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SCS 15: Continuous Lattices and Universal Algebra

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TOPIC Continuous Lattices and Universal Algebra

REFERENCE (Conversations at Oberwolfach)

1. Continuous lattices as an equational class. In his paper [1], Alan Day points out that among the complete lattices the continuous ones can be defined as those satisfying the following class of equations which we might call the continuous distributive law:

$$(1.1)\quad \prod_{i\in I}\bigsqcup_{j\in J} a_{ij} = \bigsqcup_{f\in J^I}\prod_{i\in I} a_{if(i)},$$

provided that $\{a_{ij} \mid j\in J\}$ is (upward) directed for all $i\in I$.

As he remarked to me, 1.1. can be made into a proper equation by replacing the $\bigsqcup$ by the sup of all its finite subsups — or if you like the f can go from I to finite subsets of J and $a_{if(i)}$ can mean $\bigsqcup_{j\in f(i)} a_{ij}$ — thus eliminating the proviso. This observation suggests several interesting questions, some of which can be easily answered by what we already know.

(1) It is without loss of generality in the whole class of equations that we make the set J independent of the variable i, since a sup can always be given redundant terms. However, as with the definition of complete algebras, it seems necessary to let the sets run over all cardinalities so that (1.1) represents a proper class of equations. As a special case of (1.1), of course, we have

$$(1.2) \quad a \sqcap \bigsqcup_{j \in J} b_j = \bigsqcup_{j \in J} (a \sqcap b_j),$$

provided that $\{b_j \mid j \in J\}$ is directed.

Equation (1.2) says that $\sqcap$ is continuous. But every Brouwerian algebra satisfies this (even with an arbitrary sup), and we know examples that are not continuous lattices (every complete, atomless Boolean algebra, for instance). Thus in (1.1) we cannot restrict I to being finite. I shall not investigate the question, but surely a restriction of I to countable will not work? Can we not use the known independence results for distributive laws in Boolean algebras?

(2) We note that on the right-hand side of (1.1) the sup over $f \in J^I$ is also a directed sup. (No need to say "updirected" here since a down-directed sup is of no interest.) Thus in

the equation we have only $\sqcap$ and directed $\sqcup$ (should we write $\sqcup\!\!\uparrow$?), so this suggests that from the algebraic point of view these are the fundamental operations. We make some (obvious) remarks about homomorphisms, subalgebras, and direct products in the next section. The discussion of function spaces can also profit from this approach (Section 3). What seems to be a mildly new remark concerns free algebras (Section 4). (No, not new!)

(3) The proof of equivalance of (1.1) to the "classical" definition should be extracted from Day's paper. In the one direction assume (1.1). Let x be a given element. Let I index <u>all</u> the directed sups such that

$$x \sqsubseteq \bigsqcup_{j \in J} a_{ij} .$$

There is at least one of them with all $a_{ij} = x$; hence, the lhs of (1.1) reduces to x itself. Now on the rhs of (1.1) each $\bigsqcap_{i \in I} a_{i\,f(i)}$ is <u>way below</u> x, because i runs over all directed sets over x. Thus x <u>is</u> the sup of elements way below it.

In the other direction, assume that the lattice is continuous. We only need to show that lhs $\sqsubseteq$ rhs. Suppose that $b \ll$ lhs. We only have to show $b \sqsubseteq$ rhs. Now for each $i \in I$ we have $b \ll \bigsqcup_{j \in J} a_{ij}$, so $b \sqsubseteq a_{ij}$ for some $j \in J$. Calling in the Axiom of Choice we capture a function $f \in J^I$ with $b \sqsubseteq a_{i\,f(i)}$

for all $i \in I$. This clearly gives us what we want.

2. Homomorphisms, subalgebras and direct products. As in [1] and in ATLAS [2] we define a homomorphism as a map $h: L \to M$ between continuous lattices that preserves $\sqcap$ and $\sqcup\uparrow$. A subalgebra, by the same token, should be a subset closed under $\sqcap$ and $\sqcup\uparrow$. (Since the unit, or top element, is the empty meet, homomorphisms preserve units and subalgebras contain the unit. We say nothing about the zero element, because directed sets are always nonempty.) It is obvious from the equational definition that a subalgebra of a continuous lattice is again a continuous lattice. The corresponding remark about homomorphic images requires a very small proof.

