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PARAMETER ESTIMATION FOR ORNSTEIN-UHLENBECK PROCESSES DRIVEN BY α-STABLE LÉVY MOTIONS

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Abstract. The parameter estimation theory for stochastic differential equations driven by Brownian motions or general Lévy processes with finite second moments has been well developed. In this paper, we consider the parameter estimation problem for Ornstein-Uhlenbeck processes driven by α-stable Lévy motions. The classical maximum likelihood method does not apply in this context because the likelihood ratio does not exist. We shall use the *trajectory fitting* method combined with the *weighted least squares* technique. We discuss the consistency and the asymptotic distributions of our estimators for general weights in both the ergodic and the non-ergodic cases.

1. Introduction and Notation

We consider the Ornstein-Uhlenbeck processes $X = \{X_t, t \geq 0\}$ determined by the following linear stochastic differential equation

$$\begin{cases} dX_t = -\theta X_t dt + m dt + \sigma dZ_t \,, \quad t \geq 0, \\ X_0 = x, \end{cases} \tag{1.1}$$

where θ, m and σ are given constants and Z_t is a given standard α-stable Lévy motion. The basic probability space is $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with a right continuous and increasing family $\{\mathcal{F}_t, t \geq 0\}$ of σ-algebras. The expectation on this probability space is denoted by $\mathbb{E}$. For some technical reason we also assume that $1 < \alpha < 2$.

Suppose we don't know the parameter θ (m and σ known). We have observation of the process $X = \{X_t, 0 \leq t \leq T\}$ up to the time instant T. We are interested in estimating the unknown parameter θ. When Z_t is replaced by a standard Brownian motion, the parameter estimation for θ has been extensively studied by using classical maximum likelihood method or by using the least squares technique (see Liptser and Shiryaev [8]). However, the naive classical maximum likelihood estimator (MLE) is no longer valid in our setting because the explicit density functions are not available and the Girsanov measure transformation is not well defined for the α-stable Lévy motions. In the next section we shall sketch how to use the least squares technique to study the estimator of θ when Z_t is a Brownian

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motion and explain why the naive least squares technique can no longer apply in our case. To find a consistent estimator $\hat{\theta}$ of θ for (1.1) we shall use the trajectory fitting method combined with the weighted least squares technique. The trajectory fitting method was first proposed by Kutoyants [6] as a numerically attractive alternative to the well-developed maximum-likelihood estimators for continuous diffusion processes (see Dietz and Kutoyants [2], [3], Dietz [1], and Kutoyants [7]).

To obtain our estimator we introduce

$$A_t = \int_0^t X_s ds, \ t > 0.$$

The equation (1.1) can be written as

$$X_t = x - \theta A_t + mt + \sigma Z_t.$$

Let w_t be a deterministic positive (weight) function. Multiply the above equation by w_t we have

$$w_t X_t = w_t x - \theta w_t A_t + mtw_t + \sigma w_t Z_t.$$

The weighted trajectory fitting estimate (TFE) of θ is to minimize

$$\int_0^T |w_t X_t - (w_t x - \theta w_t A_t + mtw_t)|^2 \, dt.$$

It is easy to see that the minimum is attained when θ is given by

$$\hat{\theta}_T = -\frac{\int_0^T w_t^2 (X_t - x - mt) A_t dt}{\int_0^T w_t^2 A_t^2 dt}. \tag{1.2}$$

First we shall prove the strong consistency of $\hat{\theta}_T$:

$$\lim_{T\to\infty} \hat{\theta}_T = \theta \qquad \mathbb{P} \ - a.s.$$

[In the rest of the paper, when we don't specify the convergence we always mean the almost sure convergence.] Once we have established the above result we can study the asymptotic distributions of $\hat{\theta}_T$. Namely, we will find a functional $\kappa(w,T)$ such that

$$\kappa(w,T)\left(\hat{\theta}_T - \theta\right)$$

converges in distribution to a α-stable random variable Ξ (independent of T and w). This means that

$$\hat{\theta}_T - \theta \approx \ \frac{1}{\kappa(w,T)}\Xi \quad \text{as} \ \ T \to \infty.$$

If $\frac{1}{\kappa(w,T)}$ is of the order of $T^{-\delta}$ for some positive δ, namely, $\frac{T^\delta}{\kappa(w,T)}$ converges to a functional $F(w)$ of w, then we have

$$\hat{\theta}_T - \theta \approx \ F(w)\frac{1}{T^\delta}\Xi \quad \text{as} \ \ T \to \infty.$$

$F(w)$ is called the *leading coefficient*. We prefer to have a smaller value of $F(w)$ for a given class of weight functions.

In the following we demonstrate how to view the stable process from general context of Lévy process. Namely, we consider

$$\begin{cases} dX_t = -\theta X_t dt + dL_t\,, \quad t \geq 0, \\ X_0 = x, \end{cases} \tag{1.3}$$

where θ is an unknown parameter, $\{L_t\,, t \geq 0\}$ is a one-dimensional Lévy process. Lévy processes are closely related to stable distributions. A random variable Z is said to have a stable distribution with index of stability $\alpha \in (0, 2]$, scale parameter $\sigma \in (0, \infty)$, skewness parameter $\beta \in [-1, 1]$, and location parameter $\mu \in (-\infty, \infty)$ if it has characteristic function (c.f.) of the following form:

$$\begin{aligned} \phi_Z(u) &= E\exp\{iuZ\} \\ &= \begin{cases} \exp\left\{-\sigma^\alpha |u|^\alpha \left(1 - i\beta \operatorname{sgn}(u) \tan \frac{\alpha\pi}{2}\right) + i\mu u\right\}, & \text{if } \alpha \neq 1, \\ \exp\left\{-\sigma|u| \left(1 + i\beta \frac{2}{\pi} \operatorname{sgn}(u) \log|u|\right) + i\mu u\right\}, & \text{if } \alpha = 1. \end{cases} \end{aligned}$$

For the random variable Z distributed according to the rule described above we use the notation $Z \sim S_\alpha(\sigma, \beta, \mu)$. When $\mu = 0$, we say Z is *strictly stable*. If in addition $\beta = 0$, we call Z *symmetric α-stable*. We refer to Samorodnitsky and Taqqu [10], Janicki and Weron [4] and Sato [11] for more details on stable distributions.

