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SCS 9: Commentary on Scott's Function Spaces

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SEMINAR ON CONTINUITY IN SEMILATTICES (SCS)

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TOPIC Commentary on Scott's function spaces

REFERENCE SCOTT LNM 274; ~~GIERZ~~ GIERZ and KEIMEL, A Lemma on Primes... Notably 1.4.

We quote an amplified version of a Lemma in GK (1.4)

LEMMA A. Let $L \in$ CL and $k: L \to L$ a kernel function, i.e. a function satisfying (i) $(\forall x,y)\ x \leq y \Rightarrow k(x) \leq k(y)$, (ii) $k \leq 1_L$, (iii) $k^2 = k$.

Then (I) $T = k(L)$ is a complete lattice and the corestriction $k: L \to T$ is left adjoint to the inclusion function;

(II) The following conditions are equivalent:

(iv) $k \in$ Cont where Cont is the category of ~~complete~~ continuous lattices with Scott continuous functions. In other words, $(\forall D)$ D up-directed in L $\Rightarrow \sup_L k(D) = k(\sup_L D)$.

(1) $(\forall t)\ t \in T \Rightarrow t = \sup_L \{s \in T : s \ll_L t\} = \sup_L({\Downarrow} t \cap T)$

(2) $\ll_T = \ll_L | (T \times T)$.

(3) $T \in$ CL

(4) The inclusion $T \to L$ is in CL^{op}

(5) $k \in$ CL.

NOTATION. For $L \in$ CL we write $\ker(L) \subseteq [L \to L]$ (= Cont(S,S)) for the set of all functions k satisfying (i) – (iv) (hence also (1)-(5). [See Scott, 3.11]

CH.I.

Our aim is, inter alia, to give an alternative proof os Scott's result that $[S \to T] \in$ CL. The category of posets and order preserving maps is called Poset.

~~kmmmxxxxx~~ Instead of Poset(S,T) we will write $(S \to T)$.

LEMMA 1. Let $S \in$ Poset, $T \in$ CL, then $(S \to T) \in$ CL relative to the structure induce from T^S.

Proof. $(S \to T)$ is closed in T^S in the CL-product topology, and clearly $T^S \in$ CL. □

West Germany: TH Darmstadt (Gierz, Keimel)
U. Tübingen (Mislove, Visit.)

England: U. Oxford (Scott)

USA: U. California, Riverside (Stralka)
LSU Baton Rouge (Lawson)
Tulane U., New Orleans (Hofmann, Mislove)
U. Tennessee, Knoxville (Carruth, Crawley)

DEFINITION 2. Let $S,T \in \underline{CL}$. Define $k:(S\to T)\to(S\to T)$ by

$$k(f)(s) = \sup f(\Downarrow s) = \sup_{x \ll s} f(x).$$

(Note that k is well defined: $x \leq y$ will always imply $k(f)(x) \leq k(f)(y)$!)

PROPOSITION 3. Let $S,T \in \underline{CL}$. Then $k \in \ker(S\to T)$, and k perserves arbitrary sups.

Proof. (i) Suppose $f \leq g$ in $(S\to T)$. From Def.2 it is clear that $k(f) \leq k(g)$.

(ii) Let $s \in S$. Then $k(f)(s) = \sup f(\Downarrow s) \leq \sup f(\downarrow s) = f(s)$.

(iii) Let $s \in S$. Then $k^2 f(s) = \sup_{x \ll s} k(f)(x) = \sup_{x \ll s}\sup_{y \ll x} f(y)$

$= \sup_{y \ll s} f(y) = k(f)(s)$ since for all $y \ll s$ there is an x with $y \ll x \ll s$.