(2.1) LEMMA. If L and M are complete lattices and if $h: L \to M$ preserves $\sqcap$ and $\sqcup\uparrow$, then $h(L)$ is closed under $\sqcap$ and $\sqcup\uparrow$.

Proof: Closure under $\sqcap$ is clear. Suppose then that $\{b_j \mid j \in J\} \subseteq h(L)$ is directed (as a subset of M). Define for $j \in J$:

$$a_j = \sqcap\{x \in L \mid b_j \sqsubseteq h(x)\}$$

Obviously $b_j \sqsubseteq b_k$ implies $a_j \sqsubseteq a_k$; therefore, $\{a_j \mid j \in J\}$ is directed in L. Since $b_j \in h(L)$, we see $h(a_j) = b_j$ because h preserves $\sqcap$. But $h(\bigsqcup_{j\in J} a_j) = \bigsqcup_{j \in J} b_j$, which proves the $\sqcup\uparrow$ belongs to $h(L)$.

This argument is of course a special case of the duality theory of ATLAS, but it is so elementary that it can stand by itself. Note that we need both $\sqcap$ and $\sqcup\!\!\uparrow$. The directed sup, $\sqcup\!\!\uparrow$, is not a very good algebraic operation taken alone. It follows at once from (2.1) that the homomorphic image of a continuous lattice is again such. To prove closure of the class under direct products, one needs this property of directed sets:

(2.2) LEMMA. Under projection onto a coordinate, the image of a directed subset of a cartesian product (of posets) is again a directed set.

The proof is clear. We turn now to the question of generating subalgebras. If S is a subset of a complete lattice, let $S^{\sqcap}$ be the set of all $\sqcap$ of elements of S and let $S^{\sqcup\uparrow}$ be the set of all $\sqcup\!\!\uparrow$. It is obvious that $S^{\sqcap\sqcap} = S^{\sqcap}$, and I was about to say the same for $\sqcup\!\!\uparrow$, but it does not seem to be true unless S is already closed under finite joins. However, the monotone, increasing operation on sets $S \rightsquigarrow S^{\sqcup\uparrow}$ determines a closure operation, which we denote by $S^{\sqcup\uparrow\infty}$. By definition this is the least set T where $S \subseteq T$ and $T^{\sqcup\uparrow} \subseteq T$. Since equation (1.1) lets us commute $\sqcap$ and $\sqcup\!\!\uparrow$, the following observation has a good chance of being true, because we can say:

$$S^{\sqcup\uparrow\sqcap} \subseteq S^{\sqcap\sqcup\uparrow}.$$

(2.3) False? LEMMA. In a continuous lattice L, the subalgebra generated by a set $S \subseteq L$ is the set $S^{\sqcap \sqcup\uparrow \infty}$; that is, first close S under $\sqcap$ and then under $\sqcup\uparrow$.

(Pause of 5 minutes.) But, alas, nothing so simple seems to work. When I try to define the closure of $S^{\sqcap}$ under $\sqcup\uparrow$ by transfinite recursion, I am not able to move the new $\sqcap$ over the limit stages to show that $S^{\sqcap \sqcup\uparrow \infty}$ is closed under $\sqcap$. So lets forget it for the moment. Some remarks about congruence relations will close this section. (But see Section 4.)

A <u>congruence relation</u> is an equivalence relation which is a subalgebra of $L \times L$. If $h: L \to M$ is a homomorphism of continuous lattices, then $\{(x,y) \mid h(x) = h(y)\}$ is clearly a subalgebra. (Note, we need (2.2) again.) In the other direction we have to move just a little carefully to obtain a quotient. Thus, let θ be a congruence relation. Define the kernel map $k: L \to L$ by the formula

$$k(x) = \sqcap \{y \mid x \,\theta\, y\}.$$

Since θ is a subalgebra, $k(x)$ is the least member of the equivalence class. Clearly

(i) $k(x) \sqsubseteq x$ and $k(k(x)) = x$.