Suppose that $\{L_t\,, t \geq 0\}$ is a Lévy process generated by the triplet $(0, \rho, \lambda)$, i.e. the distribution of L_t has characteristic function

$$\phi_{L_t}(u) = E[e^{iuL_t}] = \exp\left\{it\lambda u + t\int_{\mathbb{R}\ \setminus\{0\}} (e^{iux} - 1 - iux 1_D(x))\rho(dx)\right\},\ u \in \mathbb{R}\ , \tag{1.4}$$

where $D = \{x : |x| \leq 1\}$ and ρ is the Lévy measure given by

$$\rho(dx) = \frac{c_1}{x^{1+\alpha}} 1_{(0,\infty)}(x)dx + \frac{c_2}{|x|^{1+\alpha}} 1_{(-\infty,0)}(x)dx, \tag{1.5}$$

where $1 < \alpha < 2$, $c_1 \geq 0$, $c_2 \geq 0$, and $c_1 + c_2 > 0$. It is easy to see that (1.4) can be written as

$$\phi_{L_t}(u) = \exp\left\{it\left(\lambda + \int_{|x|>1} x\rho(dx)\right)u - t\sigma^\alpha |u|^\alpha \left[1 - \beta \operatorname{sgn}(u) \tan\left(\frac{\pi\alpha}{2}\right)\right]\right\},$$

where $\sigma^\alpha = -(c_1 + c_2)\Gamma(-\alpha)\cos(\pi\alpha/2)$ and $\beta = (c_1 - c_2)/(c_1 + c_2)$. Then, by the Itô-Lévy decomposition, we have

$$L_t = \lambda t + \int_0^t \int_{|x|<1} x\widetilde{N}(ds, dx) + \int_0^t \int_{|x|\geq 1} xN(ds, dx), \tag{1.6}$$

where $N(dt, dx)$ is a Poisson random measure defined by

$$N((0, t], A) = \sum_{s \leq t} 1_A(\Delta L_s)$$

for $A \in \mathcal{B}(\mathbb{R}\ \setminus \{0\})$ and $\Delta L_s = L_s - L_{s-}$ denoting the jump of L_s at time s, and the compensated Poisson random measure $\widetilde{N}(dt, dx)$ is given by

$$\widetilde{N}((0, t], A) = N((0, t], A) - t\rho(A)$$

with

$$\rho(A) = \int_A \rho(dx).$$

The Itô-Lévy decomposition can be rewritten as

$$\begin{aligned} L_t &= \lambda t + \int_0^t \int_{\mathbb{R}\setminus\{0\}} x\tilde{N}(ds,dx) + t\int_{|x|\geq 1} x\rho(dx) \\ &= \left(\lambda + \int_{|x|\geq 1} x\rho(dx)\right) t + \int_0^t \int_{\mathbb{R}\setminus\{0\}} x\widetilde{N}(ds,dx). \end{aligned} \tag{1.7}$$

Let

$$m = \lambda + \int_{|x|>1} x\rho(dx).$$

Then

$$m = \lambda + \frac{c_1 - c_2}{\alpha - 1}.$$

Denote

$$\tilde{Z}_t = \int_0^t \int_{\mathbb{R}\setminus\{0\}} x\widetilde{N}(ds,dx).$$

Then $\tilde{Z}_t$ is a α-stable Lévy motion and $\tilde{Z}_t - \tilde{Z}_s \sim S_\alpha(\sigma(t-s)^{1/\alpha}, \beta, 0)$ for any $0 \leq s < t < \infty$. We can renormalize $\tilde{Z}_t$ and define $Z_t = \tilde{Z}_t/\sigma$. Then we can easily see that $\{Z_t, t \geq 0\}$ is a standard α-stable Lévy motion (see Janicki and Weron [4]) so that Z_1 has a stable distribution $S_\alpha(1,\beta,0)$. It is clear that $L_t = mt + \sigma Z_t$ and $E[L_t] = mt$.

The paper is organized as follows. In Section 2, we sketch the least squares method in classical setting. In Section 3, we prove the strong consistency and discuss the asymptotic distribution of the weighted TFE in the ergodic case. In Section 4, we shall establish some results for the weighted TFE in the non-ergodic case.

2. Classical Least Squares Technique

Consider the Langevin equation

$$dX_t = -\theta X_t dt + dB_t, \tag{2.1}$$

where $(B_t, t \geq 0)$ is a standard Brownian motion and θ is the unknown parameter to be estimated from the observation $(X_t, 0 \leq t \leq T)$. To explain the least squares technique we write formally

$$\dot{X}_t = -\theta X_t + \dot{B}_t$$

and we minimize

$$\int_0^T |\dot{X}_t + \theta X_t|^2 dt = \int_0^T \dot{X}_t^2 dt + 2\theta \int_0^T X_t \dot{X}_t dt + \theta^2 \int_0^T X_t^2 dt\,.$$

The minimizer is given by

$$\hat{\theta}_T = -\frac{\int_0^T X_t dX_t}{\int_0^T X_t^2 dt} \tag{2.2}$$

(The meaningless term $\int_0^T \dot{X}_t^2 dt$ does not appear). To show the least squares estimator $\hat{\theta}_T$ converges to θ, we have

$$\hat{\theta}_T - \theta = -\frac{\int_0^T X_t dX_t}{\int_0^T X_t^2 dt} - \theta = \frac{\theta \int_0^T X_t^2 dt}{\int_0^T X_t^2 dt} - \frac{\int_0^T X_t dB_t}{\int_0^T X_t^2 dt} - \theta = -\frac{\int_0^T X_t dB_t}{\int_0^T X_t^2 dt}. \quad (2.3)$$

Now $\int_0^T X_t dB_t$ is a martingale with the bracket $\int_0^T X_t^2 dt$. Roughly speaking when $T \to \infty$, $\int_0^T X_t dB_t$ is the order of $\sqrt{\int_0^T X_t^2 dt}$. But when $T \to \infty$, $\frac{1}{T}\int_0^T X_t^2 dt \to \mathbb{E}\,(X_\infty^2)$. So $\frac{\int_0^T X_t dB_t}{\int_0^T X_t^2 dt}$ converges to 0 with the order $\frac{1}{\sqrt{T}}$. This argument also works if B_t is replaced by a square integrable martingale. Unfortunately the stable process Z_t is not square integrable. So this naive least squares technique does not apply directly to our case.

Another approach is the maximum likelihood ratio. In the Brownian motion case, both the maximum likelihood ratio method and the least squares method give the same estimator. But in our stable process case there is no direct extension of the Girsanov theorem because of the infinite variance property. So the maximum likelihood ratio method cannot be directly applied here neither.

3. Ergodic Case

In this section we consider the consistency and the asymptotic distribution of the weighted TFE in ergodic case. That means we assume $\theta > 0$. Then, the solution of the SDE (1.3) can be written in the following way:

$$\begin{aligned} X_t &= e^{-\theta t}x + \int_0^t e^{-\theta(t-s)} dL_s \\ &= e^{-\theta t}x + m\int_0^t e^{-\theta(t-s)} ds + \sigma \int_0^t e^{-\theta(t-s)} dZ_s. \end{aligned} \quad (3.1)$$

The general properties of generalized Ornstein-Uhlenbeck processes driven by Lévy processes have been comprehensively studied in the monograph of Sato [11]. We shall use some important results in Sato [11] freely in this paper.

