k)$ for all k $\geq$ n .

(d) If n $\geq$ 2 and $\leq(n) \subseteq \leq(n) \circ \leq(n)$ (i.e. $\leq(n)$ has the interpolation property), then $\leq(n) = \leq(n+1)$.

(e) If n $\geq$ 2 and $\Delta \subseteq \leq(n)$, then $\leq(k) = \leq(1)$ for all k $\in$ N . *Dies heißt: Alle Elemente sind kompakt, d.h. alle Ketten mit größtem El. sind endlich .*

(f) If $\leq(2)$ has the interpolation property, then $\leq(2) = \leq(3)$

Proof: (a) For all $n \in \mathbb{N}$, $n+n-1 \geq n$ and so we simply apply Proposition 1.5 and then Proposition 1.2 .

(b) It suffices to show that, for $n \geq 2$, $\Delta \cap \leq(n) = \Delta \cap \leq(n+1)$. The containment from right to left follows from 1.2 . Now, fix $x \in S$ such that $(x,x) \in \leq(n)$. Then, by 1.2 , $(x,x) \in \leq(2)$ also. Hence, by 1.5 , $(x,x) \in \leq(n+1)$.

(c) Certainly $\leq(n) = \leq(k)$ for all k such that $n \leq k \leq m$, in view of 1.2 . Hence, we need only show that if $\leq(n) = \leq(n+1)$, then $\leq(n+1) = \leq(n+2)$. Fix $x \leq(n+1) y$ and $D \subseteq S$ such that $y \leq(1) \sup D$. Then, $x \leq(n) d$ and, since we are assuming $\leq(n) = \leq(n+1)$, $x \leq(n+1) d$. Therefore, $x \leq(n+2) d$ as was to be shown.

(d) Since $n \geq 2$, $2n-1 \geq n+1$. Hence,

$$\leq(n) \subseteq \leq(n) \circ \leq(n) \subseteq \leq(2n-1) \quad \text{by } 1.5$$
$$\subseteq \leq(n+1) \quad \text{by } 1.2$$
$$\subseteq \leq(n) \quad \text{by } 1.2 .$$

(e) It suffices to show that $\leq(1) \subseteq \leq(2)$, in view of (c) above. First observe that $\Delta \subseteq \leq(2)$, in view of 1.2 . Now, $\leq(1) = \leq(1) \circ \Delta \subseteq \leq(1) \circ \leq(2) \subseteq \leq(2) \quad \text{by } 1.5$
$$\subseteq \leq(1) \quad \text{by } 1.2 .$$

(f) This follows immediately from (d) and is only isolated as a special case for emphasis. ∎

Corollary 1.7. (a) If S is compact, then $\leq(3) = \leq(n)$ for all $n \geq 3$

(b) If S is compact Lawson, then $\leq(2) = \leq(n)$ for all $n \geq 2$.

In view of Corollary 1.7 , perhaps only the pure algebraists will be interested in the general study of $\leq(n)$. However, at least the question of whether $\leq(2) = \leq(3)$ for compact S remains for the topological algebraists. Moreover, we proceed to discuss relationships between $\leq(n)$ and $\prec$ in the non-compact topological setting which should be of interest to the topological people. First we state the following proposition. For $n = 2$, this is essential in proving that $\leq(2)$ has the interpolation property in a continuous lattice (Scott - Continuous Lattices , Observation on page 110 without proof; Isbell - Meet-continuous Lattices , Istituto Nazionale di Alta Matematica Symposia Mathematica 1975, Proposition 2.3 on page 46; Scott - SCS Memo dated 3/30/76 Page 5).

Proposition 1.8. $x \leq(n) z$ and $y \leq(n) w$ imply $x \vee y \leq(n) z \vee w$.