(iv') Let $\mathcal{D} \uparrow\subseteq (S\to T)$ and set $h = \sup \mathcal{D}$ in $(S\to T)$. We claim that $\sup k(\mathcal{D}) = k(h)$. Let $s \in S$. Now $[\sup k(\mathcal{D})](s) = \sup_{\delta\in\mathcal{D}} k(\delta)(s)$

$= \sup_{\delta\in\mathcal{D}} \sup_{x \ll s} \delta(x) = \sup_{x \ll s} \sup_{\delta\in\mathcal{D}} \delta(x) = \sup_{x \ll s} h(s) = k(h)(s)$. □

PROPOSITION 4. Under the hypotheses of Prop.3,

$$k(S\to T) = [S\to T].$$

Proof. a) Let $f \in (S\to T)$. Show that $k(f) \in \underline{Cont}$. Let $D \subseteq S$ be up-directed and let $s = \sup D$. Clearly, $\sup k(f)(D) \leq k(f)(s)$. Now let $x \ll k(f)(s) = \sup f(\Downarrow s)$. By definition of $\ll$ there is an $s' \ll s$ with $x \leq f(s')$. But $s' \ll s = \sup D$ itself implies the existence of some $d \in D$ with $s' \ll d$. Thus $x \leq k(f)(d) \leq \sup k(f)(D)$.

b) Let $f \in [S\to T]$. Then $k(f)(s) = \sup f(\Downarrow s) = f(s)$ since f preserves sups of up-directed sets. □

COROLLARY 5. $[S\to T] \in \underline{CL}$ (SCOTT).

Proof. From LEMMA A, Propositions 3,4. □

CH.II

We now further investigate the kernel function k on $(S\to T)$. So far we know that it preserves arbitrary sups (hence has a left adjoint $f \mapsto \check{f}$ which we will investigate presently) and that its corestriction $k:(S\to T)\to[S\to T]$ is a $\underline{CL}$-map which is left adjoint to the inclusion map $[S\to T]\to(S\to T)$. We need to understand clearly the $\ll$-relation on $(S\to T)$. In the following we allow ourselves a slight deviation from Scott's notation.

NOTATION 6. Let $S,T \in \underline{Poset}$, $(s,t) \in S\times T$. Then $\binom{s}{t} \in (S\to T)$ is defined by $\binom{s}{t}(x) = t$ if $s \leq x$ and $= 0$ otherwise. If $S \in \underline{CL}$, then $\left[{s \atop t}\right]: S\to T$ is defined by $\left[{s \atop t}\right](x) = t$ if $s \ll x$ and $= 0$ otherwise.

Note that $\left[{s \atop t}\right] \in [S\to T]$ if $S,T \in \underline{CL}$.

Let $F \subseteq S\times T$ be finite. Then

$(F) = \sup \left\{ \binom{s}{t} : (s,t) \in F \right\}$

$[F] = \sup \left\{ \left[{s \atop t}\right] : (s,t) \in F \right\}$.

More notation: For $f,g \in (S \to T)$ we write $f \ll_p g$ iff $f(s) \ll g(s)$ for all $s \in S$. (This is the "pointwise way below relation")

LEMMA 7. Let $f \in (S \to T)$. The basic neighborhoods of f in $(S \to T)$ <u>in the Scott topology</u> are obtained by taking any finite set $F \subseteq S \times T$ such that $(F) \ll_p f$ and considering $W(F) = \{ g \in (S \to T) : (F) \ll_p g \}$.

Proof. Firstly, we note the

SUBLEMMA. $(F) \ll_p g$ iff $(\forall (s,t) \in F)\ t \ll g(s)$.

Proof of the Sublemma. $(F) \ll_p g$ iff $(\forall x)\ \sup_{(s,t) \in F} \binom{s}{t}(x) \ll g(x)$ iff $(\forall x)(\forall (s,t) \in F)\ \binom{s}{t}(x) \ll g(x)$. This clearly implies $(\forall (s,t) \in F)\ t \ll g(s)$. Conversely suppose that this latter condition is satisfied. Take $x \in S$, $(s,t) \in F$. Case 1: $s \leq x$. Then $\binom{s}{t}(x) = t \ll g(s) \leq g(x)$. Case 2. $s \not\leq x$. Then $\binom{s}{t}(x) = 0 \ll g(x)$. Hence the former condition follows. This proves the sublemma.

