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SCS 17: The Space of Lower Semicontinuous Functions into a CL-Object, Applications (Part I): Copowers in CL

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SEMINAR ON CONTINUITY IN SEMILATTICES (SCS)

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TOPIC The space of lower semicontinuous functions into a CL-object Applications (part I): Copowers in CL

REFERENCE

[0] Handwritten notes on discussions by Gierz, Hofmann, Keimel, Mislove at Darmstadt in June 1976.

[1] Hofmann, K.H. and J.D. Lawson, Irreducibility and generation in continuous lattices. Preprint.

[2] Hofmann, K.H. and A. Stralka, ATLAS, Diss. Math. 137 (1976), 1-54

In Darmstadt this summer I raised the question of calculating copowers in CL; we knew at the time that

${}^{J}2 = \Gamma(\beta J)$, where $\Gamma(X)$ for a compact space X is the $\cup$-semilattice of compact subsets and where the CL-topology is the Hausdorff topology. We had no particular idea what such simple coproducts as ${}^{J}I$ might be. Then Keimel had the insight that ${}^{J}I$ should be calculated by considering the cone with basis βJ; then the closed subsets containing the vertex and being star shaped would be the elements of the desired copowdr with $\cup$ as operation. This turned out to be correct as we proved at the time. An explicit discussion of this approach is given in an example in [1] where this information was needed and serves a useful purpose.

We thought at the time that arbitrary copowers should be calculated in an essentially similar fashion. However, there are some technical difficulties with copowers of CL-objects which are not chains. The present discussion proposes an approach which probably best accomodates these difficulties; in a philosophical way, such an ~~app~~ approach had been indicated in conversations in Darmstadt, although it was then not seriously attempted.

We actually develop a theory of function spaces of lower semicontinuous functions $f: X \longrightarrow S$, X compact, $S \in$ CL. The totality of all of these functions, which we call $LC(X,S)$ turns out to be a continuous lattice in a functorial fashion. The theory ~~of~~ around this concept is discussed in Section 1. Section 2 applies this to copowers. Further applications are to be discussed later. The result on copowers is that for any CL-object S we have

$${}^{J}S \cong LC(\beta X, S).$$

The coproejctions and the universal morphisms are explicitly given.

West Germany: TH Darmstadt (Gierz, Keimel)
U. Tübingen (Mislove, Visit.)

England: U. Oxford (Scott)

USA: U. California, Riverside (Stralka)
LSU Baton Rouge (Lawson)
Tulane U., New Orleans (Hofmann, Mislove)
U. Tennessee, Knoxville (Carruth, Crawley)

1. Lower semicontinuous functions.

1.1. LEMMA . Let X be a topological space and $S \in \underline{CL}$. Let $x \in X$ and let $\mathcal{U}$ denote the filter basis of open neighborhoods of x in X. Then the following conditions are equivalent:

(1) $\underline{\lim}\, f(x_j) \geq f(x)$ for every net x_j in X converging to x.

(2) $\uparrow f(x) \supseteq \bigcap\{ f(U)^- : U \in \mathcal{U} \}$.

(3) For each $s \ll f(x)$ there is an $U \in \mathcal{U}$ such that $f(U) \subseteq \uparrow s$.

(3') For each $s \ll f(x)$ there is a $U \in \mathcal{U}$ such that $s \ll f(u)$ for $u \in U$.

Now we denote with $G(f)$ the set $\{(x,s): f(x) \leq s\}$. Then the following conditions are equivalent:

(I) Conditions (1)-(3) above hold for all $x \in X$.

(II) $f^{-1}(\mathrm{int} \uparrow s)$ is open for all $s \in S$.

(III) $G(f)$ is closed.

Proof. (3) =>(2): For each $s \ll f(x)$ we know from (3) that $f(U) \subseteq \uparrow s$ for some $U \in \mathcal{U}$, hence $f(U)^- \subseteq \uparrow s$, and so $\bigcap_{\mathcal{U}} f(U)^- \subseteq \uparrow s$. Since $s \ll f(x)$ is arbitrary and $f(x) = \sup \Downarrow f(x)$, (2) follows.

