SCS 9: Commentary on Scott's Function Spaces

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Hofmann and Mislove: SCS 9: Commentary on Scott's Function Spaces

SEMINAR ON CONTINUITY IN SEMILATTICES (SCS)

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TOPIC Commentary on Scott's function spaces


We quote an amplified version of a Lemma in GK (1.4)

**Lemma A.** Let $L \in CL$ and $k: L \rightarrow L$ a kernel function, i.e., a function satisfying

(i) $(\forall x, y) \ x \leq y \Rightarrow k(x) \leq k(y)$,

(ii) $k \leq L$,

(iii) $k^2 = k$.

Then $(i) \ T = k(L)$ is a complete lattice and the corestriction $k: L \rightarrow T$ is left adjoint to the inclusion function.

(ii) The following conditions are equivalent:

(iii) $k \in Cont$ where $Cont$ is the category of continuous lattices with Scott continuous functions. In other words,

$(\forall D) \ D$ up-directed in $L \Rightarrow \ \sup_L k(D) = k(\sup_L D)$.

(1) $t \in T \Rightarrow t = \sup_L \{ s \in T : s \leq_L t \} = \sup_L (\uparrow t \cap T)$

(2) $\leq_L = \cup \{ t \times T \}$.

(3) $T \in CL$.

(4) $T \in CL$.

(5) The inclusion $T \rightarrow L$ is in $CL$.

**Notation.** For $L \in CL$, we write $ker(k) \in [L \rightarrow L]$ for the set of all functions $k$ satisfying (i) - (iv) (hence also (1)-(5). [See Scott, 3.11]

$CL_L$.

Our aim is, inter alia, to give an alternative proof of Scott's result that $[S \rightarrow T] \in CL$. The category of posets and order-preserving maps is called Poset.

In contrast, instead of $Poset(S,T)$ we will write $(S \rightarrow T)^c$.

**Lemma 1.** Let $S \in Poset$, $T \in CL$, then $(S \rightarrow T) \in CL$ relative to the structure induced from $T^c$.

Proof. $(S \rightarrow T)$ is closed in $T^c$ in the $CL$-product topology, and clearly $T \in CL$. □

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U. Oxford (Scott)

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U. California, Riverside (Stralka)
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Tulane U., New Orleans (Hofmann, Mislove)
U. Tennessee, Knoxville (Carruth, Crawley)
DEFINITION 2. Let $S,T \in CL$. Define $k:(S \rightarrow T) \rightarrow (S \rightarrow T)$ by

$$k(f)(a) = \sup_x f(a \cdot x) = \sup_x \leq_T f(x).$$

(Note that $k$ is well defined: $x \leq_T y$ will always imply $k(f)(x) \leq_T k(f)(y)$.)

PROPOSITION 3. Let $S,T \in CL$. Then $k \in \text{hom}(S \rightarrow T)$, and $k$ preserves arbitrary sups.

Proof. (i) Suppose $f \leq g$ in $(S \rightarrow T)$. From Def. 2 it is clear that $k(f) \leq_T k(g)$.

(ii) Let $x \leq_T y$ in $(S \rightarrow T)$. We claim that $k(x) \leq_T k(y)$.

(iii) Let $x \leq_T y$ in $(S \rightarrow T)$ and set $h = \sup f \leq_T g$. Then $k(h)(x) = \sup f(x)$.

(iv) Let $\mathcal{D}(S \rightarrow T)$ and set $h = \sup f \leq_T g$. We claim that $k(h)(x) = \sup f(x)$.

PROPOSITION 4. Under the $\text{xx}$ hypotheses of Prop. 3,

$$k(S \rightarrow T) = [S \rightarrow T].$$

Proof. a) Let $f \leq_T g$. Show that $k(f) \leq_T k(g)$. Let $D \leq_T S$ be up-directed and let $x = \sup D$. Clearly, $\sup k(f)(D) \leq_T k(f)(D)$. Now let $x \leq_T k(f)(D) = \sup f(x)$. By definition of $\leq_T$ there is an $x' \leq_T D$ with $x \leq_T f(x')$. But $x' \leq_T x = \sup D$ it itself implies the existence of some $d \in D$ with $x' \leq_T d$. Thus $x \leq_T k(d)(x) \leq_T \sup k(f)(D)$.

b) Let $f \in [S \rightarrow T]$. Then $k(f)(a) = \sup f(a \cdot x) = f(a)$ since $f$ preserves sups of up-directed sets.

COROLLARY 5. $[S \rightarrow T] \in CL$ (Scott).

Proof. From LEMMA A, Propositions 3,4.