Evidently $f \in W(F)$, and by the definition of the Scott topology, the family of sets $W(F) = \bigcap_{(s,t) \in F} \{ g : t \ll g(s) \}$ is a basis for this topology. □

Note that $\inf W(F)$ dominates (F). We follow an idea of Scott in proving

LEMMA 8. If $f \in (S \to T)$ then $f = \sup \left\{ \binom{s}{t} : t \ll f(s) \right\}$.

Proof. $\leq$ is clear. To prove the reverse, let $t \ll f(x)$. Then $\binom{x}{t}(x) = t \ll f(x)$; hence $t = \binom{x}{t}(x) \leq \sup \left\{ \binom{s}{t}(x) : t \ll f(s) \right\}$. □

PROPOSITION ~~8~~ 9. Let $S,T \in$ <u>CL</u>, $f,g \in (S \to T)$. Then the following statements are equivalent: (1) $f \ll g$. (2) There is a finite $F \subseteq S \times T$ with $f \leq (F) \ll_p g$. (3) There are finite $F,G \subseteq S \times T$ such that $f \leq (F) \ll (G) \leq g$.

Proof. The equivalence of (1) and (2) follows from Lemmas 7 and 8. The equivalence of of (2) and (3) is simple <u>CL</u>-calculus (the interpolation property of $\ll$). □

LEMMA 10. $k\binom{s}{t} = \left[{s \atop t}\right]$.

Proof. Let $x \in S$. Then $k\binom{s}{t}(x) = \sup_{u \ll x} \binom{s}{t}(u)$. Now let $s \ll x$. Then $\binom{s}{t}(u) = t$ whence $\sup_{u \ll x} \binom{s}{t}(u) = t$. Now let $s \not\ll x$. Then for any u with $u \ll x$ with have $s \not\leq u$, hence $\binom{s}{t}(u) = 0$ and so $\sup_{u \ll x} \binom{s}{t}(u) = 0$. □

Since k preserves sups we have

COROLLARY 11. Let $S,T \in$ <u>CL</u>, $f,g \in [S \to T]$. Then the following statements are equivalent: (1) $f \ll g$. (2) There is a finite $F \subseteq S \times T$ with $f \leq [F] \ll_p g$.

(3) There are finite subsets $F, G \subseteq S \times T$ such that $f \leq [F] \ll [G] \leq g$.

REMARK 12. (Scott) If $f \in [S \to T]$, then $f = \sup\{[{f \atop g}] : t \ll f(s)\}$.

Proof. Use Lemma 8 and apply k, recall Lemma 10. □

PROPOSITION 13. Let $S, T \in \underline{CL}$, $f, g \in (S \to T)$. Then $f \ll g$ implies $k(f) \ll k(g)$.

Proof. If $f \ll g$, then there is a finite set F with $f \leq (F) \ll g$. (Prop.9). If we show $k(F) \ll k(g)$ we are done. Since k preserves sups (Prop.3), it is no loss of generality to assume $f = ({s \atop t})$. By the Sublemma in Lemma 7 $f \ll g$ then means $t \ll g(s)$. We claim that $[{s \atop t}](x) \ll k(g)(x)$ for all x. For this we must show that $s \ll x$ implies $t \ll k(g)(x)$. But $s \ll x$ implies $g(s) \leq \sup g(\Downarrow x) = k(g)(x)$, and the hypothesis $t \ll g(s)$ then furnishes the claim. □

COROLLARY. 14. The left adjoint $f \to \check{f} : [S \to T] \to (S \to T)$ of the corestriction of k is a $\underline{CL}$-embedding. Thus $[S \to T]$ is isomorphic to a $\underline{CL}$ retract of $(S \to T)$.