(2) => (1). Suppose $x = \lim x_j$. Then eventually $f(x_j) \subseteq f(U)^-$ for all $U \in \mathcal{U}$. So every cluster point of $f(x_j)$ is in $\bigcap_{\mathcal{U}} f(U)^-$. Hence (1).

not (3) => not (1): Suppose there is an $s \ll f(x)$ such that for each $U \in \mathcal{U}$ we had $f(U) \not\subseteq \uparrow s$. Then there exists an $x_U \in U$ for each $U \in \mathcal{U}$ such that $f(x_U) \not\geq s$. Since $\uparrow f(x)$ is in the interior of $\uparrow s$, not cluster point of $f(x_U)$ is in $\uparrow f(x)$, which implies $\underline{\lim}\, f(x_U) \not\geq f(x)$.

(3') => (3) is trivial and (3) => (3') since for $s \ll f(x)$ there is an $s' \in S$ with $s \ll s' \ll f(x)$.

(II) <=> (2) for all $x \in X$: Clear in view of $\mathrm{int} \uparrow s = \{t \in S : s \ll t\}$

(1) for all $x \in X$ => (III): Suppose that $(x,s) = \lim (x_j, s_j)$ with $f(x_j) \leq s_j$. Then there is a subnet such that $\underline{\lim}\, (x_j, s_j) = \lim (x_{j(k)}, s_{j(k)})$. The validity of (1) for x implies $f(x) \leq \underline{\lim}\, f(x_j) = \lim f(x_{j(k)}) \leq s$.

(III) => (1) for all x. Suppose $x = \lim x_j$ in X. Let s be any cluster point of $f(x_j)$ in S, say $s = \lim f(x$ ... $)$. Then

$(x,s) = \lim (x_{j(k)}, f(x_{j(k)})$ and obviously $(x_{j(k)}, f(x_{j(k)}) \in G(f)$. Thus (III) implies $(x,s) \in G(f)$, i.e. $f(x) \leq s$. This means $f(x) \leq \underline{\lim}\, f(x_j)$ since s was an arbitrary cluster point. □

1.2. DEFINITION. A function $f: X \longrightarrow S$ is called *lower semicontinuous* iff the equivalent conditions (I) – (III) of 1.1 are satisfied. The set of all lower semicontinuous functions will be denoted with $LC(X,S)$.

1.3. LEMMA. Let $\mathcal{F} \subseteq S^X$, then $G(\sup \mathcal{F}) = \bigcap\{G(f) : f \in \mathcal{F}\}$.
Proof. Since $f \leq \sup \mathcal{F}$, we have $G(\sup \mathcal{F}) \subseteq g(f)$ for all $f \in \mathcal{F}$, whence $G(\sup \mathcal{F}) \subseteq \bigcap_{\mathcal{F}} g(f)$. If $(x,s) \in \bigcap_{\mathcal{F}} G(f)$ then $f(x) \leq s$ for all $f \in \mathcal{F}$, thus $(\sup \mathcal{F})(x) \leq s$, whence $(x,s) \in G(\sup \mathcal{F})$. □

1.4. LEMMA. Let $f,g \in LC(X,S)$. Then $fg \in LC(X,S)$ where $(fg)(x) = f(x)g(x) = f(x) \wedge g(x)$.
Proof. Let $x \in X$ and $s \ll fg(x)$. Since $f,g \in LC(X,S)$, by 1.1.(3) there is an open neighborhood U of x in X such that $f(U) \cup g(U) \subseteq \uparrow s$. Then $s \leq f(u)g(u) = fg(u)$ for all $u \in U$, which verifies 1.1.(3) for fg. □

1.5. PROPOSITION. Let X be topological space and $S \in \underline{CL}$. Then $LC(X,S)$ is a sublattice of S^X containing the identity and zero, and $LC(X,S)$ is closed under the formation of arbitrary sups. In particular, $LC(X,S)$ is a complete lattice.
Proof. In view of 1.1.(III), Lemma 1.3 shows that $LC(X,S)$ is closed under arbitrary sups. Lemma 1.4 shows that $LC(X,S)$ is closed under finite infs. □

REMARK. In general, $LC(X,S)$ is not closed under arbitrary infs: Let $x \in X$ be a non-isolated point in some topological Hausdorff space, $S = 2$. Then the inf of the characteristic functions χ_U, $U \in \mathcal{U}$ (where $\mathcal{U}$ is the set of open neighborhoods of x) is $\chi_{\{x\}}$, which is not lower semicontinuous.