Lemma 3.1. *The generalized Ornstein-Uhlenbeck processes $\{X_t, t \geq 0\}$ (generated by the triplet $(0, \rho, \lambda)$) have a unique invariant distribution μ_∞ which is self-decomposable and generated by the triplet $(0, \nu, \gamma)$ with*

$$\nu(B) = \frac{1}{\theta}\int_{\mathbb{R}} \rho(dy) \int_0^\infty 1_B(e^{-s}y) ds, \quad B \in \mathcal{B}(R)$$

and

$$\gamma = \frac{\lambda}{\theta} + \frac{1}{\theta}\int_{|y|>1} \frac{y}{|y|}\rho(dy).$$

Proof. By Theorem 17.5 of Sato [11], we only need to verify that the Lévy measure ρ satisfies the following condition

$$\int_{|x|>2} \log|x| \rho(dx) < \infty. \quad (3.2)$$

In fact, we have

$$\begin{aligned}\int_{|x|>2}\log|x|\rho(dx) &= \int_{|x|>2}\log|x|\left(\frac{c_1}{x^{1+\alpha}}1_{(0,\infty)}(x)+\frac{c_2}{|x|^{1+\alpha}}1_{(-\infty,0)}(x)\right)dx\\ &= c_1\int_2^\infty\frac{\log x}{x^{1+\alpha}}dx+c_2\int_{-\infty}^{-2}\frac{\log(-x)}{(-x)^{1+\alpha}}dx\\ &= (c_1+c_2)\int_2^\infty\log x\cdot x^{-1-\alpha}dx\\ &= \frac{c_1+c_2}{\alpha 2^\alpha}\left(\log 2+\frac{1}{\alpha}\right)<\infty.\end{aligned}$$

This completes the proof. □

We can easily find that X_t converges weakly to a random variable

$$X_\infty=\frac{m}{\theta}+\sigma\int_0^\infty e^{-\theta s}dZ_s.$$

Note that $\{Z_t\}$ is a $\mathcal{F}_t$-martingale and the random variable $\int_0^\infty e^{-\theta s}dZ_s$ has mean zero. Hence X_∞ is a α-stable random variable with mean $E[X_\infty]=\frac{m}{\theta}$. By Lemma 3.1 and ergodic theorem, we have

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T X_t dt=E[X_\infty]=\frac{m}{\theta}. \tag{3.3}$$

Remark 3.2. When we don't specify the way of convergence we always mean the almost sure convergence.

We need the following well-known integral version of the Toeplitz Lemma (see Dietz and Kutoyants [2]):

Lemma 3.3. *If φ_T is a probability measure defined on $[0,\infty)$ such that $\varphi_T([0,T])=1$ and $\varphi_T([0,K])\to 0$ as $T\to\infty$ for each $K>0$, then*

$$\lim_{T\to\infty}\int_0^\infty f_t\varphi_T(dt)=f_\infty$$

for every bounded and measurable function $f:[0,\infty)\to\mathbb{R}$ for which the limit $f_\infty:=\lim_{t\to\infty}f_t$ exists.

Denote

$$\gamma(T)=\int_0^T t^2w_t^2dt.$$

We assume that the weight function w_t is given so that $\gamma(T)$ is well defined for all $T>0$ and $\gamma(T)\to\infty$ as $T\to\infty$.

Theorem 3.4. *Assume that $\theta>0$ and $m\neq 0$. Then the weighted TFE is strongly consistent in the following sense:*

$$\lim_{T\to\infty}\hat\theta_T=\theta \qquad \mathbb{P}-a.s.$$

Proof. Since the observed process $\{X_t\}$ is the solution of (1.1), we find

$$\hat{\theta}_T = \theta - \frac{\sigma \int_0^T w_t^2 A_t Z_t dt}{\int_0^T w_t^2 A_t^2 dt}. \tag{3.4}$$

By the strong law of large numbers, Lemma 3.1 and the ergodic theorem, we have

$$\lim_{T\to\infty} T^{-1} Z_T = 0 \tag{3.5}$$

(since $E[Z_1] = 0$) and

$$\lim_{T\to\infty} T^{-1} A_T = \lim_{T\to\infty} \frac{1}{T}\int_0^T X_t dt = E[X_\infty] = \frac{m}{\theta}.$$

By the Toeplitz Lemma 3.3, we find

$$\lim_{T\to\infty} \frac{\sigma \int_0^T w_t^2 A_t Z_t dt}{\int_0^T w_t^2 A_t^2 dt} = \lim_{T\to\infty} \frac{\sigma \int_0^T \left(\frac{A_t}{t}\right)\left(\frac{Z_t}{t}\right)\frac{w_t^2 t^2}{\gamma(T)} dt}{\int_0^T \left(\frac{A_t^2}{t^2}\right)\frac{w_t^2 t^2}{\gamma(T)} dt} = 0. \tag{3.6}$$

The last identity follows from the fact that $\int_0^T \frac{w_t^2 t^2}{\gamma(T)} dt = 1$, $\lim_{T\to\infty} T^{-1} Z_T = 0$, and $\lim_{T\to\infty} T^{-1} A_T = \frac{m}{\theta}$. This proves the theorem. □

It is well known that the least squares estimator or the maximum likelihood estimator for diffusion processes driven by Brownian motion are asymptotically normal with the order of convergence $T^{-\frac{1}{2}}$. Here we shall prove that the weighted TFE is asymptotically α-stable with the order of convergence $T^{-(1-\frac{1}{\alpha})}$. When α is formally set to 2, our result coincides with the classical one.

For notational simplicity, we denote $\xi(T) = \int_0^T t w_t^2 dt$ $\tau(T) = \int_0^T \left|\int_t^T s w_s^2 ds\right|^\alpha dt$.

We assume that the weight function w_t satisfies the following condition:

(C1) $\xi(T) \to \infty$ as $T \to \infty$, $\xi(K)/\xi(T) \to 0$ as $T \to \infty$ for each $K > 0$, and

$$\frac{\xi(T) T^{1/\alpha}}{\tau(T)^{1/\alpha}} = \mathrm{O}(1). \tag{3.7}$$

Sufficient conditions on w_t satisfying (C1) will be provided later on.

Theorem 3.5. *If the generalized Ornstein-Uhlenbeck process is ergodic with $\theta > 0$ and $m \neq 0$, then the weighted TFE is asymptotically α-stable under the condition (C1), i.e.*

$$\frac{\gamma(T)}{\tau(T)^{1/\alpha}}\left(\hat{\theta}_T - \theta\right) \Rightarrow -\frac{\sigma\theta}{m}\kappa \tag{3.8}$$

as $T \to \infty$, where κ is a α-stable random variable with distribution $S_\alpha(1, \beta, 0)$.