Proof. As a left adjoint of a surjection, it is an inf preserving injection. Since k respects $\ll$, its left adjoint is a $\underline{CL}$-morphism (see ATLAS). □

We finally identify $\check{f}$.

PROPOSITION 15. Let $f \in [S \to T]$. Then $\check{f} \in (S \to T)$ is defined by

$$\check{f}(s) = \inf f(\operatorname{int} \uparrow s) = \inf\{f(x) : s \ll x\}.$$

Proof. Let $g \in (S \to T)$. We must show that $\check{f} \geq g$ iff $f \geq k(g)$. If $f \geq k(g)$; let $x \ll v$, then by assumption $g(x) \leq f(v)$ which shows $\check{f} \geq g$. Conversely, assume $\check{f} \geq g$. Take any $u \ll x$, then $g(u) \leq \check{f}(u) \leq f(x)$. So $f \geq k(g)$. □

PROPOSITION 16. Let $S, T \in \underline{CL}$. Then the map $r : (S \to T) \to (S \to T)$ given by $r(f) = k(f)^\vee$ is a $\underline{CL}$ retraction onto the set of all $f \in (S \to T)$ with

$$(*) \qquad f(s) = \inf f(\operatorname{int} \uparrow s) = \inf_{s \ll x} f(x) \text{ for all } s \in S.$$

Moreover, im $r \cong [S \to T]$.

Proof. It follows from the preceding that r is a $\underline{CL}$-retraction onto a subobject of $(S \to T)$ which is isomorphis to im $k \cong [S \to T]$. Remains to identify the image of r. If $g \in$ im r, then $g = \check{g}$ for some $g \in [S \to T]$. Then $\inf_{s \ll x} g(x) = \inf_{s \ll x} \inf_{x \ll y} g(y) = \inf_{s \ll y} g(y) = \check{g}(s) = g(s)$. Conversely, suppose that f satisfies $(*)$. We claim $f = k(f)^\vee$, and since $f \leq k(f)^\vee$ (by adjunction theory) we have to show that $k(f)^\vee(x) \leq f(x)$ for any x. Let $a \ll k(f)^\vee(x)$. Since $T \in \underline{CL}$ and $f(x) = \inf_{x \ll y} f(y)$ we take an arbitrary y with $x \ll y$, and we must show that $a \leq f(y)$. From $a \ll k(f)^\vee(x)$ we now have $a \ll k(f)(y) = \sup_{s \ll y} f(s)$. So there is an $s \ll y$ with $a \leq f(s)$. But $f(s) \leq f(y)$ implying the claim. □

We note that the elements of $[S\to T]$ in $(S\to T)$ are characterized in a dual fashion by what have done before:

COROLLARY 17. Let $S,T \in \underline{CL}$. Then a map $f\in(S\to T)$ is in $[S\to T] \subseteq (S\to T)$ iff

$$(**)\quad f(s) = \sup f(\downarrow s) = \sup_{x\ll s} f(x), \text{ for all } s\in S.$$

Proof. We recall $\sup f(\downarrow s) = k(f)(s)$. Thus $(**)$ is equivalent to $k(f) = f$, and since k is a retraction, this is equivalent to $f\in \operatorname{im} k$ $=[S\to T]$ (Prop.4).□

One may consider Corollary 17 as a characterisation of the Scott continuous maps among the monotone maps $S\to T$. Recall that $K(L)$ is the set of compact elements c of L, i.e. elements characterized by $c \ll c$.

We make a few comments on the compact elements in $(S\to T)$ and $[S\to T]$.

PROPOSITION 18. Let $S,T \in \underline{CL}$ and $f\in(S\to T)$. Then the following are equivalent:

(1) $f\in K(S\to T)$ (2) $f=(F)$ for some finite set $F\subseteq S\times T$ such that $(s,t)\in F$ implies $t\in K(T)$.

