SCS 10: Points with Small Semilattices

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(1) First of all I would like to call attention to a pre-
print I have just submitted for publication entitled "Spaces
which force a basis of subsemilattices." In this paper it is
shown that a topological semilattice has small semilattices at
a point $p$ if $p$ has a compact, finite-dimensional, "well-
fitted" neighborhood, where "well-fitted" is a technical term
describing the behavior of components in a neighborhood of a
point. It is defined below. Points in $\mathfrak{m}$ locally connected,
totally disconnected, and locally connected $X$ totally discon-
ected spaces have well-fitted neighborhoods. In fact a rather
far-reaching class of finite-dimensional spaces are included.

It is convenient for our purposes to introduce a component
operator. Let $X$ be a topological space, $A \subseteq X$, and $p \in A$.
Then $C_p(A)$ denotes the component (i.e., maximal connected set)
of $p$ in the subspace $A$.

**Definition.** Let $S$ be a topological space. If $A \subseteq B \subseteq S$,
then $A$ is said to be fitted within $B$ if for each $p \in A$,

$$C_p(A) = C_p(B) \cap A.$$ 

A neighborhood $W$ of a point $p \in A$ is a fitted neighborhood

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of $p$ if $W$ is compact and $p$ has a basis of compact neighborhoods, each of which is fitted within $W$.

A neighborhood $W$ of a point $p \in S$ is a well-fitted neighborhood of $p$ if (i) for each $q$ in the interior of $W$, $W$ is a fitted neighborhood of $q$, and (ii) for any $A \subseteq W$, if $C_p(W) \cap (\bigcup_{a \in A} C_a(W))^\ast \neq \emptyset$, then $p \in (\bigcup_{a \in A} C_a(W))^\ast$.

(2) Let me at this point throw in a couple of conjectures.

First a definition. The space $X$ is said to have local component convergence (l.c.c.) at $p$ if for any neighborhood $W$ of $p$, there exist neighborhoods $V$ and $U$ of $p$ such that

1. $V \subseteq U \subseteq W$,
2. If $Q \subseteq V$ and $C_p(U) \cap (\bigcup_{q \in Q} C_q(U)) \neq \emptyset$, then $p \in (\bigcup_{q \in Q} C_q(W))^\ast$. Roughly speaking, we are requiring that if components approach the component of $p$ locally, then they actually approach $p$.

Conjecture 1. Let $S \in CS$. If $p \in S$, $S$ is l.c.c. at $p$, and $p$ has a finite-dimensional neighborhood in which components are locally connected, then $S$ has small semilattices at $p$.

Conjecture 2. Let $S \in CS$. $S$ finite-dimensional, and suppose the peripheral points in $S$ are closed. Then $S \in CL$. 
Proofs or counter-examples are not easily forthcoming on such problems if past experience is any guide.

(3) Let \( S \in \text{CS} \). Let \( \Lambda(S) \subseteq S \) be all elements of \( S \) at which \( S \) has small semilattices.

**Proposition 1.** \( \Lambda(S) \) is a sup-subsemilattice of \( S \) containing 0 closed under arbitrary sup \( S \). Hence in its own order, \( \Lambda(S) \) is a complete lattice.

**Proof.** Let \( x, y \in \Lambda(S) \). Then \( x = \sup\{a: x \in (\uparrow a)^{\circ}\} \) and \( y = \sup\{b: y \in (\uparrow b)^{\circ}\} \), and both of these are up-directed sets. Hence \( x y = \sup\{a \vee b: x \in (\uparrow a)^{\circ} \text{ and } y \in (\uparrow b)^{\circ}\} \) and

\[
x y \in (\uparrow a)^{\circ} \cap (\uparrow b)^{\circ} = (\uparrow a \vee b)^{\circ}.
\]

Thus \( x y \in \Lambda(S) \).

Now suppose \( x_\alpha \) is an up-directed net in \( \Lambda(S) \) and \( x = \sup x_\alpha \). If \( U \) is open, \( x \in U \), \( \exists x \in U \). Since \( x_\beta \in \Lambda(S) \), \( \exists y \in U \) such that \( x_\beta \in (\uparrow y)^{\circ} \). Hence \( x \in (\uparrow y)^{\circ} \). □

Note that this proposition applies nicely to some of the considerations of \( H \) and \( M \), Memo 6-28-76, e.g. Proposition 11.

**Question:** Is \( \Lambda(S) \in \text{CL} \)?

(4) **Definition.** Let \( \Lambda \) be a topological semilattice. \( x \in S \). \( \{U_n: n=1,2,\ldots\} \) is a fundamental system for \( x \) if

1. Each \( U_n \) is open;
2. \( U_n \cdot U_n \subseteq U_{n-1}, \overline{U_n} \subseteq U_{n-1} \)
(3) \( x \in U_n \) for each \( n \).

**Proposition 2.** (1) If \( \{U_n\}_{n=1}^{\infty} \) is a fundamental system for \( x \), \( \bigcap_{n=1}^{\infty} U_n \) is a closed semilattice containing \( x \).

(2) Each neighborhood of \( x \) contains a fundamental system for \( x \).

**Proposition 3.** If \( S \in CS \), then for \( x \in S \) and each fundamental system \( \lambda = \{U_n\}_{n=1}^{\infty} \), let \( x_{\lambda} = \inf(\bigcap_{n=1}^{\infty} U_n) \). Then if the fundamental systems are ordered by inclusion, \( x_{\lambda} \) is a net converging upwards to \( x \).

**Definition.** \( y \ll\ll x \) if whenever \( \forall \lambda \geq x \), there exists \( \text{finite} \subset \lambda \exists y \ll y' \).

**Proposition 4.** Let \( S \in CS \). Then \( y \ll\ll \emptyset \iff x \in (\uparrow y)^o \).

**Proof.** \( \square \) Straightforward.

\( \square \) By Prop. 3 \( x = \sup x_{\lambda} \) where \( \lambda \) is a fundamental system. 

Hence \( \exists \lambda = \{U_n\}_{n=1}^{\infty} \\exists y \ll x_{\lambda} \).

For each \( U_i \) in \( \lambda \), pick \( x_i \in U_i \setminus \uparrow y \) (we can do this if \( x \not\in (\uparrow y)^o \)). 

Now

\[
x_i \ x_{i+1} \ldots x_{i+j} \in U_i \ U_{i+1} \ldots U_{i+j} \\
\subseteq U_i \ U_{i+1} \ldots (U_{i+j-1})^2 \\
\vdots \\
\subseteq U_i \subseteq U_{i-1}
\]

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Hence $w_i = \bigwedge_{j \geq i} x_j \in \overline{U}_{i-1} \subseteq U_{i-2}$.

Now $w_i$ is an increasing sequence which must converge up to some $w$. Since $w_i \in U_{i-2}$, $w \in \bigcap_{i=1}^{\infty} U_i$. Thus $w \supseteq x_\lambda$.

Since $y \ll x_\lambda$, $y \in W_j \supseteq y \leq w_j$.

But $w_j \supseteq x_j$ and $y \not\in x_j$, a contradiction.

So $x \in (\uparrow y)^\circ$. □

**Corollary 6.** If $x \ll y \ll z$, then $x \ll z$. Hence $w \in \Lambda(S)$ if $w = \sup\{x : x \ll w\}$. 