Proof. By (3.4), it follows that

$$\begin{aligned}\hat{\theta}_T-\theta &= -\frac{\sigma\int_0^T w_t^2 A_t Z_t dt}{\int_0^T w_t^2 A_t^2 dt}\\ &= \frac{\left[-\sigma\int_0^T\left(\frac{A_t}{t}-\frac{m}{\theta}\right)w_t^2 Z_t t dt-\frac{m\sigma}{\theta}\int_0^T w_t^2 Z_t t dt\right]}{\int_0^T\left(\frac{A_t}{t}\right)^2\cdot w_t^2 t^2 dt}\\ &= \frac{\phi_1(T)+\phi_2(T)}{\phi_3(T)}.\end{aligned}$$

Then, we have

$$\frac{\gamma(T)}{\tau(T)^{1/\alpha}}\left(\hat{\theta}_T-\theta_0\right)=\frac{\tau(T)^{-1/\alpha}\phi_1(T)+\tau(T)^{-1/\alpha}\phi_2(T)}{\gamma^{-1}(T)\phi_3(T)}. \tag{3.9}$$

By the ergodic property and the Toeplitz Lemma 3.3, we find

$$\lim_{T\to\infty}\frac{\phi_3(T)}{\gamma(T)}=\lim_{T\to\infty}\int_0^T\left(\frac{A_t}{t}\right)^2\cdot\frac{w_t^2t^2}{\gamma(T)}dt=\left(\frac{m}{\theta}\right)^2\quad \mathbb{P}-a.s. \tag{3.10}$$

Now, we consider $\tau(T)^{-1/\alpha}\phi_1(T)$. Note that

$$|\tau(T)^{-1/\alpha}\phi_1(T)|\le\sigma\tau(T)^{-1/\alpha}Z_T^*\int_0^T\left|\frac{A_t}{t}-\frac{m}{\theta}\right|tw_t^2dt, \tag{3.11}$$

where

$$Z_T^*=\sup_{0\le t\le T}|Z_t|.$$

Denote $\tilde{\phi}_1(T)=\int_0^T\left|\frac{A_t}{t}-\frac{m}{\theta}\right|tw_t^2dt$. By ergodic property and the Toeplitz Lemma 3.3, it follows that

$$R_T:=\frac{|\tilde{\phi}_1(T)|}{\xi(T)}=\int_0^T\left|\frac{A_t}{t}-\frac{m}{\theta}\right|\frac{tw_t^2}{\xi(T)}dt\to 0\quad \mathbb{P}-a.s.$$

as $T\to\infty$. Then, for $\varepsilon>0$ and $\delta>0$, we have

$$\begin{aligned}\mathbb{P}(|\tau(T)^{-1/\alpha}\phi_1(T)|\ge\varepsilon) &\le \mathbb{P}\left(\sigma\tau(T)^{-1/\alpha}\xi(T)Z_T^*R_T\ge\varepsilon\right)\\ &\le \mathbb{P}(R_T\ge\delta)+\mathbb{P}\left(Z_T^*\ge\frac{\varepsilon\tau(T)^{\frac{1}{\alpha}}}{\sigma\delta\xi(T)}\right).\end{aligned} \tag{3.12}$$

By Markov inequality, we find

$$\begin{aligned}\mathbb{P}\left(Z_T^*\ge\frac{\varepsilon\tau(T)^{\frac{1}{\alpha}}}{\sigma\delta\xi(T)}\right) &\le \frac{\sigma\delta\xi(T)\mathbb{E}[Z_T^*]}{\tau(T)^{\frac{1}{\alpha}}\varepsilon}\\ &\le \frac{\sigma\delta\xi(T)}{\tau(T)^{\frac{1}{\alpha}}\varepsilon}\cdot C\left(\int_0^T dt\right)^{\frac{1}{\alpha}}=\frac{C\sigma\delta\xi(T)T^{1/\alpha}}{\varepsilon\tau(T)^{1/\alpha}},\end{aligned} \tag{3.13}$$

where C is a positive constant depending only on α. By the condition (C1), for any fixed $\varepsilon > 0$, we can choose δ arbitrarily small so that

$$\lim_{T\to\infty} \mathbb{P}\left(Z_T^* \geq \frac{\varepsilon\tau(T)^{\frac{1}{\alpha}}}{\sigma\delta\xi(T)}\right) = 0. \tag{3.14}$$

Obviously, $\mathbb{P}(R_T \geq \delta) \to 0$ as $T \to \infty$. Therefore, we have

$$\lim_{T\to\infty} \mathbb{P}(|\tau(T)^{-1/\alpha}\phi_1(T)| \geq \varepsilon) = 0. \tag{3.15}$$

Next, we turn to consider the limiting behavior of $\tau(T)^{-1/\alpha}\phi_2(T)$. By integration by parts, it follows that

$$\int_0^T Z_t t w_t^2 dt = \int_0^T \left[\int_t^T s w_s^2 ds\right] dZ_t.$$

By the inner clock property for the α-stable stochastic integral (see Rosinski and Woyczynski [9], Kallenberg [5], and Zanzotto [12]), we know easily that $\int_0^T \left[\int_t^T s w_s^2 ds\right] dZ_t$ has the same distribution as $Z'_{\tau(T)}$, where Z' has the same law as Z. Thus, $Z'_{\tau(T)}$ has a α-stable distribution $S_\alpha((\tau(T))^{1/\alpha}, \beta, 0)$.

By basic properties of α-stable random variable (see Janicki and Weron [4]), we find that $\frac{\phi_2(T)}{\tau(T)^{1/\alpha}}$ converges weakly to a stable distribution $S_\alpha(\frac{\sigma|m|}{\theta}, \mathrm{sgn}(-m)\beta, 0)$. Summarizing we have that

$$\frac{\gamma(T)}{\tau(T)^{1/\alpha}}\left(\hat{\theta}_T - \theta\right)$$

converges weakly to $-\frac{\sigma\theta}{m}\kappa$, where κ is a random variable with α-stable distribution $S_\alpha(1, \beta, 0)$. □

Remark 3.6. We shall specify some conditions on w_t so that the condition (C1) is satisfied. Of course, we always assume that w_t is continuous on $[0, \infty)$. We further assume that there exists a nonnegative function $g(s)$ such that

$$\lim_{T\to\infty} \frac{w_{sT}^2}{w_T^2} = g(s), \forall s \in [0, 1]. \tag{3.16}$$

and $s^2 g(s) \leq 1$.

Now, we can verify that (C1) holds if (3.16) is satisfied. Indeed, we have

$$\begin{aligned}
\lim_{T\to\infty} \frac{\xi(T)^\alpha T}{\tau(T)} &= \lim_{T\to\infty} \frac{\xi(T)^\alpha T}{\int_0^T (\xi(T) - \xi(t))^\alpha dt} \\
&= \lim_{T\to\infty} \frac{T}{\int_0^T \left(1 - \frac{\xi(t)}{\xi(T)}\right)^\alpha dt} \\
&= \lim_{T\to\infty} \frac{1}{\int_0^1 \left(1 - \frac{\xi(sT)}{\xi(T)}\right)^\alpha ds}.
\end{aligned} \tag{3.17}$$

Also, by L'Hospital's rule, it follows that for each $s \in [0,1]$

$$\lim_{T\to\infty} \frac{\xi(sT)}{\xi(T)} = \lim_{T\to\infty} \frac{\int_0^{sT} r w_r^2 dr}{\int_0^T r w_r^2 dr}$$

$$= \lim_{T\to\infty} \frac{sTw_{sT}^2 s}{Tw_T^2} = s^2 \lim_{T\to\infty} \frac{w_{sT}^2}{w_T^2} = s^2 g(s). \tag{3.18}$$

Therefore, we have

$$\lim_{T\to\infty} \frac{\xi(T)^\alpha T}{\tau(T)} = \frac{1}{\int_0^1 (1-s^2 g(s))^\alpha ds}, \tag{3.19}$$

which implies that the condition (C1) is satisfied.

