Seminar on Continuity in Semilattices

Volume 1 | Issue 1 Article 20

10-23-1976

SCS 20: More on the Coproduct. Errata and Addenda

Karl Heinrich Hofmann
Technische Universitat Darmstadt, Germany, hofmann@mathematik.tu-darmstadt.de

SEMINAR ON CONTINUITY IN SEMILATTICES (SCS)

NAME(S) K.H.Hofmann	DATE	M	D	Y
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TOPIC More on the coproduct. Errata and addenda.

REFERENCE SCS 9- 20-76 Hofmann ... lower semicontinuous functions..
SCS 10- 5-76 Keimel GIERz (to Hofmann only)

Gierz and Keimel point out that the proof of Lemma 2.4 ~~and~~ in Hofmann SCS 9-20-76 and provide a counterexample to the proof and the word " continuous" in the statement of Lemma 2.4.

Here is a new attack on the ~~xxxxxx~~ coproduct. It is not as ~~x~~ explicit as one would like it to see (says JDL), but for the moment it is the best I can say. A good deal of this was presented to DS on a marble table of the Café du Monde near a mound of beignet

DEFINITION 1. Let POT be the category of partially ordered topological spaces (with partial orderes having closed graphs) and order presering continuous functions. POC is the full subcategory of compact Hausdorff partially ordered spaces.

LEMMA 2. The inclusion functor POC ---> POT has a left adjoint ω : POT---> POC. I.e. for each pospace X there is a unique compact pospace $\omega(X)$ and a POT -morphism $\pi_X:X \longrightarrow \omega(X)$ such that for every POT-morphism $f:X \longrightarrow Y$ into a compact pospace Y there is a unique $f':\omega(X) \longrightarrow Y$ with $f = f'\pi_X$. (In the words of the Herrlich - Strecker crowd, POC is an ~~xxx~~reflective subcategory.)

Proof. Freyd's theorem.∎

In the following I am running by an alternate version of Gierz-Keimel ,Sectic 3, " A LEMMA ON PRIMES" Houston J.Math.

For a compact pospace X let $\Gamma^u(X) \subseteq \Gamma(X)$ be the subset of closed upper sets of X, i.e. $\Gamma^u(X) = \{ A \in \Gamma(X): \uparrow A = A\}$. Then

West Germany: TH Darmstadt (Gierz, Keimel)
U. Tübingen (Mislove, Visit.)

England: U. Oxford (Scott)

USA: U. California, Riverside (Stralka)
LSU Baton Rouge (Lawson)
Tulane U., New Orleans (Hofmann, Mislove)
U. Tennessee, Knoxville (Carruth, Crawley)

$\Gamma^u(X)$ is closed under arbitrary intersections (sups in $\Gamma(X)$!) and finite unions (infs in $\Gamma(X)$!); as a sublattice of $\Gamma(X)$ it is distributive. Then function $A \mapsto \uparrow A$ is well defined (use closedness of the graph of $\leq$) and is a kernel operator. Its image is $\Gamma^u(X)$. It preserves directed sups: Indeed for a filterbasis $\{A_j\}$ of compact subsets of X we note $\uparrow \cap_J A_j = \cap_J \uparrow A_i$ (use closedness of graph $\leq$). Hence by a Lemma due to Gierz and Keimel, loc. cit. and expanded in SCS Keimel 8-1-76 ,1.13, we know that $\Gamma^u(X) \in$ CL and that $A \mapsto \uparrow A: \Gamma(X) \longrightarrow \Gamma^u(X)$ is a CL-morphis

LEMMA 3. For $X \in$ POC we have $\Gamma^u(X) \in$ CL and $A \mapsto \uparrow A$: $\Gamma(X) \longrightarrow \Gamma^u(X)$ is a CL -quotient.□

PROPOSITION 4. (Gierz, Keimel, A LEMMA). The assignment $X \mapsto \Gamma^u(X)$: POC $\longrightarrow$ CL is left adjoint to the inclusion. Specifically, for each POC-morphism $f: X \longrightarrow S$ into a CL-object there is a unique $f': \Gamma^u(X) \longrightarrow S$ in CL with $f(x) = f'(\uparrow x)$, namely, the morphism given by $f'(A) = \inf f(A)$.