Proof. (1) $\Rightarrow$ (2): Suppose $f\ll f$. Then by Proposition 9, there is a finite set $G\subseteq S\times T$ such that $f\leq (G)\ll f$, whence $f=(G)\ll(G)$. We define

$$F = \{(s,t(s)) : (s,t')\in G \text{ for some } t' \text{ and } t(s) = \max\{t'' : (s'',t'')\in F \text{ with } s''\leq s\}\}.$$

SUBLEMMA. $(F) = (G)$.

Proof of the Sublemma. Note $(G) = \sup\{\binom{s}{t(s)} : (s,t)\in F\}$. Consider $(F)(s)$ for any $(s,t)\in F$. Indeed, $(F)(s) = \max\{\binom{s'}{t'}(s) : (s',t')\in F\} = \max\{t' : (s',t')\in F$ and $s'\leq s\} = t(s) = \max\{t(s') : s'\leq s\} = \max\{\binom{s'}{t(s')}(s) : (s',t')\in F\} =$ $=(G)(s)$. This shows $(F)=(G)$.□

We now observe that for $(s,t)\in F$ we have $(F)(s) = t$. From $(F)=(G)=f\ll f$ we have $(F)\ll_p(F)$, hence $t=(F)(s)\ll(F)(s)=t$, i.e. $t\in K(T)$.

(2) $\Rightarrow$ (1): If $t\in K(T)$, then $t\ll t$, whence $\binom{s}{t}\ll_p\binom{s}{t}$ and so $\binom{s}{t}\ll\binom{s}{t}$. Since $K(S\to T)$ is a sup-subsemilattice of $(S\to T)$, (2)$\Rightarrow$(1) follows.□

COROLLARY 19. Let $S,T\in\underline{CL}$, $f\in[S\to T]$. Then the following are equivalent: (1) $f\in K[S\to T]$. (2) $f=[F]$ for some finite $F\subseteq S\times T$ with $t\in K(T)$ for $(s,t)\in F$.

Proof. By LEMMA A (2), for any $L\in\underline{CL}$ and any $k\in \operatorname{Ker}(S)$ one has $k(K(L))\subseteq K(L)$. If k happens to respect $\ll$, then we conclude $k(K(L)) = K(k(L))$. By Prop.13 this is the case for $L=(S\to T)$ and the kernel function k of Def.2. Hence, by Prop.4 we have $K[S\to T] = k(K(S\to T))$. In view of Lemma 10, the Corollary now follows from Prop. 18. □

Remark. $\binom{s}{t} = \left[\begin{smallmatrix}s\\t\end{smallmatrix}\right]$ iff $s\in K(S)$.

3a.

Recall that the category $\underline{Z}$ of compact zero dimensional semilattices and continuous semilattice morphisms is isomorphic to the category of (complete) algebraic lattices and maps presering all infs and sup of supdirected sets. (See HMS Duality).

PROPOSITION 20. Let $S, T \in \underline{CL}$. Then the following statements are equivalent:

(1) $T \in \underline{Z}$. (2) $T^S \in \underline{Z}$. (3) $(S \to T) \in \underline{Z}$ (4) $[S \to T] \in \underline{Z}$.

Proof. (1)$\Rightarrow$(2)$\Rightarrow$(3) is clear (Consider the $\underline{CL}$-topologies). (3)$\Rightarrow$(4): By Prop. 3, k preserves sups, and after Prop. 13, k preserves compact elements. (4)$\Rightarrow$(1): T is a $\underline{CL}$-retract of $[S \to T]$ under the map $f \mapsto f(0)$. $\square$

CH III.

Scott raised the question whether ker(S) was in $\underline{CL}$ for any S $\underline{CL}$-object We wish to comment on this question. The inclusion map $\ker(S) \to [S \to S]$ preserves sups, so that ker S is a complete lattice in its own right, and we know from Lemma A that $\ker(S) \in \underline{CL}$ iff

(§) For all $f \in \ker S$ we have $f = \sup\{g \in \ker S : g \ll_{[S \to S]} f\}$.