We provide two classes of functions which satisfy the given condition.

(i) Let $w_t = t^p$. Then, it is seen that $\lim_{T\to\infty} \frac{w_{sT}^2}{w_T^2} = s^{2p}$. Thus,

$$\lim_{T\to\infty} \frac{\xi(T)^\alpha T}{\tau(T)} = \frac{1}{\int_0^1 (1-s^{2+2p})^\alpha ds}.$$

Here, we need to assume that $2+2p > 0$ or $p > -1$. By some basic calculation, we find

$$\int_0^1 (1-s^{2+2p})^\alpha ds = \frac{1}{2+2p} \int_0^1 (1-z)^\alpha z^{\frac{1}{2+2p}-1} dz = \frac{1}{2+2p} B(\frac{1}{2+2p}, 1+\alpha).$$

So,

$$\lim_{T\to\infty} \frac{\xi(T)T^{1/\alpha}}{\tau(T)^{1/\alpha}} = \left(\lim_{T\to\infty} \frac{\xi^\alpha(T)T}{\tau(T)}\right)^{1/\alpha} = \left(\frac{2+2p}{B(1/(2+2p), 1+\alpha)}\right)^{1/\alpha}.$$

(ii) Let $w_t = e^{qt},\ q > 0$. Then, it is easy to see that

$$\lim_{T\to\infty} \frac{w_{sT}^2}{w_T^2} = g(s) = \begin{cases} 0, & \text{if } s \in [0,1) \\ 1, & \text{if } s = 1. \end{cases}$$

So,

$$\lim_{T\to\infty} \frac{\xi^\alpha(T)T}{\tau(T)} = \frac{1}{\int_0^1 (1-s^2 g(s))^\alpha ds} = 1.$$

Remark 3.7. Suppose that the condition (C1) is satisfied. As in Remark 3.6, we still assume that there exists a nonnegative function $g(s)$ such that

$$\lim_{T\to\infty} \frac{w_{sT}^2}{w_T^2} = g(s), \forall s \in [0,1]. \tag{3.20}$$

and $s^2 g(s) \le 1$. Under this condition, as shown in Remark 3.6, we have

$$\lim_{T\to\infty} \frac{\xi(sT)}{\xi(T)} = s^2 g(s).$$

We claim that

$$\frac{\gamma(T)}{\tau^{1/\alpha}(T)} = O(T^{1-\frac{1}{\alpha}}). \tag{3.21}$$

This is equivalent to

$$\frac{\gamma(T)}{\tau^{1/\alpha}(T)T^{1-\frac{1}{\alpha}}} = \mathrm{O}(1)$$

or

$$\frac{\gamma^{\alpha}(T)}{\tau(T)T^{\alpha-1}} = \mathrm{O}(1).$$

By some basic calculation, we find

$$\begin{aligned}
\lim_{T\to\infty}\frac{\gamma^{\alpha}(T)}{\tau(T)T^{\alpha-1}} &= \lim_{T\to\infty}\frac{\left(\int_0^T t d\xi(t)\right)^{\alpha}}{\int_0^T(\xi(T)-\xi(t))^{\alpha}dt\cdot T^{\alpha-1}} \\
&= \lim_{T\to\infty}\frac{\left(\int_0^T(\xi(T)-\xi(t))dt\right)^{\alpha}}{\int_0^T(\xi(T)-\xi(t))^{\alpha}dt\cdot T^{\alpha-1}} = \lim_{T\to\infty}\frac{\left(\int_0^1(\xi(T)-\xi(sT))Tds\right)^{\alpha}}{\int_0^1(\xi(T)-\xi(sT))^{\alpha}Tds\cdot T^{\alpha-1}} \\
&= \lim_{T\to\infty}\frac{\left(\int_0^1(1-\xi(sT)/\xi(T))ds\right)^{\alpha}}{\int_0^1(1-\xi(sT)/\xi(T))^{\alpha}ds} = \frac{\left(\int_0^1(1-s^2g(s))ds\right)^{\alpha}}{\int_0^1(1-s^2g(s))^{\alpha}ds}. \qquad (3.22)
\end{aligned}$$

As in Remark 3.6, we consider the following two special classes of weight functions:

(i) Let $w_t = t^p$. Then, $g(s) = s^{2p}$ with $p > -1$. It is easy to find

$$\lim_{T\to\infty}\frac{\gamma(T)}{\tau^{1/\alpha}(T)T^{1-1/\alpha}} = \left(\frac{2+2p}{3+2p}\right)\left[\frac{1}{2+2p}B(1/(2+2p),1+\alpha)\right]^{-1/\alpha}. \qquad (3.23)$$

It is not hard to verify that the limit function of p in the right hand side of (3.23) is increasing and is bounded by constant 1.

(ii) When $w_t = e^{qt}, q > 0$. It is known from Remark 3.6 that

$$g(s) = \begin{cases} 0, & \text{if } s \in [0,1), \\ 1, & \text{if } s = 1. \end{cases}$$

It follows that

$$\lim_{T\to\infty}\frac{\gamma(T)}{\tau^{1/\alpha}(T)T^{1-1/\alpha}} = 1. \qquad (3.24)$$

Remark 3.8. The convergence result in Theorem 3.5 means that there is a stable random variable Ξ such that

$$\hat{\theta}_T - \theta \approx \frac{\tau(T)^{1/\alpha}}{\gamma(T)}\Xi.$$

Thus the leading coefficient is proportional to $\frac{\tau(T)^{1/\alpha}}{\gamma(T)}$. In fact, when $w_t = t^p$ with $p > -1$, it is easy to see that for all $T > 0$

$$\frac{\tau^{1/\alpha}(T)T^{1-1/\alpha}}{\gamma(T)} = \left(\frac{3+2p}{2+2p}\right)\left[\frac{1}{2+2p}B(1/(2+2p),1+\alpha)\right]^{1/\alpha} =: f(p).$$

4. Non-Ergodic Case

In this section, we consider the non-ergodic case, i.e., $\theta < 0$. The solution of the SDE (1.1) is given by

$$X_t = e^{-\theta t} X_0 + m e^{-\theta t} \int_0^t e^{\theta s} ds + \sigma e^{-\theta t} \int_0^t e^{\theta s} dZ_s. \tag{4.1}$$

Let $\eta_t = X_0 + m \int_0^t e^{\theta s} ds + \sigma \int_0^t e^{\theta s} dZ_s$. Then,

$$e^{\theta t} X_t = \eta_t. \tag{4.2}$$

Let $\xi_t = \int_0^t e^{\theta s} dZ_s$. Then, $\{\xi_t\}_{t \geq 0}$ is a L^p-bounded cadlag $\mathcal{F}_t$-martingale ($1 < p < \alpha$). Moreover, ξ_t is a α-stable random variable with distribution $S_\alpha(\tau_t^{1/\alpha}, \beta, 0)$, where

$$\tau_t = \int_0^t |e^{\theta s}|^\alpha ds = \frac{1}{\alpha\theta}\left(e^{\alpha\theta t} - 1\right).$$