It is very convenient for the following to consider characteristi functions of subsets of X:

1.6. NOTATION . If X is a set and $S \in$ CL , the for each $Y \subseteq X$ the function $\chi_Y : X \longrightarrow S$ is defined by $\chi_Y(x) = 1$ if $x \in Y$ and $= 0$ otherwise. If $s \in S$ we identify s with the constant function $X \longrightarrow S$ with value s and write $s\chi_Y$ for the function taking the value s on Y and 0 elsewhere. []

Note that $s\chi_U \in LC(X,S)$ for a topological space X and an open subset $U \subseteq X$.

1.7. PROPOSITION. Let X be a compact topological space and $S \in$ CL. If $f,g \in LC(X,S)$ then the following statements are equivalent:

(1) $f \ll g$. (2) For each $x \in X$ there is an open neighborhood $U=U(x)$ of x in X and an $s = s(x) \in S$ such that

$$f(u) \leq s \ll g(u) \text{ for all } u \in U$$

(i.e. $f(U) \subseteq \downarrow s$ and $g(U) \subseteq \operatorname{int} \uparrow s$)

(3) $G(g) \subseteq \operatorname{int} G(f)$.

Proof. (3) <=> (2) : (3) means that for every $x \in X$ there is a basic open set of the special form $U \times \operatorname{int} \uparrow s$ containing $g(x)$ and being contained in $G(f)$ .But this is precisely (2).

(2) => (1): Suppose that h_j is an up-directed net in $LC(X,S)$ whose sup h dominates g. Then for each x we have $g(x) \leq h(x) = \lim h_j(x)$. Thus there is a $j = j(x)$ with $s(x) \ll h_j(x)$, and since h_j is lower semicontinuous there is an open set $V=V(x) \subseteq U(x)$ such that $s \ll h_j(v)$ for all $v \in V$. By the compactness of X we find finitely many $x_1,\ldots,x_n$ such that $X = V(x_1) \cup \ldots \cup V(x_n)$. Let k be an index with $k \geq j(x_1),\ldots,j(x_n)$. Since h_j is up-directed, we conclude that for each $x \in X$ there is an $i \in \{1,\ldots,n\}$ with $f(x) \leq s(x_i) \ll h_k(x)$

In particular $f \leq h_k$. This proves (1).

(1) =>(2): Let $\mathcal{F}(g)$ be the set of all functions $s\chi_U$ such that (i) U is open in X, (ii) $s \ll g(x)$ for all $x \in \bar{U}$ (!!). By (i) we have $\mathcal{F}(g) \subseteq LC(X,S)$. Since X is regular and 1.1.(3') applies to g, we know that (iii) $g = \sup \mathcal{F}(g)$ in LC(X,S). Hence, by the definition of $f \ll g$ there is a finite collection $\{s_i\chi_{U_i}: i=1,\ldots,n\} \subseteq \mathcal{F}(g)$ with (iv) $f \leq \sup_i s_i\chi_{U_i}$. Now let us take an arbitrarz $x \in X$. Let $I(x)=\{i: i \in \{1,\ldots,n\}$ and $x \in \bar{U}_i\}$. Since $s_i \ll g(y)$ for all $y \in \bar{U}_i$ by (ii) above, $i \in I(x)$ implies $s_i \ll g(x)$. If we set $s(x) = \sup\{s_i: i \in I(x)\}$ then also (v) $s(x) \ll g(x)$ since $\downarrow g(x)$ is closed under finite sups. ~~There is an s' ∈ S with s(x) << s' << g(x).~~ The set $V(x) = X \setminus \bigcup\{\bar{U}_i: i \in \{1,\ldots,n\} \setminus I(x)\}$ is an open neighborhood of x. By 1.1.(3') we find an open neighborhood $U(x) \subseteq V(x)$ such that $u \in U(x)$ implies ~~s' ≤ g(u), hence~~ $s(x) \ll g(u)$. But $u \in U(x)$ implies that $u \notin U_i$ for $i \notin I(x)$ whence $f(u) \leq \sup\{s_i\chi_{U_i}(u): i=1,\ldots,n\} = \sup\{s_i\chi_{U_i}(u): i \in I(x)\}$ $= s(x)$. This proves condition (3).$\square$

Note that it is possible that $I(x) = \emptyset$.