LEMMA 21 . $[{a \atop b}][{u \atop v}] = [{u \atop [{a \atop 1}][v].\, b}]$.

Proof. i) $[{a \atop b}][{u \atop v}](x) = [{a \atop b}](v)$ [if $u \ll x$, and $= 0$ otherwise] $= b$[if $u \ll x$ and $a \ll v$, and $= 0$ otherwise].

ii) $[{u \atop [{a \atop 1}](v)b}](x) = [{a \atop 1}](v)\, b$. [if $u \ll x$, and $= 0$ otherwise]

$= 1.b$ [if $u \ll x$ and $a \ll v$, and $= 0$ otherwise] . □

LEMMA 22 . Let $F,G \subseteq S \times S$ be finite. Then there is some $H \subseteq S \times S$ finite such that $[F][G] = [H]$.

Proof. We have $[F][G] = (\sup_{(a,b)\in F} [{a \atop b}])(\sup_{(u,v)\in G} [{u \atop v}]) =$

$\sup \{[{u \atop [{a \atop 1}](v)b}] : (a,b) \in F,\ (u,v) \in G\} = [H]$ for

$H = \{(u, [{a \atop 1}](v)b) : (a,b)\in F,\ (u,v)\in G\}$, by LEMMA 21 . □

~~PROPOSITION 23~~

PROPOSITION 23. Let $S \in$ CL and $F \subseteq S \times S$ finite. Then there is a natural number $n = n(F)$ and a finite set $F' = F'(F) \subseteq S \times S$ such that

(i) $[F]^n = [F']$ ~~(ii) ...~~ (ii) $[F']^2 = [F']$.

Proof. Let X be the finite set $[F](S)$. ~~For each $x \in X$ we have $y = [F](x) \in X$ and $y \leq x$, hence by the finiteness of X for each x there is a natural number $n(x)$ such that $[F]^{n(x)+1}(x) = [F]^{n(x)}(x)$. Let us define $n = \max\{n(x): x \in X\} + 1$ and take $s \in S$. Then $x = [F](s) \in X$ whence $[F]^{n+1}(s) = [F]^n(x) = [F]^{m+n(x)+1}(x) = [F]^{m+n(x)}(x) = [F]^{n-1}(x) = [F]^n(s)$.~~ By Lemma 22 (and induction) there is some finite set $F' \in S \times S$ such that $[F]^n = [F']$. Then ~~(i)~~ (i) and (ii) are clear. ~~and (iii) is a consequence of (ii).~~ □

Then $[F]|X$ generates a finite semigroup in X^X, thus $[F]$ generates a finite semigroup in S^S which contains an idempotent $[F]^n$ in its minimal ideal

COROLLARY 24. Let $S \in$ CL . For $f \in \ker(S)$ the following statements are equivalent:

(1) $f = \sup\{g \in \ker S : g \ll f\} = \sup\{\downarrow f \cap \ker S\}$.

(2) $f = \sup\{[F] \in \ker S : F \in S \times S \text{ finite},\ [F] \ll f\}$.

Proof. Trivially (2)⇒(1) . In order to prove (1)⇒(2) , let $g \ll f$.

By Corollary 11 , there is a finite $G \subseteq S \times S$ such that $g \leq [G] \ll f \leq 1$.

then $[G]^m \leq [G]$ for all $m = 1, 2, \ldots$.

Let $F = G'$ according to Prop.23. Then $g = g^{n(G)} \leq [G]^{n(G)} = [F] \leq [G] \ll f$, and $[F] \in \ker S$ by Prop.23. □

The following is an observation which is dual to one made by Scott.