Letting $t \to \infty$, we find that ξ_t converges to a α-stable random variable with distribution $S_\alpha((-\alpha\theta)^{-1/\alpha}, \beta, 0)$. Therefore, by martingale convergence theorem, it follows that

$$\lim_{t\to\infty} e^{\theta t} X_t = X_0 - \frac{m}{\theta} + \sigma \int_0^\infty e^{\theta s} dZ_s := \eta_\infty, \quad \mathbb{P} - a.s. \tag{4.3}$$

Similar to Section 3, we can define the weighted TFE of θ as follows

$$\hat{\theta}_T = \frac{\int_0^T w_t^2 (X_t - X_0 - mt) A_t dt}{\int_0^T w_t^2 A_t^2 dt},$$

where $A_t = \int_0^t X_s ds$. We also have

$$\hat{\theta}_T - \theta = -\frac{\sigma \int_0^T w_t^2 A_t Z_t dt}{\int_0^T w_t^2 A_t^2 dt}.$$

Denote

$$h_1(T) = \int_0^T w_t^2 e^{-2\theta t} dt \quad \text{and} \quad h_2(T) = \int_0^T w_t^2 e^{-\theta t} dt\,.$$

In this section, we always assume that the weight function w_t is given so that $h_i(T) \to \infty$ and $h_i(K)/h_i(T) \to 0$ as $T \to \infty$ for each $K > 0$ and $i = 1, 2$.

We first prove the consistency of the weighted TFE.

Theorem 4.1. *If $\theta < 0$, then*

$$\lim_{T\to\infty} (\hat{\theta}_T - \theta) = 0 \quad \mathbb{P} - a.s. \tag{4.4}$$

Proof. By the Toeplitz Lemma 3.3, it follows that

$$\begin{aligned} \lim_{t\to\infty} \frac{A_t}{e^{-\theta t}} &= \lim_{t\to\infty} \frac{\int_0^t (e^{\theta s} X_s - \eta_\infty) e^{-\theta s} ds + \eta_\infty \int_0^t e^{-\theta s} ds}{e^{-\theta t}} \\ &= \frac{\lambda(t)}{e^{-\theta t}} \int_0^t (e^{\theta s} X_s - \eta_\infty) \frac{e^{-\theta s}}{\lambda(t)} ds + \eta_\infty \frac{\int_0^t e^{-\theta s} ds}{e^{-\theta t}} \\ &= \frac{\eta_\infty}{-\theta}, \quad \mathbb{P} - a.s., \end{aligned} \tag{4.5}$$

where $\lambda(t) = \int_0^t e^{-\theta_0 s} ds$. Then, by the Toeplitz Lemma 3.3 again, we have

$$\begin{aligned}\lim_{T\to\infty}(\hat{\theta}_T - \theta) &= \lim_{T\to\infty} -\frac{\sigma \int_0^T w_t^2 A_t Z_t dt}{\int_0^T w_t^2 A_t^2 dt} \\ &= \lim_{T\to\infty} -\frac{\sigma \int_0^T \left(\frac{A_t}{e^{-\theta t}}\right)\cdot\left(\frac{Z_t}{e^{-\theta t}}\right)\frac{w_t^2 e^{-2\theta t}}{h_1(T)} dt}{\int_0^T \left(\frac{A_t}{e^{-\theta t}}\right)^2 \frac{w_t^2 e^{-2\theta t}}{h_1(T)} dt} \\ &= 0, \quad \mathbb{P}-a.s. \end{aligned} \tag{4.6}$$

since $\lim_{t\to\infty} Z_t/e^{-\theta t} = 0$, $\mathbb{P}-a.s.$ (from (3.5)). This completes the proof. □

Next, we are going to discuss the asymptotic distribution of $\hat{\theta}_T$. We assume that the weight function w_t satisfies the following condition:

(C2) There exist constants $C_0 > 0$, $b < 0$ such that when T is large enough,

$$\frac{w_t^2 e^{-\theta t}}{h_2(T)} \le C_0 e^{b(T-t)}, \forall t \in [0, T]. \tag{4.7}$$

We have the following result:

Theorem 4.2. *If $\theta < 0$ and (C2) holds, then*

$$\frac{h_1(T)}{h_2(T)T^{1/\alpha}}(\hat{\theta}_T - \theta) \Rightarrow -\sigma|\theta|\frac{\zeta}{\eta_\infty}, \tag{4.8}$$

where ζ is a random variable with α-stable distribution $S_\alpha(1, \beta, 0)$ independent of η_∞.

Proof. We have

$$\begin{aligned} &\frac{h_1(T)}{h_2(T)T^{1/\alpha}}(\hat{\theta}_T - \theta) \\ &= -\frac{\sigma h_2^{-1}(T) T^{-\frac{1}{\alpha}} \int_0^T w_t^2 A_t Z_t dt}{h_1^{-1}(T)\int_0^T w_t^2 A_t^2 dt} \\ &= \frac{\eta_T^2}{h_1^{-1}(T)\int_0^T w_t^2 A_t^2 dt}\Bigg(\frac{(-\sigma) h_2^{-1}(T) T^{-\frac{1}{\alpha}} Z_T \int_0^T w_t^2 A_t dt}{\eta_T^2} \\ &\qquad + \frac{\sigma h_2^{-1}(T) T^{-\frac{1}{\alpha}} \int_0^T w_t^2 A_t (Z_T - Z_t) dt}{\eta_T^2}\Bigg) \\ &:= F_T(G_T + H_T). \end{aligned} \tag{4.9}$$

From the Toeplitz Lemma 3.3, it follows that

$$\lim_{T\to\infty} h_1^{-1}(T)\int_0^T w_t A_t^2 dt = \lim_{T\to\infty}\int_0^T \left(\frac{A_t}{e^{-\theta t}}\right)^2 \frac{w_t^2 e^{-2\theta t}}{h_1(T)} dt = \frac{\eta_\infty^2}{|\theta|^2}. \ \mathbb{P}-a.s. \tag{4.10}$$

Thus, we have

$$\lim_{T\to\infty} F_T = |\theta|^2 \quad \mathbb{P}-a.s. \tag{4.11}$$

Now let us consider G_T. Note that

$$G_T = \frac{(-\sigma)h_2^{-1}(T)\int_0^T w_t^2 A_t dt}{\eta_T} \cdot \frac{T^{-\frac{1}{\alpha}} Z_T}{\eta_T}. \tag{4.12}$$