1.8. LEMMA. Let X be compact and $f \in LC(X,S)$. Then

$$f = \sup\{g \in LC(X,S): g \ll f\}.$$

Proof. As was observed earlier, f is the sup of the family of all $s\chi_U \in LC(X,S)$ such that $s \in S$, U is open in X and $s \ll f(u)$ for all $u \in U$. (Use 1.1.(3).) But by Proposition 1.7 every such $s\chi_U$ satisfies the relation $s\chi_U \ll f$. This proves the Lemma.$\square$

1.9. RECALL. Let $T \in$ CL and $t \in T$, and t_j a net. Then the following statements are equivalent: (1) $t = \lim t_j$. (2) $t = \sup_j \inf\{t_k: j \leq k\}$.$\square$

1.10. THEOREM. Let X be a compact Hausdorff space and S a CL-object. Then

(i) LC(X,S) is a CL-object;

(ii) $f \ll g$ iff for each $x \in X$ there is an open set U and an $s \in S$ such that $f(u) \leq s \ll g(u)$ for all $u \in U$;

(iii) if $f \in LC(X,S)$ and f_j is a net in LC(X,S) then $f = \lim f_j$ iff $f = \sup_j \inf^{LC}\{f_k: j \leq k\}$.

If X is zero dimensional, then $f \ll g$ iff there is a locally constant function h with $f(x) \leq h(x) \ll g(x)$ for all $x \in X$.

Proof. (i) follows from 1.5 and 1.8. (ii) is a portion of 1.7. (iii) follows from 1.9. If X is zero dimensional, then there is a cover of X by disjoint compact open sets $V_1, \ldots, V_n$ which refines the cover $\{U(x): x \in X\}$ (notation as in 1.7 and its proof). For each $j \in \{1,\ldots,n\}$ find x so that $V_j \subseteq U(x)$ and set $s_j = s(x)$. Then $v \in V_j$ implies $f(v) \leq s_j \ll g(v)$. Define $h: X \longrightarrow S$ by $h(x) = s_j$ iff $x \in V_j$. ■

We add ~~conclude the section with~~ some remarks on the functorial properties of $(X,S) \longmapsto LC(X,S)$: Comp x CL $\longrightarrow$ CL

1.11. LEMMA. Let $\varphi: X \longrightarrow Y$ be a continuous function of compact spaces. For every $f \in LC(Y,S)$ the function $f \circ \varphi: X \longrightarrow S$ is lower semicontinuous. Let $\varphi^*: LC(Y,S) \longrightarrow LC(X,S)$ be the function defined by $\varphi^*(f) = f \circ \varphi$. Then $\varphi^* \in \underline{CL}^{op}$.

Proof. From 11.(II), $f \circ \varphi$ is lower semicontinuous if f is. Thus φ^* is well-defined. Since sups are calculated pointwise in $LC(Y,S)$ and $LC(X,S)$, clearly φ^* preserves arbitrary sups. ■ It remains to show that $f \ll g$ in $LC(Y,S)$ implies $\varphi^*(f) \ll \varphi^*(g)$ in $LC(X,S)$. We consider the commutative diagram

$$\begin{tikzcd}
LC(Y,S) \arrow[r, "\varphi^*"] \arrow[d, "G_Y"'] & LC(X,S) \arrow[d, "G_X"] \\
\Gamma(Y \times S) \arrow[r, "A \mapsto (\varphi \times 1_S)^{-1}(A)"'] & \Gamma(X \times S)
\end{tikzcd}$$

$G_Y(f) = \{(y,s) \in Y \times S : f(y) \leq s\}$,

G_X defined similarly.

(Indeed $(x,s) \in (\varphi \times 1_S)^{-1}(G_Y(f))$ iff $(\varphi(x), s) \in G_Y(f)$ iff $f(\varphi(x)) \leq s$ iff $(x,s) \in G_X(\varphi^*(f))$!)