Proof. Let $f: X \longrightarrow S$ be given. There is a unique CL-morphism $f_o: \Gamma(X) \longrightarrow S$ such that $f(x) = f_o(\{x\})$, namely the morphism given by $f_o(A) = \inf f(A)$. Since f preserves order, $\uparrow f(A) \supseteq f(\uparrow A)$ whence $f_o(A) = \inf f(A) = \inf \uparrow f(A) \leq \inf f(\uparrow A) \leq$ $\inf f(A)$. Thus $f_o(A) = f_o(\uparrow A)$, i.e. f_o factors through $A \mapsto \uparrow A$: $\Gamma(X) \longrightarrow \Gamma^u(X)$ with f' as the second factor. The uniqueness of f' is clear.□

COROLLARY 5 . $\Gamma^u \circ \omega$: POT $\longrightarrow$ CL is left adjoint to the inclusion functor.□

Now let $p: F \longrightarrow B$ be a surjective POT -map with the identity relation as partial order on B, .(This means that elements $x, y \in E$ with $x \leq y$ have to lie in one and the same fiber $E_b = p^{-1}(b$ Clearly $\omega(B) = \beta(B)$ with the finest partial order on $\beta(B)$. We thus obtain a POC -map $\omega(p): \omega(E) \longrightarrow \beta(B)$ which is clearly surjective. Once again the partial order of $\omega(E)$ goes along the fibers. If $b \in B$ is isolated, then $\omega(p)^{-1}(b') \cong \omega(E_b)$ where b' is the image of b in βB: If $B = \{b\}$, this is clear, if

not, fix $x_o \in E_b$ and make a POT-retraction $F: E \longrightarrow E_b$ over $B \longrightarrow \{$
by letting $F|E_b : E_b \longrightarrow E_b$ be the identity and setting $F(E\setminus E_b) = \{x_o\}$.
~~If~~ Then we contemplate for a little while the commuting diagram

$$\begin{tikzcd}
E_b \arrow[rrr] \arrow[drr] \arrow[ddr, "1"'] \arrow[ddd] & & & \omega(E_b) \arrow[drr] \arrow[ddr, "1"] \arrow[ddd] & & \\
& & E \arrow[rrr] \arrow[dl] \arrow[ddd, "p"] & & & \omega(E) \arrow[dl] \arrow[ddd, "\omega(p)"] \\
& E_b \arrow[rrr] \arrow[ddd] & & & \omega(E_b) \arrow[ddd] & \\
\{b\} \arrow[rrr] \arrow[drr] \arrow[ddr] & & & \{\pi(b)\} \arrow[drr] \arrow[ddr] & & \\
& & B \arrow[rrr] \arrow[dl] & & & \beta(B) \arrow[dl] \\
& \{b\} \arrow[rrr] & & & \{\pi(b)\} &
\end{tikzcd}$$

and conclude the assertion. Suppose that B is dicrete, and that E_b is compact for all b, then we remark from the preceding discussion that E is topologically and order theoretically embedded in $\omega(E)$ under π_E. In a way which we should perhaps make more precise some other time, the fiber $\omega(E)_c$ over a point $c \in \beta B \setminus B$ is the space of all liftings (to E) of the unique ultrafilter on B representing c.

LEMMA 6. Let J be a set and $\{T_j : j \in J\}$ a family in CL. Let $E = \cup\{T_j \times \{j\} : j \in J\}$ be the topological and order theoretical sum (i.e. the coproduct) of the family in POT) of all T_j and let $X = \omega(E)$. By the preceding discussion, X contains E as an open dense subset which is order theoretically embedded.
Define $c_j : T_j \longrightarrow \Gamma^u(X)$ by $c_j(t) = \uparrow t \times \{j\}$. Then for any family $f_j : T_j \longrightarrow S$ of CL-morphisms there is a unique CL-morphis $F: \Gamma^u(X) \longrightarrow S$ such that $f_j = F \circ c_j$ for all $j \in J$.
Proof. We define $f: E \longrightarrow S$ by $f(t,j) = f_j(t)$. Then $f \in$ POT. The map $c: E \longrightarrow \Gamma^u(X)$, $c(t,j) = c_j(t) = \uparrow t \times \{j\}$ is the front adjunction of the left adjoint $\Gamma^u \circ \omega$

at the POT-object E. Thus there is a unique $\underline{CL}$-map $F: \Gamma^u(X) \longrightarrow S$ with $F \circ c = f$, i.e. with $F(\uparrow t \times \{j\}) = f_j(t)$ for all $t \in T_j$, $j \in J$.□

We can identify F more explicitly if we wish: f induces a unique POC-morphism $f': X = \omega(E) \longrightarrow S$; then $F(A) = \inf f'(A)$ for all $A \in \Gamma^u(X)$.□