If $x \sqsubseteq y$, then $x \sqcap y = x$; so by the usual sort of argument $k(x) \sqsubseteq k(x) \sqcap k(y)$ because $x \sqcap y \,\theta\, k(x) \sqcap k(y)$. Thus, k is monotone. Hence, if $\{x_i \mid i \in I\}$ is

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directed, then so is $\{k(x_i) \mid i \in I\}$. By the subalgebra property of Θ we have

$$k(\bigsqcup_{i\in I} x_i) \sqsubseteq \bigsqcup_{i\in I} k(x_i) .$$

The inclusion in the other direction follows by monotonicity; hence,

(ii) k is continuous.

(That is, k preserves $\bigsqcup\uparrow$.) A kernel operation in general is not a homomorphism from L into L; however, the image $k(L)$, as a lattice in itself, __is__ a homomorphic image. We can see that L/Θ is a complete lattice because it has $\sqcap$ owing to the subalgebra property of Θ. But L/Θ is isomorphic to $k(L)$, so $k(L)$ is a complete lattice. This makes the map $k: L \to k(L)$ a $\sqcap$-preserving map. By (ii) it also preserves $\bigsqcup\uparrow$; thus, we have a homomorphism and $k(L) \cong L/\Theta$ is continuous. We can make this more explicit by noting the equation:

$$\text{(iii)} \quad k(\prod_{i\in I} x_i) = k(\prod_{i\in I} k(x_i)) .$$

But (iii) easily follows from (i) and (ii) (or even monotonicity), and it means that $k(L)$ is complete by the inf defined in (iii). That is to say, __every__ kernel operation gives a continuous quotient and the k's are in a one-to-one correspondence with the Θ's.

These facts are also contained in ATLAS

and reviewed by Keimel in [3]. However, it seems useful to point out the direct connection with the standard notion of congruence relation so that we see the generality of the construction. Also in this discussion we do not have to do any calculations with the $\ll$-relation. The use of k seems also simpler than Day [1, pp. 53-4].

3. *Function spaces.* Given two continuous lattices L and M, there are several function spaces associated with them about which we can derive properties in a somewhat simpler way than in Hofmann and Mislove [4]. It is not good to _always_ avoid using $\ll$, because to understand the topology of these spaces it is very helpful; but for the general properties it seems quicker to proceed from first principles as we have done above.

We define:

$\mathcal{F}(L,M) = M^L =$ _all_ functions from L into M

$\mathcal{M}(L,M) =$ _monotone_ functions

$\mathcal{C}(L,M) =$ _continuous_ functions $= [L \to M]$

$\mathcal{H}(L,M) =$ _homomorphisms_

We also consider the lattice of _congruence relations_ of L which we can write as $\Theta(L)$ or as $\mathcal{K}(L) \subseteq \mathcal{C}(L,L)$, the kernel functions on L into L. Some other spaces will be mentioned below.

(1) All the function spaces will be given the pointwise partial ordering. Thus $\mathcal{F}(L, M)$ is nothing more than a direct power of M where no use is made of the structure of L. We know at once that $\mathcal{F}(L, M)$ is a continuous lattice with the pointwise lattice operations on functions.

(2) The pointwise sup and inf of monotone functions is again a monotone function, so $\mathcal{M}(L, M)$ is a subalgebra of $\mathcal{F}(L, M)$; this makes it a continuous lattice. (In [4] a topological reason is given in the proof of Lemma 1 of [4], but the algebraic reason seems more elementary.)

(3) As in [4], we note that $\mathcal{C}(L, M)$ is a quotient of $\mathcal{M}(L, M)$, and so it is also a continuous lattice. (The quotient business requires proof.)

(Pause for lunch.) Perhaps the following observation provides a more general approach:

(3.1) LEMMA. Suppose L is a complete lattice and $a : L \to L$ is continuous. Then $L_a = \{x \mid a(x) \sqsubseteq x\}$ is a $\sqcap, \sqcup\!\uparrow$ subalgebra.