Now if M,N are compact spaces and $\varphi: M \longrightarrow N$ is a continuous map Then the function $\varphi': \Gamma(N) \longrightarrow \Gamma(M)$ given by $\varphi'(A) = \varphi^{-1}(A)$ satisfies the condition $\varphi'(A) \ll \varphi'(B)$, whenever $A \ll B$ where $A \ll B$ means $B \subseteq \text{int } A$; ~~since~~ for $\varphi^{-1}(B) \subseteq \varphi^{-1}(\text{int } A) \subseteq \text{int } \varphi^{-1}(A)$ by the continuity of φ. Since the ~~diagra~~ maps G_Y and G_X are injective and preserve $\ll$, by 1.7 we conclude that φ^* preserves $\ll$. □

1.12. NOTATION. In the context of 1.11 the left adjoint of φ^*, which is given by $f \longmapsto \sup \{ g \in LC(Y,S) : g \circ \varphi \leq f \}$, $f \in LC(X,S)$, will be denoted by $LC(\varphi,S): LC(X,S) \longrightarrow LC(Y,S)$ (somewhat contrary to the customary notation used in the case of the cofunctor $C(-,Z)$.)

1.13. LEMMA. Let X be a compact space and $\pi: S \longrightarrow T$ a CL-morphism. Then $LC(X,\pi): LC(X,S) \longrightarrow LC(X,T)$, $LC(X,\pi)(f) = \pi \circ f$ is well -defined and a CL-morphism.

Proof. Let $\delta: T \longrightarrow S$ be the right adjoint of π. Then $\pi(s) \geq t$ iff $s \geq \delta(t)$ for $(s,t) \in S \times T$; hence $\pi \circ f \geq g$ iff $f \geq \delta \circ g$ for $(f,g) \in S^X \times T^X$. Now $\delta: T \longrightarrow S$ is lower semicontinuous [2, ATLAS 1.20, p.15]. Hence $\delta \circ g \in LC(X,S)$ for all $g \in LC(X,T)$. Since δ preserves sups, so does $g \longmapsto \delta \circ g$. Since δ preserves $\ll$, by 1.7.(2), $g \longmapsto \delta \circ g$ also preserves the way below relation $\ll$. Hence $g \longmapsto \delta \circ g: LC(X,T) \longrightarrow LC(X,S)$ is a CL^{op}-morphism [2, ATLAS 1.20] Thus its left adjoint $LC(X,\pi)$ is a CL-morphism. □

As a consequence of 1.11-1.13 we record:

1.14. PROPOSITION. $LC(-,-)$: Comp $\times$ CL $\longrightarrow$ CL is a functor. □

Note that it is a bit curious that we have COVARIANCE in both arguments; you would normally expect contravariance in the left hand argument.

One further remark:
The map $G: LC(X,S) \longrightarrow \Gamma(X \times S)$, $G(f) = \{(x,s): f(x) \leq s\}$ preserves arbitrary sups by 1.3 and the way below relation $\ll$ by 1.7. Hence it is a CL^{op}-morphism, for what it is worth. What is its left adjoint? Let $A \subseteq \Gamma(X \times S)$. Define $L(A): X \longrightarrow S$ by $L(A) = \sup\{f \in LC(X,S): G(f) \supseteq A\}$. According to [2, ATLAS], this is the required left adjoint. Thus

1.15. PROPOSITION. The map $L: \Gamma(X \times S) \longrightarrow LC(X,S)$ given by $L(A)(x) = \sup\{f(x): f \in LC(X,S) \text{ with } A \subseteq G(f)\}$, $x \in X$ is a surjective CL -morphism. []

1.16 LEMMA. Let $S, T \in$ CL, then any monotone Scott continuous function $f: S \longrightarrow T$ is lower-semicontinuous.
Proof. Let $x \in S$ and $t \ll f(x)$. Since $x = \sup \Downarrow x$ and f preserves sups of up-directed sets we have $f(x) = \sup\{f(y): y \ll x\}$. By the definition of $\ll$ there is a $y \ll x$ with $t \leq f(y)$. Let U be the open set $\operatorname{int} \uparrow y$. Then U is a neighborhood of x and $u \in U$ implies $t \leq f(y) \leq f(u)$. Thus by 1.1(3) the assertion follows. []

1.17 COROLLARY. $[S \longrightarrow T] \subseteq LC(S,T)$. □

In a later memo we should discuss this inclusion further and resolve such questions as the following: Is $[S \longrightarrow T]$ closed in $LC(S,T)$? There are probably linkts to such matters as the random unit interval (SCS Hofmann and Liukkonen 9-1-76).