Proof. The proof is by induction on n , is not difficult, and is omitted. ▨

Proposition 1.9. Let $(S, \mathcal{T}, \leq)$ have the property that $(S, \mathcal{T})$ is a topological space and $(S, \leq)$ is a complete lattice. We define, as always, $x \prec y$ iff $y \in (\uparrow x)^{o}$. Then:

(a) $(\uparrow x)^{o} \cap (\uparrow y)^{o} = [\uparrow (x \vee y)]^{o}$.

(b) For all $y \in S$, $D_y = \{ s \in S \mid s \prec y \}$ is closed

under finite sups and hence is up-directed;

(c) $\leq(n) \circ \prec \subseteq \prec$ for all $n \in \mathbb{N}$.

If m is separately continuous, then:

(d) $\prec \circ \leq(n) \subseteq \prec$ for all $n \in \mathbb{N}$.

(e) $(\uparrow x)^o$ is an upper end; i.e. $\uparrow \left[(\uparrow x)^o \right] \subseteq (\uparrow x)^o$.

If $(S, \mathcal{T})$ is compact T_2 and m is continuous, then:

(f) $\prec \subseteq \leq(n)$ for all $n \in \mathbb{N}$. (Previously observed.)

Proof. Once again, the proofs are straightforward and are omitted. ▨

Ron Wilson observes that (f) of 1.9 can also be proved

for Scott's relation $\prec$. In fact, both results follow from a more

general relation theoretic result in the spirit of Mike Smith,

and the Gierz, Hofmann, Keimel, Mislove SCS Memo dated 8/1/76 .

Proposition 1.10. Let $\sqsubset$ be a binary relation on S such that:

(a) $\sqsubset \subseteq \leq(1)$;

(b) $\sqsubset \circ \leq(1) \subseteq \sqsubset$;

(c) $a \sqsubset \sup D$ implies $a \sqsubset d$.

Then, $\sqsubset \subseteq \leq(n)$ for all $n \in \mathbb{N}$.

Proof. The case $n = 1$ is simply (a) . Assume for $n = k$.

Fix $x \sqsubset y$ and $D \subseteq S$ such that $y \leq(1) \sup D$. Then, by (b),

$x \sqsubset \sup D$ and, by (c), $x \sqsubset d$. Now, by the I.H., $x \leq(k) d$

and so $x \leq(k+1) y$. ▨

Along these same lines, consider the category <u>CSRIP</u>. If we modify the assumptions somewhat, while retaining the interpolation property, we can obtain $\sqsubset\ \subseteq\ \leqslant(n)$ for all $n \in \mathbb{N}$.

<u>Proposition 1.11</u>. Suppose $(S, \sqsubset)$ satisfies:

(1) S is a complete lattice; and

(2) $\sqsubset$ is a binary relation on S such that:

(2.1) $\sqsubset\ \subseteq\ \sqsubset \circ \sqsubset$; and

(2.2) $\sqsubset\ \subseteq\ \leqslant(2)$.

Then, $\sqsubset\ \subseteq\ \leqslant(n)$ for all $n \in \mathbb{N}$.

<u>Proof</u>. For $n = 1$ use 2.2 and 1.2 . For $n = 2$ use 2.2 . Assume for k with $k \geqslant 2$. Fix $x \sqsubset y$ and $D \subseteq S$ such that $y \leqslant(1)$ supD . By two applications of 2.1 we obtain elements a and b such that $x \sqsubset a \sqsubset b \sqsubset y \leqslant(1)$ supD . Hence, $x \leqslant(k)\ a \leqslant(k)\ b \leqslant(1)$ supD by the I.H. and it follows that $x \leqslant(k)\ a \leqslant(k-1)\ d$. Therefore, $x \leqslant(2k-2)\ d$ by 1.5 and, since $k \geqslant 2$, $2k-2 \geqslant k$ which implies $x \leqslant(k)\ d$. Thus, $x \leqslant(k+1)$

This seems to be a good place to make an observation due to Ron Wilson. The reference is the Gierz, Hofmann, Keimel, Mislove SCS Memo dated 3/1/76 . On page 20 , the question "Is $w_S = v_S$?" is asked. Unless we are missing something, the answer is yes and it follows from the following:

<u>Observation 1.12</u>. $\ll*\ =\ \prec*$ for each compact S .