Proof: We recall that continuous functions are monotone. Suppose each $x_i \in L_a$ for $i \in I$. By monotonicity, $a(\sqcap_{i \in I} x_i) \sqsubseteq a(x_i) \sqsubseteq x_i$ for $i \in I$. It follows that the meet belongs to L_a. If $\{x_i \mid i \in I\}$ is now assumed directed, then $a(\bigsqcup x_i) = \bigsqcup a(x_i) \sqsubseteq \bigsqcup x_i$ by continuity. $\square$

Consider now a continuous lattice L and a retract $a = a \circ a \in \mathcal{C}(L, L)$. Every kernel map is a retract, but the converse does not hold. However L_a, being a subalgebra of L, is a continuous lattice by (3.1). Clearly, by monotonicity, L_a is closed under a. BUT, restricted to L_a, the map a is a kernel map. Since $a(L) =$ the fixed-point set of a, we see $a(L) = a(L_a)$. Thus by previous work, $a(L)$ is a continuous lattice: the retract of a continuous lattice is also one. That is a good, general result proved before with the aid of $\ll$ and the topology; so we only have to show that $\mathcal{C}(L, M)$ is a retract of $\mathcal{F}(L, M)$, provided L and M are continuous lattices.

I do not see how to do this without $\ll$. Is there a more direct "algebraic" reason? In [4], Hofmann and Mislove use a definition that I have often applied: an arbitrary function is turned into a continuous function; specifically, we define a map $\gamma : \mathcal{F}(L, M) \to \mathcal{C}(L, M)$ by the equation:

$$\gamma(f)(x) = \bigsqcup \{ f(y) \mid y \ll x \}.$$

Recalling these properties of $\ll$: (i) $\{y \mid y \ll x\}$ is directed; (ii) $x = \bigsqcup \{y \mid y \ll x\}$; (iii) $y \ll \bigsqcup S$ iff $y \ll z$, some $z \in S$ whenever S is directed; we see that $\gamma(f)$ does indeed belong to $\mathcal{C}(L, M)$. (This only seems to use the continuity of L.) Since γ is defined by sups, and since $\mathcal{C}(I,M)$ is closed under point-wise sups, it is clear that γ preserves arbitrary sups — so it is certainly continuous. Now a function f is continuous iff $f=\gamma(f)$; so we conclude that $\gamma(\gamma(f))=\gamma(f)$, which means that γ is a retract with image $\mathcal{C}(I,M)$. If we assume that M is continuous, then so is $\mathcal{F}(I,M)$ and hence $\mathcal{C}(I,M)$ is too.

Well, at least we didn't have to use the $\ll$ of $\mathcal{C}(I,M)$.

(4) What can we say about $\mathcal{H}(I,M)$? Since finite meet is continuous, we can easily show that $\mathcal{H}(I,M)$ is closed under pointwise $\sqcap$ of functions; thus, it is a semigroup and a subsemigroup of $\mathcal{C}(I,M)$. It does not seem to be a complete lattice or any kind of an infinitary subalgebra of $\mathcal{C}(I,M)$; nor do there seem to be any lattice-theoretic properties of $\mathcal{H}(I,M)$ within $\mathcal{C}(I,M)$. Right?

(5) An interesting space is the class of all closure operations:

$$\mathcal{A}(I)=\{a\in\mathcal{C}(I,I) \mid id\sqsubseteq a=a\circ a\}$$

It is somewhat opposite to the kernel operations $\mathcal{K}(I)$. Both are easily seen to be complete lattices; indeed they are intervals of the complete lattice of all retracts:

$$\mathcal{R}(I)=\{a\in\mathcal{C}(I,I) \mid a=a\circ a\}.$$

To see that $\mathcal{R}(\mathbb{L})$ is a complete lattice, one notes that composition ($\circ$) is a continuous operation on $\mathcal{C}(\mathbb{L},\mathbb{L})$ and that the fixed points of a continuous (even: monotone) function always form a complete lattice. By continuity the fixed points are closed under $\bigsqcup\uparrow$ but usually not $\sqcap$. Thus it is not always clear when the fixed-point set is a <u>continuous</u> lattice.