By the Toeplitz Lemma 3.3 again, we get

$$\lim_{T\to\infty} h_2^{-1}(T)\int_0^T w_t^2 A_t dt = \lim_{T\to\infty}\int_0^T \left(\frac{A_t}{e^{-\theta t}}\right)\frac{w_t^2 e^{-\theta t}}{h_2(T)}dt = \frac{\eta_\infty}{|\theta|} \tag{4.13}$$

almost surely. Consequently

$$\lim_{T\to\infty} \frac{(-\sigma)h_2^{-1}(T)\int_0^T w_t^2 A_t dt}{\eta_T} = -\frac{\sigma}{|\theta|} \quad \mathbb{P}-a.s. \tag{4.14}$$

For the second factor in G_T, we have

$$\frac{T^{-\frac{1}{\alpha}} Z_T}{\eta_T} = \frac{T^{-\frac{1}{\alpha}}(Z_T - Z_{T^{\frac{1}{\alpha}}}) + T^{-\frac{1}{\alpha}} Z_{T^{\frac{1}{\alpha}}}}{\eta_{T^{\frac{1}{\alpha}}} + (\eta_T - \eta_{T^{\frac{1}{\alpha}}})} \tag{4.15}$$

We have the following claims.
(1) The random variable $T^{-\frac{1}{\alpha}}(Z_T - Z_{T^{\frac{1}{\alpha}}})$ has a α-stable distribution $S_\alpha(\sigma(1 - T^{\frac{1}{\alpha}-1})^{1/\alpha}, \beta, 0)$, which converges weakly to a random variable ζ with stable distribution $S_\alpha(1,\beta,0)$ as $T\to\infty$.
(2) By strong law of large numbers, we have

$$\lim_{T\to\infty} T^{-\frac{1}{\alpha}} Z_{T^{\frac{1}{\alpha}}} = 0 \quad \mathbb{P}-a.s.$$

(3) It is clear that

$$\lim_{T\to\infty} \eta_{T^{\frac{1}{\alpha}}} = \eta_\infty \quad \mathbb{P}-a.s.$$

(4) $T^{-\frac{1}{\alpha}}(Z_T - Z_{T^{\frac{1}{\alpha}}})$ and $\eta_{T^{\frac{1}{\alpha}}}$ are independent.
(5) We have that $\eta_T - \eta_{T^{\frac{1}{\alpha}}}$ converges to zero in probability as $T\to\infty$.
Proof of (5). By the definition of η_t, we find

$$\eta_T - \eta_{T^{\frac{1}{\alpha}}} = m\int_{T^{\frac{1}{\alpha}}}^T e^{\theta s}ds + \int_{T^{\frac{1}{\alpha}}}^T e^{\theta s}dZ_s.$$

It follows that

$$\begin{aligned} |\eta_T - \eta_{T^{\frac{1}{\alpha}}}| &\le |m|\int_{T^{\frac{1}{\alpha}}}^T e^{\theta s}ds + \left|\int_{T^{\frac{1}{\alpha}}}^T e^{\theta s}dZ_s\right| \\ &\le |m|\cdot\frac{e^{\theta T} - e^{\theta T^{\frac{1}{\alpha}}}}{\theta} + \left|\int_{T^{\frac{1}{\alpha}}}^T e^{\theta s}dZ_s\right|. \end{aligned} \tag{4.16}$$

The first term on the right hand side of (4.16) converges to zero as $T\to\infty$. The second term converges to zero in probability as $T\to\infty$, since

$$\mathbb{P}\left\{\left|\int_{T^{\frac{1}{\alpha}}}^T e^{\theta s}dZ_s\right| > \varepsilon\right\} \le \frac{\mathbb{E}\left|\int_{T^{\frac{1}{\alpha}}}^T e^{\theta s}dZ_s\right|}{\varepsilon}$$

$$\leq \frac{C}{\varepsilon}\left(\int_{T^{\frac{1}{\alpha}}}^{T} e^{\alpha\theta s}ds\right)^{\frac{1}{\alpha}} \leq \frac{C}{\varepsilon}\left(\frac{e^{\alpha\theta T}-e^{\alpha\theta T^{\frac{1}{\alpha}}}}{\alpha\theta}\right)^{\frac{1}{\alpha}}, \tag{4.17}$$

which tends to zero as $T \to \infty$ for any given $\varepsilon > 0$ and some constant $C > 0$. From all the claims (1)-(5), we conclude that

$$\frac{T^{-\frac{1}{\alpha}}Z_T}{\eta_T} \Rightarrow \frac{\zeta}{\eta_\infty} \tag{4.18}$$

where ζ and η_∞ are independent. Combining (4.14) and (4.18), we find

$$G_T \Rightarrow -\frac{\sigma}{|\theta|}\frac{\zeta}{\eta_\infty} \tag{4.19}$$

as $T \to \infty$. Finally, we shall prove that $H_T \to 0$ in probability as $T \to \infty$. We have

$$\begin{aligned}
&\left|\sigma T^{-\frac{1}{\alpha}}h_2^{-1}(T)\int_0^T w_t^2 A_t(Z_T - Z_t)dt\right| \\
&\leq \sigma T^{-\frac{1}{\alpha}}h_2^{-1}(T)\int_0^T w_t^2\left|\int_0^t X_s ds\right||Z_T - Z_t|dt \\
&\leq \sigma T^{-\frac{1}{\alpha}}h_2^{-1}(T)\int_0^T \left(\int_0^t |e^{\theta s}X_s|e^{-\theta s}ds\right)|Z_T - Z_t|w_t^2 dt \\
&\leq \frac{\sigma}{|\theta|}\sup_{t\geq 0}|e^{\theta t}X_t| \cdot T^{-\frac{1}{\alpha}}h_2^{-1}(T)\int_0^T |Z_T - Z_t|w_t^2 e^{-\theta t}dt.
\end{aligned} \tag{4.20}$$

It is easy to see that $\sup_{t\geq 0}|e^{\theta t}X_t|$ is almost surely finite. We claim that the last factor $T^{-\frac{1}{\alpha}}h_2^{-1}(T)\int_0^T |Z_T - Z_t|w_t^2 e^{-\theta t}dt$ in the above inequality converges to zero in probability. Indeed, we have for T large enough

$$\begin{aligned}
&\mathbb{E}\left[T^{-\frac{1}{\alpha}}h_2^{-1}(T)\int_0^T |Z_T - Z_t|w_t^2 e^{-\theta t}dt\right] \\
&\leq T^{-\frac{1}{\alpha}}\int_0^T \mathbb{E}[|Z_T - Z_t|]\frac{w_t^2 e^{-\theta t}}{h_2(T)}dt \\
&\leq C_0 T^{-\frac{1}{\alpha}}\int_0^T C(1,\alpha)(T-t)^{\frac{1}{\alpha}}e^{b(T-t)}dt \\
&= C_0 T^{-\frac{1}{\alpha}}\int_0^T C(1,\alpha)u^{\frac{1}{\alpha}}e^{bu}du \\
&\leq C(1,\alpha)C_0 T^{-\frac{1}{\alpha}}\int_0^\infty u^{\frac{1}{\alpha}}e^{bu}du \\
&\leq C(1,\alpha)C_0 T^{-\frac{1}{\alpha}}\Gamma(1+\frac{1}{\alpha})|b|^{-(1+\frac{1}{\alpha})},
\end{aligned} \tag{4.21}$$