2. APPLICATIONS I. The copowers.

2.1. DEFINITION. Let X be a compact space and $S \in \underline{CL}$, $T \in \underline{CL}$. A hemimorphism $F\colon X \times S \longrightarrow T$ is a lower semi-continuous function such that $s \longmapsto F(x,s)\colon S \longrightarrow T$ is in $\underline{CL}$ for all $x \in X$.

For each pair (x,s) we denote with $\Delta(x,s)\colon X \longrightarrow S$ the function given by

$$\Delta(x,s)(y) = \begin{cases} s & \text{if } y = x \\ 1 & \text{otherwise} \end{cases}.$$

2.2. REMARK. $\Delta\colon X \times S \longrightarrow LC(X,S)$ is a hemimorphism.

Proof. We have $G(\Delta(x,s)) = (X \times \{1\}) \cup (\{x\} \times \uparrow s)$. Clearly $(x,s) \longmapsto G(\Delta(x,s))\colon X \times S \longrightarrow \Gamma(X \times S)$ is lower semi-continuous. Now $LG(\Delta(x,s)) = \Delta(x,s)$ where L is as in 1.15. Since L is continuous, Δ is continuous. The rest is clear. $\square$

2.3. PROPOSITION. Let X be a compact space, S, $T \in \underline{CL}$. For each hemimorphism $F\colon X \times S \longrightarrow T$ and each $f \in LC(X,S)$ we write $\phi(f) = \phi_F(f) = \inf_{x \in X} F(x, f(x)) \in T$. Then

(i) $\phi\colon LC(X,S) \longrightarrow T$ is a $\underline{CL}$-morphism,

(ii) the diagram

$$\begin{tikzcd} X \times S \arrow[r, "\Delta"] \arrow[dr, "F"'] & LC(X,S) \arrow[d, "\phi_F"] \\ & T \end{tikzcd}$$

commutes,

(iii) ϕ is the only $\underline{CL}$-morphism making the diagram in (ii) commutative.

In other words, there is a canonical bijection $F \longmapsto \phi_F$: $\mathrm{Hem}(X \times S, T) = \underline{CL}(LC(X,S), T)$.

Proof. First we prove (ii): $F(y, \Delta(s,x)(y)) = F(x,s)$ if $y = x$

and $= 1$ if $y \not\leq x$. Thus $\phi(\Delta(x,s) = F(x,s)$. Assertion (iii) is clear from the fact that $\{\Delta(x,s): (x,s) \in X \times S\}$ is an order generating set of $LC(X,S)$ (and in particular a generating set). Remains to show (i): We calculate the left adjoint $d: T \longrightarrow LC(X,S)$ of ϕ. Let $f \in LC(X,S)$, $t \in T$. Then $\phi(f) \geq t$ iff $\inf_{x \in X} F(x,f(x)) \geq t$ iff $F(x,f(x)) \geq t$ for all x, iff $f(x) \geq \inf\{s \in S: F(x,s) \geq t\}$ for all $x \in X$ [since $s \longmapsto F(x,s)$ is in $\underline{CL}$]. So we define $d(t)(x) = \inf\{s \in S: F(x,s) \geq t\}$. Since $G(d(t)) = \{(x,s): d(t)(x) \leq s\} = F^{-1}(\uparrow t)$ and since F is lower semi-continuous, $G(d(t))$ is closed, whence $d(t) \in LC(X,S)$ by 1.1.(III). Since $\phi(f) \geq t$ iff $f \geq d(t)$, d is the left adjoint of ϕ. By [2, ATLAS] it suffices to show now that $t \ll t'$ implies $d(t) \ll d(t')$, which, according to 1.7 is equivalent to $G(d(t') \subseteq \operatorname{int} G(d(t))$. i.e. to $F^{-1}(\uparrow t') \subseteq F^{-1}(\uparrow t)$. But this follows from the upper semi-continuity of F in view of $t \ll t'$ iff $t' \in \operatorname{int} \uparrow t$, i.e. $\uparrow t' \subseteq \operatorname{int} \uparrow t$ (See 1.1.(II)). $\square$