In [5] I had asked whether $\mathcal{R}(\mathbb{L})$ was a continuous lattice in general. For the case of $\mathbb{L}$ = the lattice of subsets of an infinite set, Ershov has given a negative answer and the argument is reproduced in [6]. In [7], Ershov showed that $\mathcal{K}(\mathbb{L})$ is not continuous, a result independently found by Hofmann and Mislove [4]. For $\mathcal{A}(\mathbb{L})$, however, things go the other way: it <u>is</u> a retract of $\mathcal{C}(\mathbb{L},\mathbb{L})$. This was discovered by Hancock and Martin-Löf (thinking of continuous closure operations as abstract deductive systems ala Tarski) and is proved in [6] for $\mathbb{L}$ = sets of integers. Hence, if $\mathbb{L}$ is continuous, then so is $\mathcal{A}(\mathbb{L})$. The retraction can be defined thus:

$$\alpha(f)(x) = \sqcap\{y \in \mathbb{L} \mid x \sqcup f(y) \sqsubseteq y\}.$$

The argument in [6] for showing that α is continuous in f and x is quite general. Clearly $\alpha(f) = f$ iff $f \in \mathcal{A}(\mathbb{L})$. It is odd that $\alpha \in \mathcal{A}(\mathcal{C}(\mathbb{L},\mathbb{L}))$.

We should take note of the fact that the lattice $\Theta(I)$ of congruence relations is anti-isomorphic to the lattice $\mathcal{K}(I)$ of kernel operators, since the larger the congruence, the smaller the minimal element of the congruence class. In an ordinary finitary algebra, the lattice of congruences (under inclusion) is algebraic. For continuous lattices this is not so, because congruence is an infinitary notion (cf. our remarks about generating subalgebras). I do not have an example at hand, but it seems very likely that $\Theta(I)$ is generally not a continuous lattice either. The same must be true of the lattice $\mathcal{S}(I)$ of all subalgebras of I. I suppose we could ask how to characterize $\Theta(I)$ and $\mathcal{S}(I)$, but I won't.

4. **Free algebras**. Since we have an equational class with easy to understand products, subalgebras and homomorphisms, we should ask whether free algebras exist. The reason why this is a question is that in the case of complete Boolean algebras there are no free algebras with infinitely many generators (a well-known result of Gaifman and Hales with a simple proof by Solovay). The trouble seems to be the failure of the distributive laws. Since the class of continuous lattices has some distributive laws, there is a chance that free algebras exist. From the false lemma (2.3), I thought I had a proof, but after

simplification I found I had a correct proof. For the sake of notation, let $\Phi(I)$ denote the algebraic lattice of all filters in the Boolean algebra of all subsets of the set I. We recall that $\Phi(I)$ is closed under infinite intersection and directed union.

(4.1) PROPOSITION. In the category of continuous lattices and homomorphisms, the free continuous lattice with I generators is the lattice $\Phi(I)$ with the principal ultrafilters as generators.

Proof: Let L be an arbitrary continuous lattice and let $a \in L^I$ be an arbitrary function. We must show that a determines a unique homomorphism $\bar{a} : \Phi(I) \to L$. We define:

$$\bar{a}(F) = \bigsqcup_{S \in F} \prod_{i \in S} a_i$$

for filters $F \in \Phi(I)$. This mapping is obviously continuous. To show it is a homomorphism, compute:

$$\begin{aligned}
\prod_{j \in J} \bar{a}(F_j) &= \prod_{j \in J} \bigsqcup_{S \in F_j} \prod_{i \in S} a_i \\
&= \bigsqcup_{\sigma \in \mathsf{X}_{j \in J} F_j} \prod_{j \in J} \prod_{i \in \sigma(j)} a_i \\
&= \bigsqcup_{\sigma \in \mathsf{X}_{j \in J} F_j} \prod_{i \in \bigcup_{j \in J} \sigma(j)} a_i \\
&= \bigsqcup_{S \in \bigcap_{j \in J} F_j} \prod_{i \in S} a_i = \bar{a}\Big(\bigcap_{j \in J} F_j\Big)
\end{aligned}$$

The last step is valid because $\bigcup_{j\in J}\sigma(j) \in \bigcap_{j\in J} F_j$ and every set in the intersection of the filters is of this form. The middle step was valid because L is continuous.