which tends to zero as $T \to \infty$, where $C(1,\alpha) = \frac{4\Gamma(-1/\alpha)}{\alpha\sqrt{\pi}\Gamma(-1/2)}$ (see Zolotarev [13]). This implies that $\sigma T^{-\frac{1}{\alpha}}h_2^{-1}(T)\int_0^T w_t^2 A_t(Z_T - Z_t)dt$ converges to zero in probability as $T \to \infty$. So, $H_T \to 0$ in probability as $T \to \infty$. From (4.9), (4.11), (4.19), (4.20) and (4.21), we conclude that

$$\frac{h_1(T)}{h_2(T)T^{\frac{1}{\alpha}}}(\hat{\theta}_T - \theta) \Rightarrow -\sigma|\theta|\frac{\zeta}{\eta_\infty}, \tag{4.22}$$

where ζ is a $S_\alpha(1, \beta, 0)$ random variable independent of η_∞. This completes the proof. □

Remark 4.3. We consider two special classes of weight functions.

(i) Let $w_t = t^p, p \geq 0$. Some basic calculation yields

$$h_2(T) = \int_0^T w_t^2 e^{-\theta t} dt = \int_0^T t^{2p} e^{-\theta t} dt \geq C_1 T^{2p} e^{-\theta T}, \tag{4.23}$$

for each $T \geq T_0$ with some $T_0 > 0$ and some $C_1 > 0$. It follows that for T large enough

$$\frac{w_t^2 e^{-\theta t}}{h_2(T)} \leq \frac{t^{2p} e^{-\theta t}}{C_1 T^{2p} e^{-\theta T}} \leq \frac{1}{C_1} e^{\theta(T-t)}, \ t \in [0, T]. \tag{4.24}$$

This implies that (C2) is satisfied with $b = \theta$.

(ii) Let $w_t^2 = e^{rt}$. Then,

$$h_2(T) = \int_0^T e^{rt} e^{-\theta t} dt = \frac{e^{(r-\theta)T} - 1}{r - \theta}. \tag{4.25}$$

We have

$$\frac{w_t^2 e^{-\theta t}}{h_2(T)} \leq C_2 e^{(\theta - r)(T-t)}, \tag{4.26}$$

for T large enough and $b = \theta - r < 0$. Thus, if $r > \theta$, then the condition (C2) is satisfied.

Remark 4.4. We claim that

$$\frac{h_1(T)}{h_2(T)T^{1/\alpha}} = O(e^{-\theta T} T^{-1/\alpha}). \tag{4.27}$$

This is equivalent to

$$\frac{h_1(T)}{h_2(T)} = O(e^{-\theta T}).$$

We assume that w_t is a continuously differentiable function satisfying

$$\lim_{T \to \infty} \frac{w_T'}{w_T} = a, \tag{4.28}$$

where a is a constant such that $-2\theta+2a > 0$ and $2a-\theta > 0$. Then, by L'Hospital's rule, we have

$$\begin{aligned}\lim_{T\to\infty}\frac{h_1(T)e^{\theta T}}{h_2(T)} &= \lim_{T\to\infty}\frac{w_T^2 e^{-2\theta T}e^{\theta T}+h_1(T)e^{\theta T}\theta}{w_T^2 e^{-\theta T}} \\ &= 1+\theta\lim_{T\to\infty}\frac{h_1(T)}{w_T^2 e^{-2\theta T}} \\ &= 1+\theta\lim_{T\to\infty}\frac{w_T^2 e^{-2\theta T}}{w_T^2 e^{-2\theta T}(-2\theta)+2w_T w_T' e^{-2\theta T}} \\ &= 1+\frac{\theta}{2(a-\theta)}=\frac{2a-\theta}{2(a-\theta)}. \end{aligned} \tag{4.29}$$

Two special cases: (i) Let $w_t = t^p, p \geq 0$. Then,

$$\lim_{T\to\infty}\frac{w_T'}{w_T}=\lim_{T\to\infty}\frac{pT^{p-1}}{T^p}=0.$$

So,

$$\lim_{T\to\infty}\frac{h_1(T)e^{\theta T}}{h_2(T)}=1/2. \tag{4.30}$$

(ii) Let $w_t = e^{rt/2}, r > \theta$. Then,

$$\lim_{T\to\infty}\frac{w_T'}{w_T}=\lim_{T\to\infty}\frac{\frac{r}{2}e^{rT/2}}{e^{rT/2}}=\frac{r}{2}.$$

So,

$$\lim_{T\to\infty}\frac{h_1(T)e^{\theta T}}{h_2(T)}=\frac{r-\theta}{r-2\theta}. \tag{4.31}$$

References

1. Dietz, H. M.: Asymptotic behavior of trajectory fitting estimators for certain non-ergodic SDE; *Statistical Inference for Stochastic Processes* **4** (2001) 249–258.
2. Dietz, H. M. and Kutoyants, Yu. A.: A class of minimum-distance estimators for diffusion processes with ergodic properties; *Statistics and Decisions* **15** (1997) 211–227.
3. Diets, H. M. and Kutoyants, Yu. A.: Parameter estimation for some non-recurrent solutions of SDE; *Statistics and Decisions* **21** (2003) 29–45.
4. Janicki, A. and Weron, A.: *Simulation and Chaotic Behavior of α-stable Stochastic Processes.* Marcel Dekker, 1994.
5. Kallenberg, O.: Some time change representations of stable integrals, via predictable transformations of local martingales; *Stochastic Process. Appl.* **40** (1992) 199–223.
6. Kutoyants, Yu. A.: Minimum distance parameter estimation for diffusion type observations; *C. R. Acad. Sci. Paris, Serie I*, **312** (1991) 637–642.
7. Kutoyants, Yu. A.: *Statistical Inference for Ergodic Diffusion Processes.* Springer-Verlag, London, Berlin, Heidelberg, 2004.
8. Liptser, R. S. and Shiryaev, A. N.: *Statistics of Random Processes: II Applications.* Second Edition, Applications of Mathematics, Springer-Verlag, Berlin, Heidelberg, New York, 2001.
9. Rosinski, J. and Woyczynski, W. A.: On Itô stochastic integration with respect to p-stable motion: inner clock, integrability of sample paths, double and multiple integrals; *Ann. Probab.* **14** (1986) 271–286.
10. Samorodnitsky, G. and Taqqu, M. S.: *Stable non-Gaussian Random Processes: Stochastic Models with Infinite Variance.* Chapman & Hall, New York, London, 1994.

11. Sato, K. I.: *Lévy Processes and Infinitely Divisible Distributions.* Cambridge University Press, 1999.
12. Zanzotto, P. A.: Representation of a class of semimartingales as stable integrals; *Theory Probab. Appl.* **43** (1998) 666–676.
13. Zolotarev, V. M.: *One-dimensional Stable Distributions.* American Mathematical Society, 1986.

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