The preceding Lemma almost gives us the coproduct, but the maps $c_j: T_j \longrightarrow \Gamma^u(X)$, although being continuous and preserving products, fail to be CL-morphisms (if card $J > 1$) since they do not respect identities. Define L (for lid) to be $\inf\{c_j(1_j): j \in J\} = \{(1_j, j): j \in J\}^- \in \Gamma^u(X)$. If the $f_j: T_j \longrightarrow S$ are as in Lemma 6, then $F(\{(1_j, j)\}) = f_j(1_j) = 1$ (since $f_j \in \underline{CL}$), Hence $F(L) = 1$. This means that $F: \Gamma^u(X) \longrightarrow S$ factors uniquely through the CL-morphism $A \longmapsto A \cup L: \Gamma^u(X) \longrightarrow \Gamma^u(X) \cup L$. The maps $copr_j: T_j \longrightarrow \Gamma^u(X) \cup L$ given by $copr_j(t) = c_j(t) \cup L$ are CL-morphisms. By Lemma 6, for each family $f_j: T_j \longrightarrow S$ of CL-morphisms there is a unique CL- morphism $\varphi: \Gamma^u(X) \cup L \longrightarrow S$ such that $f_j = \varphi \circ copr_j$ for all $j \in J$, and $\varphi(A) = \inf f'(A)$, where $f': X \longrightarrow S$ is the unique extension of f. Thus

THEOREM 7. Let $\{T_j: j \in J\}$ be a family of CL-objects. Let E be the coporduct in POT of the underlying pospaces $|T_j|$ (i.e. the disjoint union with the sum topology and the sum order) and let L be the closure in $\omega(E)$ of the set of maximal elements in $E \subseteq \omega(E)$. Then $\Gamma^u(\omega(E)) \cup L$ is the coproduct in CL of the family $\{T_j: j \in J\}$ with coprojections $t \longmapsto (\text{image of } \uparrow t \text{ in } \omega(E)) \cup L: T_j \longrightarrow \Gamma^u(\omega(E)) \cup L$.□

In view of the construction of $\omega(E)$ for $E = \bigcup_J (T_j \times \{j\})$ one needs to understand for a ultrafilter c on J the ultraproduct $\prod_J^c T_j = \operatorname{colim}_{U \in c} \prod_U T_j$ (the colimit maps being the restrictions). The colim is to be taken in CL-./

Direct limits are not easily described in CL ; in fact their description is probably on the level of the descritpion of a coproduct. It is true that direct limits could be calculated as inverse limits in CL^{op} ,but products in that category are bad (in fact they are coproducts in CL).

There follows a brief report on the visit of Dana Scott to Tulane. (Submitted in gratitude for the detailed accounts we all received from the east side of the Atlantic on the supposedly considerable activity in Darmstadt during September.)

Dana Scott received this years A.M. Turing Award of the Association of Computing Machinery which he shared with Michael Rabin. The award was presented this past week at Houston. Despite his many commitments at home in Oxford and his very tight travelling schedule he was so kind as to accept an invitation to stop over at Tulane. Dana arrived on Thursday evening; we checked in at the Cornstalk in the French quarter and sampled a few oysters at Acme's and retreated to the Cafe du Monde, where we talked about coproducts,among other things. I had notified Jimmie Lawson, who fortunately could come over on Friday morning. From 11 to 2 (including the Lunch hour at the faculty dining room) we discussed continuous lattices,partly on the basis of earlier memos,notably SCS Scott 8-23-76. Jimmie pointed out that that the existence (now secured) of convex compact sets without extreme points opened up a wide class of non-CL CS -objects in the form of the semilattice of closed convex subsets of such objects under the formation of closed convex hulls. We also talked about Smyth's work described in an earlier letter of Scott's. At 3 p.m. Dana spoke in the colloquium on " Tiling the plane with dominoes" complete with projected pictures and gadgets from a mathematical puzzle factory. Subsequently we had a colloquium lecture from Henryk Toruncyk from the Academy in Warsaw on 1^2 -manifolds in which recent developments in infinite dimensional topology were surveyed. The department then adjourned to my house for cocktails, and the evening concluded with a dinner at the Versailles(recently included into the list of acceptable places to go among the local gourmets). Saturday morning Dana invited me to breakfast at the Four Seasons, after which we went to Tulane where he lectured for some two and a half hours to a group of aficionadas on his own route to algebraic and continuous lattices and the rudiments of the lambda calculus. At 2 p.m. his all too brief visit came to an end with his departure for Miami and London.-

The Associate Secretary of the South Eastern Section of the American Mathematical Society approved the organisation of a Special Session on Topological Algebra and Lattice Theory at the Huntsville Meeting. Jimmie and I will begin the organisation and let you know as soon as things develop.