<u>Proof</u>. It suffices to show that $\ll$ dyadic chains are $\prec$ dyadic chains, and conversely. If φ is a $\ll\ =\ \leqslant(2)$ dyadic chain, then

$r < s$ implies $\varphi(r) \leq(2) \varphi(\frac{r+s}{2}) \leq(2) \varphi(s)$ which implies

$\varphi(r) \leq(3) \varphi(s)$ by 1.5 . Now, by Lawson's observation that

$\leq(3) = \prec$, we have $\varphi(r) \prec \varphi(s)$. That each $\prec$ dyadic chain

is a $\leq(2)$ dyadic chain follows from 1.9(f) . ∎

This last observation indicates, it would seem, that the

relations $\leq(n)$ can be useful, even though they collapse to various

degrees in the compact setting. As to the purely algebraic setting,

the next proposition and its corollary, due to C. E. Clark, are of

interest.

<u>Proposition 1.13</u>. Let S be a complete lattice and fix $x \in S$.

Let $T = (\mathbb{N} \times S) \cup \{0,p,1\}$ and define a partial order on T by:

(i) $\uparrow(i,y) = \left[\{i\} \times (\uparrow y)\right] \cup \{(j,1) \mid j \geq i\} \cup \{1\}$;

(ii) $\uparrow 0 = T$; $\uparrow 1 = \{1\}$;

(iii) $\uparrow p = \overset{\infty}{\underset{i=1}{\cup}} \left[\{i\} \times (\uparrow x)\right] \cup \{p\} \cup \{1\}$.

Then, $(T,\leq)$ is a complete lattice and the following properties hold:

(1) $x \leq(n) y$ in S implies $p \leq(n) (i,y)$ in T $\forall i \in \mathbb{N}$;

(2) $x \not\leq(n) y$ in S implies $p \not\leq(n) (i,y)$ in T $\forall i \in \mathbb{N}$.

(3) $x \leq(n) 1$ in S and $x \not\leq(n+1) 1$ in S imply

$p \leq(n+1) 1$ in T and $p \not\leq(n+2) 1$ in T .

<u>Proof</u>. First we show that T is a complete lattice. Fix $A \neq \square$

in T . (a) If $1 \in A$, then $\sup A = 1$. In the remaining cases

we assume $1 \notin A$.

(b) If $A \cap (\{i\} \times S) \neq \square$ for infinitely many i , then $\sup A = 1$.

(c) If $A \subseteq \{0,p\}$, then $\sup A \in \{0,p\}$.

(d) If $A \subseteq \{0,p\} \cup (\{i\} \times S)$ for some i and $A \cap (\{i\} \times S) \neq \square$, then $\sup A = (i, \sup \pi_2(A))$ if either $\sup \pi_2(A) \geq x$ or $p \notin A$, and $\sup A = (i, (\sup \pi_2(A)) \vee x)$ otherwise.

(e) If $A \cap (\{i\} \times S) \neq \square \neq A \cap (\{j\} \times S)$ with $i \neq j$, then $\sup A = (n,1)$, where $n = \max \{k \in \mathbb{N} \mid A \cap (\{k\} \times S) \neq \square\}$.

These cases are exhaustive and so $(T, \leq)$ is a complete lattice.

