SCS 20: More on the Coproduct. Errata and Addenda

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Gierz and Keimel point out that the proof of Lemma 2.4 in Hofmann SCS 9-20-76 and provide a counterexample to the proof and the word "continuous" in the statement of Lemma 2.4.

Here is a new attack on the coproduct. It is not as explicit as one would like it to see (says JDL), but for the moment it is the best I can say. A good deal of this was presented to DS on a marble table of the Café du Monde near a mound of beignet.

**DEFINITION 1.** Let \( \text{POT} \) be the category of partially ordered topological spaces (with partial orders having closed graphs) and order preserving continuous functions. \( \text{FOC} \) is the full subcategory of compact Hausdorff partially ordered spaces.

**LEMMA 2.** The inclusion functor \( \text{FOC} \rightarrow \text{POT} \) has a left adjoint \( \omega : \text{POT} \rightarrow \text{FOC} \). I.e. for each pospace \( X \) there is a unique compact pospace \( \omega(X) \) and a \( \text{POT} \)-morphism \( \pi_X : X \rightarrow \omega(X) \) such that for every \( \text{POT} \)-morphism \( f : X \rightarrow Y \) into a compact pospace \( Y \) there is a unique \( f' : \omega(X) \rightarrow Y \) with \( f = f' \pi_X \). (In the words of the Herrlich–Strecker crowd, \( \text{FOC} \) is an reflective subcategory.)

Proof. Freyd's theorem. []

In the following I am running by a version of Gierz–Keimel, Section 3, "A LEMMA ON PRIMES" Houston J.Math.

For a compact pospace \( X \) let \( \Gamma^U(X) \subseteq \Gamma(X) \) be the subset of closed upper sets of \( X \), i.e. \( \Gamma^U(X) = \{ \; A \in \Gamma(X) : \uparrow A = A \} \). Then

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West Germany:  
TH Darmstadt (Gierz, Keimel)  
U. Tübingen (Mislove, Visit.)

England:  
U. Oxford (Scott)

USA:  
U. California, Riverside (Stralka)  
LSU Baton Rouge (Lawson)  
Tulane U., New Orleans (Hofmann, Mislove)  
U. Tennessee, Knoxville (Carruth, Crawley)
\( \Gamma^U(X) \) is closed under arbitrary intersections (sup in \( \Gamma(X) \)) and finite direct unions (inf in \( \Gamma(X) \)); as a sublattice of \( \Gamma(X) \) it is distributive. Then function \( A \mapsto \Gamma^U(A) \) is well defined (use closedness of the graph of \( \leq \)) and is a kernel operator. Its image is \( \Gamma^U(X) \). It preserves directed sups: Indeed for a filterbasis \( \{ A_j \} \) of compact subsets of \( X \) we note \( \bigcup_j A_j = \bigcap_j \Gamma^U(A_j) \) (use closedness of graph \( \leq \)). Hence by a Lemma due to Gierz and Keimel, loc. cit. and expanded in a SCS Keimel & l-76 l.1.13 we know that \( \Gamma^U(X) \subseteq \mathcal{CL} \) and that \( A \mapsto \Gamma^U(A) : \Gamma(X) \rightarrow \Gamma^U(X) \) is a \( \mathcal{CL} \)-morphism.

**Lemma 3.** For \( X \in \text{POC} \) we have \( \Gamma^U(X) \subseteq \mathcal{CL} \) and \( A \mapsto \Gamma^U(A) : \Gamma(X) \rightarrow \Gamma^U(X) \) is a \( \mathcal{CL} \)-quotient. \[ \]

**Proposition 4.** (Gierz, Keimel, A Lemma). The assignment \( X \mapsto \Gamma^U(X) : \text{POC} \rightarrow \mathcal{CL} \) is left adjoint to the inclusion. Specifically, for each \( \text{POC} \)-morphism \( f : X \rightarrow S \) into a \( \mathcal{CL} \)-object there is a unique \( f' : \Gamma^U(X) \rightarrow S \) in \( \mathcal{CL} \) with \( f(x) = f'(\Gamma(X)) \), namely, the morphism given by \( f'(A) = \inf f(A) \).