2.4. LEMMA. Let J be a set and $S \in \underline{CL}$. Suppose that $\{f_j : j \in J\}$ is a family of morphisms $f_j: S \longrightarrow T$. Then there is a unique continuous hemimorphism $F: J \times S \longrightarrow T$ such that the diagram

$$\begin{tikzcd}
J \times S \arrow[r] \arrow[dr, "(j,s) \mapsto f_j(s)"'] & \beta J \times S \arrow[d, "F"] \\
& T
\end{tikzcd} \qquad (D)$$

commutes.

Proof. The existence of a continuous function F making (D) commutative is immediate from the fact that for a compact S the space $\beta J \times S$ is canonically homeomorphic to $\beta(J \times S)$. Let $x \in \beta J$, then there is a net $j_\alpha \in J$ converging to x (where we identify J with a subset of βJ in the obvious fashion). If $s,t \in S$ then $F(x,s)F(x,t) = \lim F(j_\alpha,s) \lim F(j_\alpha,t) = \lim f_{j_\alpha}(s) f_{j_\alpha}(t) = \lim f_{j_\alpha}(st) = F(x,st)$. $\square$

Now we are ready to calculate arbitrary copowers of an arbitrary CL-object S.

2.5 . THEOREM . Let J be a set and $S \in CL$. Then the copower ${}^J S$ in CL is canonically isomorphic to $LC(\beta J, S)$, and the j-th coprojection is given by $s \longmapsto \Delta(j,s): X \longrightarrow LC(X,S)$.

Specifically, let $\{\varphi_j : j \in J\}$ be a family of morphisms $\varphi_j : S \longrightarrow T$ in CL. Then there is a unique morphism $\phi : LC(X,S) \longrightarrow T$ such that $\varphi_j(s) = \phi(\Delta(j,s))$ for all $j \in J$ and $s \in S$; moreover ϕ is given by $\phi(f) = \inf_{j \in J} \varphi_j(f(j))$.

Proof. By 2.4 we obtain a unique hemimorphism $F: J \times S \longrightarrow T$ extending the function $(j,s) \longmapsto \varphi_j(s)$. By 2.3 there is a unique morphism $\phi = \phi_F : LC(X,S) \longrightarrow T$ with $F = \phi\Delta$. Thus ϕ is a unique morphism satisfying $\varphi_j(s) = \phi(\Delta(j,s))$ for all $(j,s) \in J \times S$. By 2.3 we have $\phi(f) = \inf_{x \in \beta J} F(x, f(x))$. Since J is dense in βJ, we conclude $\phi(f) = \inf_{j \in J} F(j, f(j)) = \inf_{j \in J} \varphi_j(f(j))$, since f and hence $x \longmapsto F(x, f(x))$ is lower semicontinuous.□

We should remember that knowing co-powers give us a pretty good hold on co-products in general. If J is a set, then $\{S_j : j \in J\} \longrightarrow \coprod_J S_j : CL^J \longrightarrow CL$ is a functor. The retraction diagram

$$\begin{tikzcd} S_k \arrow[r, "can_k"] \arrow[dr, "1_{S_k}"'] & \prod_J S_j \arrow[d, "pr_k"] \\ & S_k \end{tikzcd}$$

then induces a retraction diagram

$$\begin{tikzcd} \coprod_J S_k \arrow[r, tail] \arrow[dr, "1"'] & {}^J(\prod_J S_j) \arrow[d] \\ & \coprod_J S_k \end{tikzcd}$$

which pretty much reduces the question of coproducts to products and copowers. In fact in such categories as CL the co-product $\coprod_J S_k$ is the identified with that subobject of ${}^J(\prod_J S_j)$ which is generated by the iamges of $S_k \xrightarrow{can_k} \prod_J S_j \xrightarrow{copr_k} {}^J(\prod_J S_j)$. At this point we do not elaborate further what this means in 2.5.