If we let $\hat{i} = \{S \subseteq I \mid i \in S\}$. Then clearly $\bar{a}(\hat{i}) = a_i$. Also $F = \bigcup_{S\in F} \bigcap_{i\in S} \hat{i}$, so the homomorphism $\bar{a}$ is indeed uniquely determined by what it does to the generators. The proof is complete. □

Now I am very puzzled! Does not (4.1) make (2.3) true? In fact, the subalgebra generated by a set $\{a_i \mid i \in I\} \subseteq L$, should be just $\{a_i \mid i\in I\}^{\sqcap\sqcup}$. The reason is that the set indicated is the image of $\Phi(I)$ under the homomorphism $\bar{a}$, and by (2.1) $\bar{a}(\Phi(I))$ must be the subalgebra generated by the a_i. I suppose that in the proof I tried to give of (2.3) I did not use the proper indexing of the sups and infs, which is produced automatically by $\Phi(I)$ via (4.1) and (2.1). Good! In any case it still does not seem true that $S^{\sqcup\sqcap\sqcup} = S^{\sqcap\sqcup}$ for all subsets S of a continuous lattice.

In [8] Gierz and Keimel say that (4.1) is well known (cf. p. 12), but they give no reference. Their proof makes it a simple special case of a result about compact partially ordered spaces. The argument above seems pretty minimal. Certainly Day [1] contains similar calculations, but I did not find the conclusion explicitly drawn in his paper.

Free algebras are a special case of free products (coproducts), but I am unable to determine whether the category of continuous lattices and homomorphisms has coproducts. I would guess not, because the dual category of Hofman and Stralka [2] surely does not have products. The projections are OK, but if you take maps $f_i : I \to M_i$, then the combination $f^*: I \to \prod_{i \in I} M_i$ does not preserve $\ll$. Or am I wrong again? In any case it seems like a question that ought to be asked. I would like to know the same about the category of continuous lattices and continuous functions.

A final remark: even when I is countable, $\Phi(I)$ has cardinality $2^{2^{\aleph_0}}$; thus, $\Phi(I)$ does not have a countable basis. (Well, anyway as an algebraic lattice it has too many finite (compact) elements.) Thus we get outside the category of retracts of the lattice of sets of integers, whereas other useful constructions preserve countable bases. Note that $\Phi(I)$ is a subalgebra of a direct power of the two-element lattice, so it follows that there are no other interesting equational classes of continuous lattices (i.e. equation completeness). Enough!

REFERENCES

[1] Alan Day. Filter monads, continuous lattices and closure systems. Can. J. Math. vol. 27 (1975), pp. 50-59.

[2] Karl H. Hofmann & A. Stralka. The algebraic theory of compact Lawson semilattices [ATLAS]. Diss. Math. (to appear)

[3] Klaus Keimel. Relations with the interpolation property and continuous lattices. (Report on work of Gierz, Hofmann, Keimel & Mislove). SCS, 1 VIII 76.

[4] K. H. Hofmann & M. Mislove. Commentary on Scott's function spaces. SCS, 7 VII 76.

[5] D. Scott. Continuous lattices. Springer LNM, vol. 274, pp 97-136.

[6] D. Scott. Data types as lattices. Springer LNM, vol. 499; revised version SIAM Jour. (to appear).

[7] Yu. L. Ershov. The theory of A-spaces. Algebra i logika, vol. 12 (1973), pp. 369-416 (Russian); translated pp 209-232.

[8] G. Gierz & K. Keimel. A lemma on primes appearing in algebra and analysis. ~~THD~~ Preprint Nr. 273 (1976).