**Proof.** Let \( f : X \rightarrow S \) be given. There is a unique \( \mathcal{CL} \)-morphism \( f_0 : \Gamma(X) \rightarrow S \) such that \( f(x) = f_0([x]) \), namely the morphism given by \( f_0(A) = \inf f(A) \). Since \( f \) preserves order, \( |f(A)| \leq |f(A)| \) whence \( \mathcal{CL} \) \( f_0(A) = \inf f(A) = \inf |f(A)| \leq \inf f(A) \). Thus \( f_0(A) = f_0([A]) \), i.e. \( f_0 \) factors through \( A \mapsto \Gamma^U(A) : \Gamma(X) \rightarrow \Gamma^U(X) \) with \( f' \) as the second factor. The uniqueness of \( f' \) is clear. \[ \]

**Corollary 5.** \( \Gamma^U \circ \omega : \text{POT} \rightarrow \mathcal{CL} \) is left adjoint to the inclusion functor. \[ \]

Now let \( p : E \rightarrow B \) be a surjective \( \text{POT} \)-map with the identity relation as partial order on \( B \). (This means that \( \text{POT} \) elements \( x, y \in E \) with \( x \leq y \) have to lie in one and the same fiber \( F = p^{-1}(x) \). Clearly \( \omega(E) = \beta(B) \) with the finest partial order on \( \beta(B) \). We thus obtain a \( \text{POC} \)-map \( \omega(p) : \omega(E) \rightarrow \beta(B) \) which is clearly surjective. Once again the partial order of \( \omega(E) \) goes along the fibers. If \( b \in B \) is isolated, then \( \omega(p)^{-1}(b') = \omega(F_b) \)

where \( b' \) is the image of \( b \) in \( \beta(B) \). If \( B = \{ b \} \), this is clear, if
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not. Fix \( x \in E_b \) and make a \( \text{POT-retraction} \) \( F: E \to E_b \) over \( \eta \) by \( \delta E_b \to E_b \) be the identity and setting \( F(E \setminus E_b) = \{ x \} \).

If then we contemplate for a little while the commuting diagram

and conclude the assertion. Suppose that \( B \) is discrete, and that \( E \) is compact for all \( b \), then we remark from the preceding discussion that \( E \) is topologically and order theoretically embedded in \( \omega(E) \) under \( \eta_E \). In a way which we should perhaps make more precise some other time, the fiber \( \omega(E)_c \) over a point \( c \in B \setminus B \) is the space of all liftings of the unique ultrafilter \( \mathfrak{F} \) on \( B \) representing \( c \).

**Lemma 6.** Let \( J \) be a set and \( \{ T_j : j \in J \} \) a family in \( \mathcal{CL} \).

Let \( E \) be the topological and order theoretical sum (i.e. the coproduct of the family in \( \mathbb{POT} \)) of all \( T_j \) and let \( X = \omega(E) \). By the preceding discussion, \( X \) contains \( \text{maximal} \) \( E \) as an open dense subset which is order theoretically embedded.

Define \( c_j: T_j \to \Gamma^U(X) \) by \( c_j(t) = \uparrow t \times \{ j \} \). Then for any family \( f_j: T_j \to S \) of \( \mathcal{CL} \)-morphisms there is a unique \( \mathcal{CL} \)-morphism \( F: \Gamma^U(X) \to S \) such that \( f_j = F \circ c_j \) for all \( j \in J \).

**Proof.** We define \( f: E \to S \) by \( f(t,j) = f_j(t) \). Then \( f \in \mathbb{POT} \). The map \( c: E \to \Gamma^U(X) \), \( c(t,j) = c_j(t) = \uparrow t \times \{ j \} \) is the front adjunction of the left adjoint \( \Gamma^U \circ \omega \).
at the \( \text{POT-object} \ E \). Thus there is a unique \( \mathbb{CL} \)-map
\[
F: \Gamma^U(X) \longrightarrow S \text{ with } F \circ \mu = f, \ \text{i.e. with } F(t^t \times \{j\}) = f_j(t)
\]
for all \( t \in T_j \), \( j \in J \). \([9]

We can identify \( F \) more explicitly if we wish: \( f \) induces a unique \( \mathbb{CL} \)-morphism \( f': X = \omega(E) \longrightarrow S \); then \( F(A) = \inf f'(A) \) for all \( A \in \Gamma^U(X) \). \([9]