PROPOSITION 25. Let $S \in \underline{CL}$ and $f \in \ker S$. Set $f' = \sup\{g \in \ker S : g \ll f\}$. For each $g \ll f$ in ker S we have $g(S) \in \underline{CL}$. Let $\varphi_g : S \longrightarrow g(S)$ be the corestriction of g. Now suppose $g \leq h \ll f$. Then $g(S) \subseteq h(S)$; let $\varphi_{gh} : h(S) \longrightarrow g(S)$ be its left adjoint, which is in $\underline{CL}$. ~~Sin~~ For each $g \ll f$ there is a commutative diagram

$$\begin{tikzcd} S \arrow[rr, "f''"] \arrow[dr, "\varphi_g"'] & & f'(S) \arrow[dl, "\psi_g"] \\ & g(S) & \end{tikzcd}$$

$f'' =$ corest f

Thus there is a natural morphism $\psi : f'(S) \longrightarrow \lim_{g \leq h \ll f} (g(S), \varphi_{gh})$.

This map is an isomorphism.

Proof. The relation $g(S) \subseteq h(S)$ is proved by Scott (p.121). Hence the inverse system $(g(S), \varphi_{gh}, g \leq h \ll f)$ is well defined as is its ~~L~~ limit in $\underline{CL}$. All maps φ_g are surjective, hence so are all ψ_g. It follows that ψ is surjective. We must show that ψ separates points. Suppose that $s, t \in S$ are such that $f(s) \not\leq f(t)$. We then find an $a \in S$ with ~~f~~ $a \not\leq f(s)$ but $a \ll f(t)$. Going back to the definition of f' as the sup of the up-directed set of all $g \in \ker S$ with $g \ll f$ we find a $g \ll f$ in ker S such that $a \leq g(t)$. Then $g(s) \leq f(s)$ implies ~~g(s)~~ $a \not\leq g(s)$. Thus $g(s) \neq g(t)$, which implies ~~thaaa~~ that $\psi(f(s)) \neq \psi(f(t))$. $\square$

COROLLARY 26. If the conditions of Corollary 24 are all satisfied, then $f(S)$ $\underline{Z}$.

Proof. We apply Prop. 25 to condition (2) in Coroll.24 and conclude that $f(S)$ is profinite, hence in $\underline{Z}$. $\square$

LEMMA 27 . Let $f \in \ker S$ such that $f(S) \in \underline{Z}$. Then

$$f = \sup\left\{\left[{c \atop c}\right] : c \in K(f(S))\right\} .$$

Proof. We must show $f(s) = \sup\left\{\left[{c \atop c}\right](s) : c \in K(f(S))\right\}$. But $\left[{c \atop c}\right](s) = c$ if $c \leq s$ and $= 0$ otherwise, since $c \in K(f(S)) \subseteq K(S)$,because the inclusion $f(S) \hookrightarrow S$ respects $\ll$ as a CL^{op} map (LEMMA A). But ~~f~~ the corestriction $\tilde{f} : S \longrightarrow f(S)$ is left adjoint to the inclusion, whence $c \leq s$ iff $c \leq f(s)$, so that $\left[{c \atop c}\right] = \left[{c \atop c}\right] \circ f = j \circ \left[{c \atop c}\right]_{f(S)} \circ \tilde{f}$.

But on any $\underline{Z}$-object T we clearly have $1_T = \sup\{[{}^c_c]: c \in K(T)\}$, and j preserves arbitrary sups. Thus $\boxed{\cancel{xxxxxxxfx}}$ $\sup\{[{}^c_c](s)\ x : c \in K(f(S))\}$ $= j(\ \sup\{[{}^c_c]_{f(s)}(\tilde{f}(s)): c \in K(f(S))\}\ ) = j(\tilde{f}(s)) = f(s)$. □

A compact semigroup is dimensionally stable iff it topological dimension dominates that of all of its quotients.

THEOREM I . Let $S \in \underline{CL}$. Then the following ~~xxxxx~~ statements are equivalent :

(1) ker $S \in \underline{CL}$ (Scott's parlance: J_S is a continuous lattice).

(2) S is a dimensionally stable compact zerodimensional semilattice.

