Seminar on Continuity in Semilattices

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SCS 14: SCS Memo of Lawson Dated 7-12-76

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SEMINAR ON CONTINUITY IN SEMILATTICES (SCS)

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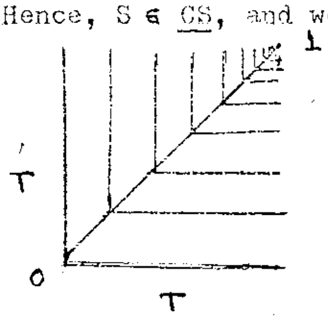
In the above mentioned memo, Lawson defines the subset L(S) for a compact semilattice S to be those points of S where S has small semilattices. He shows that L(S) is closed under arbitrary sups, and that, for all $s \in S$, we have $s \in L(S)$ iff $s = \sup \Downarrow s$. The question is then raised as to whether $L(S) \in$ <u>CL</u>. The following example shows that such need not be the case:

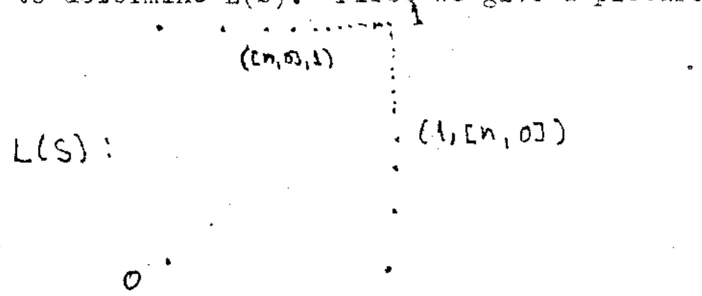
Step 1. Let $R \in$ <u>CS</u> such that, for all subsemilattices S of R, int $S \neq \emptyset$ implies that $0 \in S$ (such examples exist; see, e.g., Lawson, "Lattices with no interval homomorphisms" Pac. Jour. 32, 459-465). We let $R' = \mathbb{N} \times R$ in the lexicographic order, where $\mathbb{N}$ has its natural total order. Then, in the product topology, R' is a locally compact semilattice. Moreover, any compact subset of R' intersects at most finitely many of the sets $\{n\} \times R$. Hence, if $R'' = R' \cup \{1\}$ is the one-point compactification of R', it follows that the sets of the form $\uparrow(n,0)$ form a neigborhood basis for the topology at 1. Thus, if we let 1 act as an identity for R", we have that R" is a compact semilattice. It is readily verified that $Q = \{((n,0),(n+1,1)) : n \in \mathbb{N}\} \cup \Delta(R'')$ is a closed congruence on R", so that $T = R''/Q$ is a compact semilattice with identity. In "picturesque" language, T is a stack of countably many copies of R with an identity at the top. We wish to determine L(T). Clearly $0 \in L(T)$, and since the sets of the form $\uparrow(n,0)$ form a neighborhood basis at 1 in R", the sets $\uparrow[n,0]$ form a neighborhood basis at 1 in T (where $[n,0]$ denotes the Q-class of $(n,0)$ in T), and so we conclude that $1 \in L(T)$ also. Now, let $r \in R$, and consider $[n,r]$. If U is any semilattice neighborhood of $[n,r]$ in T, then $U \cap (\{n\} \times R)$ is a semilattice neighborhood of $[n,r]$ in the subsemilattice $\{n\} \times R$, which is isomorphic to R. Hence, $[n,0] \in U$ by the defining property of R. Furthermore, if $n > 0$, then $[n,0] = [n-1,1]$, and the same argument as that just given for $[n,r]$ shows that any semilattice neighborhood of $[n,0]$ must also contain $[n-1,0]$. From these two facts we conclude that, for any $n > 0$, any semilattice neighborhood of $[n,r]$ must also contain $[0,0]$, the zero of T. Thus, $L(T) = \{0,1\}$.