Proof of (1): For $n = 1$ the assertion follows immediately from the definition of $\uparrow p$. Assume for $n = k$. Suppose $x \leq(k+1) y$. Fix $i \in \mathbb{N}$ and $D \subseteq T$ such that $(i,y) \leq(1) \sup D$. If $1 \in D$, then $x \leq(k+1) y \implies x \leq(k) y \implies p \leq(k) (i,y)$ by the I.H. $\implies p \leq(k) 1$ by 1.5 $\implies p \leq(k) d$. If $1 \notin D$, then $D \cap (\{j\} \times S) \neq \square$ for some $j \geq i$ (since $\sup(\{1,\ldots,i-1\} \times S) = (i-1,1)$). If D meets at least one additional slices, then $(j,1) \in D$ for some $j \geq i$. Hence, again, $x \leq(k+1) y \implies x \leq(k) y \implies p \leq(k) (j,y)$ by the I.H. $\implies p \leq(k) (j,1)$ by 1.5 $\implies p \leq(k) d$. Finally, if D meets only the j- slice, then $y \leq(1) \sup(\pi_2[(\{j\} \times S) \cap D])$. (If $j = i$, this is easy, whereas if $j > i$, $\sup(\pi_2[(\{j\} \times S) \cap D]) = 1 \geq(1) y$.) Now, since $x \leq(k+1) y$, we have $x \leq(k) \pi_2(d)$ for some $d \in \{j\} \times S$. Hence, $p \leq(k) (j, \pi_2(d)) = d$. Therefore, in any case, $p \leq(k) d$ for some $d \in D$, so that $p \leq(k+1) (i,y)$ and we are finished.

Proof of (2): For $n = 1$ the assertion follows immediately from the definition of $\uparrow p$. Assume for $n = k$. Suppose $x \nleq(k+1) y$. Fix $i \in \mathbb{N}$. Then there exists D such that $y \leq(1) \sup D$ but

$x \not\leq(k)$ d for any d $\in$ D .   Now, $\{i\} \times$ D is an up-directed set

in T ,   (i,y) $\leq(1)$ (i,supD) = sup $\left[\{i\} \times D\right]$.   But, x $\not\leq(k)$ c

for all d $\in$ D implies, by the I.H., p $\not\leq(k)$ (i,d) for all d $\in$ D .

Hence, p $\not\leq(k+1)$ (i,y) .

<u>Proof of (3)</u>: First we show that p $\leq(n+1)$ 1 in T . Let D be

an up-directed set in T such that supD = 1 . If 1 $\in$ D , then

x $\leq(n)$ 1 in S implies (by (1)) p $\leq(n)$ (i,1) for all i $\in$ ℕ

and hence p $\leq(n)$ 1 . *(by 1.5)* If 1 $\notin$ D , then there exist i,j with i $\neq$ j

such that D $\cap$ ($\{i\} \times$ S) $\neq \square \neq$ D $\cap$ ($\{j\} \times$ S) . It follows that

D contains an element of the form d = (ℓ,1) for some $\ell \in$ ℕ .

But, x $\leq(n)$ 1 implies p $\leq(n)$ (ℓ,1) by (1) . Hence,

p $\leq(n+1)$ 1 .   Next we show that p $\not\leq(n+2)$ 1 in T . Let

D = $\left\{(i,1) \mid i \in ℕ\right\}$. Then supD = 1 . But x $\not\leq(n+1)$ 1 in S

implies p $\not\leq(n+1)$ (i,1) for all i $\in$ ℕ by (2) . Therefore,

p $\not\leq(n+2)$ 1 in T .   ∎

<u>Corollary 1.14</u>. There exists a complete lattice S such that

$\leq(n+1) \neq \leq(n)$ for all n $\in$ ℕ .

<u>Proof</u>. According to 1.13 and 1.4 , there is, for each n $\in$ ℕ ,

a complete lattice S_n such that $\leq(n+1) \neq \leq(n)$. Stack such
                                      S_n S_n

a collection almost any way you like (adding 0 and/or 1 if necessary)

and a suitable example is obtained. For example, if we stack them

up (with no identifications) and add a 1 to obtain T , then one

can show that $\leq(n) \cap (S_i \times S_i) = \leq(n)$ for all i,n $\in$ ℕ from
                T S_i

which it follows that $\leq(n+1) \neq \leq(n)$ for all n $\in$ ℕ . ∎
                        T   T