(3) S is a (complete) algebraic lattice such that the set K(S) of compact elements does not contain any non-degenerate order dense chain.

(4) $\boxed{\cancel{xxxxxxx}}$ ker $S \in \underline{Z}$ (ker S is a(complete)algebraic lattice).

Proof. (1) is equivalent to (§) preceding Lemma 21. Corollary 11, PRoposition 25 and Lemma 27 then show ~~x~~ that (1) iff

(1') For all f ker S the image f(S) $\underline{Z}$. $\boxed{\cancel{xxxxxxxxx(2)xxxxxxx}}$

A totally ordered set C is ~~order dense~~ iff
C is non-degenerate and x < y in C
implies the existence of a c ∈ C with
x < c < y.

Evidently, (2) implies (1'). Conversely suppose that S is not dimensionally stable. Then there is a $\underline{CL}$-surgection $g:S \to I = [0,1]$. Let $d:I \to S$ be its right adjoint and set $f:S \to S$, $f= gd$. Then $\boxed{\cancel{xxxxxxx}}$ $f^2 = f \_ 1$ and since $g,d \in \underline{Cont}$, then f Cont. Thus $f \in$ ker S , but $f(S) = I$, so (1') is violated.

The equivalence of (2) and (3) is not entirely trivial; it was proved in DIMENSION RAISING (Hofmann,Mislove, Stralka, Math.~~M~~ Z. 135 (1973) 1-36).

(4) ⇒ (1) is trivial. If (1) -(3) are satisfied, then for each $f \in$ ker S we have $f = \sup\{[{}^c_c] : c \in K(f(S))\}$ and all $[{}^c_c]$ are compact in ker S . □

In another memo (to emanate from Darmstadt) it is shown that ~~xxxx~~ the set of relations $\prec$ on a complete lattice L which satisfy a few axioms describing basic properties of $\ll$ and occuring in L letter from Scott to Hofmann of 3-30-76 ,pp.7-8 (being attributed there to Mike Smyth) is order isomorphic to ker PL where PL is the $\underline{Z}$-object of lattice ideals of L considered in ATLAS. The questione whetherz the totality of $\prec$ -relations on a complete lattice L is a continuous lattice is then answered in the following

COROLLARY 28. Let L be a complete lattice. Then ker $PL \in \underline{CL}$ iff L does not contain any ~~non-degenerate~~ orderz dense chains.

Proof. We have $K PL = (L,\vee)$, where $(L,\vee)$ is the ~~discrete~~ sup semilattice underlying L. ~~Since~~ The assertion then follows from THEOREM I □

One might ask the question which $\underline{Z}$-objects can occur as $\ker S$.

PROPOSITION 29. Let $S \in \underline{Z}$ be dimensionally stable. Then $\ker S$ is dimensionally stable.

Proof. We must show that no ~~non-degenerate~~ chain C in $K(\ker S)$ can be order dence. If $f \in K(\ker S)$, then $f \ll f$ and so by Lemma 27 there are elements $c_1, \ldots, c_n$ such that $f \leq \begin{bmatrix} c_1 \\ c_1 \end{bmatrix} \vee \ldots \vee \begin{bmatrix} c_n \\ c_n \end{bmatrix} \ll f$. Hence $f = \sup_j \begin{bmatrix} c_j \\ c_j \end{bmatrix}$. Thus $f(S)$ is finite. If now C is a chain in $K(\ker S)$ then $f \leq g \leq h$ in C is equivalent to $f(S) \subseteq g(S) \subseteq h(S)$. Since $h(S)$ is finite, there are only finitely many g with $f \leq g \leq h$. This shows that C cannot be ordere dense. □

We note in conclusion that $\ker(S)$ is isomorphic to the lattice of of CL^{op}-subobjects of S and thus isomorphic to $\mathrm{Cong}(S)^{op}$, the opposite of the lattice of closed ($\underline{CL}$) congruences of S.