Step 2. The semilattice in which we are interested is a subsemilattice of $T \times T$. Let $S = \bigcup_n \Big(\{([n,0],[m,r]) : n \leq m \in \mathbb{N}\} \cup \{([m,r],[n,0]) : n \leq m \in \mathbb{N}\} \Big) \cup \Delta(T)$

$= \bigcup_n \Big((p_1^{-1}([n,0]) \cap \uparrow([n,0],[n,0]) \cup (p_2^{-1}([n,0]) \cap \uparrow([n,0],[n,0])\Big) \cup \Delta(T)$

where $p_i: T \times T \longrightarrow T$ is the projection on the i^{th} factor. Then, from the second definition, it is clear that S is a countable union of closed subsemilattices of $T \times T$. Moreover, for each $n \in \mathbb{N}$, all but finitely many of these subsemilattices are contained in $\uparrow([n,0],[n,0])$, and these upper sets form a neighborhood basis at $(1,1)$. It then follows that S is a closed subset of $T \times T$. To see that S is a subsemilattice, we choose $s,s' \in S$, and we assume that $s = ([n,0],[m,r])$ and $s' = ([k,t],[j,0])$ with $n \leq m$ and $j \leq k$. Either $j \leq n$ or $n \leq j$, and we assume the former. Then, $j \leq n \leq m$, so that $[m,r][j,0] = [j,0]$. Also, $j \leq k,n$ implies that $[j,0] \leq [n,0][k,r]$. Hence, $ss' = ([i,0],[j,0])$ with $j \leq \inf k,n = i$, so that $ss' \in S$ in this case. Similar arguments take care of the other possibilities, so that S is indeed a subsemilattice of $T \times T$. Hence, $S \in \underline{CS}$, and we want to determine $L(S)$. First we give a picture of S :





Now, clearly $(0,0) \in L(S)$, and since $(1,1) = \sup_n ([n,0],[n,0])$ and $([n,0],[n,0]) \ll_S (1,1)$ (this is not true in $T \times T$!) for each $n \in \mathbb{N}$, we have that $(1,1) \in L(S)$ also. Now, if $n \in \mathbb{N}$, then it follows that $([n,0],[n,0]) \ll_S ([n,0],1)$, and so $([n,0],[m,0]) \ll_S ([n,0], 1)$ for each $m \geq n$. Hence $([n,0],1) = \sup \Downarrow_S ([n,0],1)$, so that $([n,0],1) \in L(S)$, and a similar argument shows that $(1,[n,0]) \in L(S)$ for each $n \in \mathbb{N}$. Finally, for each $n \in \mathbb{N}$, $p_1^{-1}([n,0]) \cap \uparrow([n,0],[n,0])$ is isomorphic to T, so that the only possible points of $L(S)$ in this subsemilattice are $([n,0],1)$ and $([n,0],0)$. We have already seen that $([n,0],1) \in L(S)$, and since $([n,0],0) = ([n,0],[n,0]) \in \Delta(T)$, and $\Delta(T)$ is also isomorphic to T, we conclude that $([n,0],0) \in L(S)$ implies that $n = 0$. A similar argument works for the points of the form $([m,r],[n,0])$ with $n \leq m$, and we conclude that $L(S) = \{([n,0],1),(1,[n,0]) : n \in \mathbb{N}\} \cup \{(1,1),(0,0)\}$. Fix $m \in \mathbb{N}$. Then, for all $n \in \mathbb{N}$, $([n,0],1)(1,[m,0]) = (0,0)$ (this product is in $L(S)$). Hence, since $(1,1) = \sup_n ([n,0],1)$, it follows that $L(S)$ is not lower continuous, and so $L(S)$ cannot have a compact semilattice topology.

<u>Question</u>: Let $S \in \underline{CS}$, and suppose that $L(S) \in \underline{CS}$. Is then $L(S) \in \underline{CL}$?