The preceding Lemma almost gives us the coproduct, but the maps \( c_j: T_j \longrightarrow \Gamma^U(X) \), although being continuous and preserving products, fail to be \( \mathbb{CL} \)-morphisms (if card \( J > 1 \)) since they do not respect identities. Define \( L \) (for lid) to be
\[
\inf\{c_j(l_j): j \in J\} = \{\langle l_j, j \rangle: j \in J\} \overset{\text{inf}}{\longrightarrow} S \text{ are as in Lemma 6, then } f_j: \Gamma^U(T_j) \longrightarrow S \text{ are unique extension of } f. \text{Thus }
\]
\[
\text{THEOREM 7. Let } \{T_j: j \in J\} \text{ be a family of } \mathbb{CL}-\text{objects. Let } E \text{ be the coproduct in } \text{POT} \text{ of the underlying pospaces } |T_j|
\}
\text{ (i.e. the disjoint union with the sum topology and the sum order) and let } L \text{ be the closure in } \omega(E) \text{ of the set of maximal elements in } E \subseteq \omega(E). \text{Then } \Gamma^U(\omega(E)) \cup L \text{ is the coproduct in } \mathbb{CL} \text{ of the family } \{T_j: j \in J\} \text{ with coprojections } t \longrightarrow (\text{image of t in } \omega(E)) \cup L: T_j \longrightarrow \Gamma^U(\omega(E)) \cup L. \]
\[
\text{In view of the construction of } \omega(E) \text{ for } E = \bigcup_j (T_j \times \{j\}) \text{ one needs to understand for a ultrafilter c on J the ultraproduct } \prod^C_j T_j \coloneqq \colim_U \prod^C_j T_j \text{ (the colimit maps being the restrictions). The colim is to be taken in } \mathbb{CL}. \]

Direct limits are not easily described in $\mathbb{CL}$; in fact their description is probably on the level of the description of a coproduct. It is true that direct limits could be calculated as inverse limits in $\mathbb{CL}^{\text{op}}$, but products in that category are bad (in fact they are coproducts in $\mathbb{CL}$).

There follows a brief report on the visit of Dana Scott to Tulane. (Submitted in gratitude for the detailed accounts we all received from the east side of the Atlantic on the supposedly considerable activity in Darmstadt during September.)

Dana Scott received this year's Turing Award of the Association of Computing Machinery which he shared with Michael Rabin. The award was presented this past week at Houston. Despite his many commitments at home in Oxford and his very tight travelling schedule he was so kind as to accept an invitation to stop over at Tulane. Dana arrived on Thursday evening; we checked in at the Cornstalk in the French quarter and sampled a few oysters at a Acme's and retreated to the Cafe du Monde, where we talked about coproducts, among other things. I had notified Jimmie Lawson, who fortunately could come over on Friday morning. From 11 to 2 (including the Lunch hour at the faculty dining room) we discussed continuous lattices, partly on the basis of earlier memos, notably $\mathbb{C}$ SCS Scott 8-23-76. Jimmie pointed out that the existence (now secured) of convex compact sets without extreme points opened up a wide class of non-$\mathbb{CL}$ objects in the form of the semilattice of closed convex subsets of such objects under the formation of closed convex hulls. We also talked about Smyth's work described in an earlier letter of Scott's. At 3 p.m. Dana spoke in the colloquium on "Tiling the plane with dominoes" complete with projected pictures and gadgets from a mathematical puzzle factory. Subsequently we had a colloquium lecture from Henryk Torunczyk from the Academy in Warsaw on $\mathbb{R}^2$ manifolds in which recent developments in infinite dimensional topology were surveyed. The department then adjourned to my house for cocktails, and the evening concluded with a dinner at the Versailles (recently included into the list of acceptable places to go among the local gourmets). Saturday morning Dana invited me to breakfast at the Four Seasons, after which we went to Tulane where he lectured for some two and a half hours to a group of aficionados on his own route to algebraic and continuous lattices and the rudiments of the lambda calculus. At 2 p.m. his all too brief visit came to an end with his departure for Miami and London. The Associate Secretary of the South Eastern Section of the American Mathematical Society approved the organisation of a Special Session on Topological Algebra and Lattice Theory at the Huntsville Meeting. Jimmie and I will begin the organisation and let you know as soon as things develop.