We now further investigate the kernel function $k$ on $(S \rightarrow T)$. So far we know that it preserves arbitrary sups (hence has a left adjoint $f \mapsto f^-$ which we will investigate presently) and that its corestriction $k(S \rightarrow T) \\ [S \rightarrow T]$ is a $CL$ map which is left adjoint to the inclusion map $[S \rightarrow T] \rightarrow (S \rightarrow T)$. We need to understand clearly the $\leq_T$ relation on $(S \rightarrow T)$.

In the following we allow ourselves a slight deviation from Scott's notation.

NOTATION 6. Let $S,T \in \text{Pos}$. Then $\leq_T (x,y) \in S \times T$. Then $(x \leq_T y)$ is defined by $(x \leq_T y) \equiv (x \leq_T y)$.

Note that $\leq_T (x,y) \in (S \rightarrow T)$ if $S,T \in CL$.

Let $F \leq S \times T$ be finite. Then
\((\forall t) = \sup \left\{ \left( \frac{t}{t} \right) : (s,t) \in \mathcal{I} \right\} \)

\((\exists t) = \sup \left\{ \left( \frac{t}{t} \right) : (s,t) \in \mathcal{I} \right\} \).

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Lem. 7. Let \(F \subseteq (S \rightarrow T)\). The basic neighborhoods of \(F\) in \((S \rightarrow T)\) in the Scott topology are obtained by taking any finite set \(F \subseteq S \times T\) such that \((\forall s \in S) f(s) < < g(s)\) for all \(s \in S\). (This is the "pointwise way below relation".)

**Proof.** Firstly, we note the

**Sublemma.** \((\forall s \in S) f \leq g\) iff \((\forall (s,t) \in \mathcal{I}) t \leq g(s)\).

**Proof of the Sub lemma.** \((\forall s \in S) f \leq g\) iff \((\forall x) (\forall (s,t) \in \mathcal{I}) (s,t)(x) \leq g(x)\). This clearly implies \((\forall x) (\forall (s,t) \in \mathcal{I}) (s,t)(x) \leq g(x)\). Conversely suppose that this latter condition is satisfied. Take \(x \in S\), \((s,t) \in \mathcal{I}\). Case 1: \(s \leq x\). Then \((s,t)(x) = t \leq g(s) \leq g(x)\).

Case 2: \(s \not\leq x\). Then \((s,t)(x) = 0 \leq g(x)\). Hence the former condition follows.

This proves the sub lemma.

Evidently \(f \in \mathcal{W}(F)\), and by the definition of the Scott topology, the family of sets \(\mathcal{W}(F) = \left\{ (s,t) \in \mathcal{I} : g : t \leq g(s) \right\}\) is a basis for this topology.

Note that \(\inf \mathcal{W}(F)\) dominates \((F)\). We follow an idea of Scott in proving

Lem. 8. If \(f \in (S \rightarrow T)\) then \(f = \sup \left\{ \left( \frac{t}{t} \right) : t \leq f(s) \right\}\).

**Proof.** It is clear. To prove the reverse, let \(t \leq f(s)\). Then \((t)(x) = t \leq f(s)\); hence \(\inf \left\{ \left( \frac{t}{t} \right) : t \leq f(s) \right\}\).

**Proposition 9.** Let \(S,T \subseteq \mathcal{C}L\), \(f,g \in (S \rightarrow T)\). Then the following statements are equivalent: \(1\) \(f \leq g\), \(2\) There is a finite \(F \subseteq S \times T\) with \(f(F) \leq g\).

(1) There are finite \(F,G \subseteq S \times T\) such that \(f \leq F\) and \(G \leq g\).

**Proof.** The equivalence of (1) and (2) follows from Lemmas 7 and 8. The equivalence of (1) and (2) is simple \(\mathcal{C}L\)-calculus (the interpolation property of \(\leq\)).

Lem. 10. \(k \left( \frac{t}{t} \right) = \left[ \frac{t}{t} \right]\).

**Proof.** Let \(x \in S\). Then \(k(t)(x) = \sup \left\{ \left( \frac{t}{t} \right) : t \leq x \right\}\).

\(\left( \frac{t}{t} \right)(x) = t\) whence \(\sup \left\{ \left( \frac{t}{t} \right) : t \leq x \right\}\).

Then for any \(u \leq x\) with \(u \leq x\) with \(s \not\leq u\), hence \(\left( \frac{t}{t} \right)(u) = 0\). And so \(\sup \left\{ \left( \frac{t}{t} \right) : t \leq x \right\} = 0\).

Since \(k\) preserves \(\sup\) s we have

Corollary 11. Let \(S,T \subseteq \mathcal{C}L\), \(f,g \in (S \rightarrow T)\). Then the following statements are equivalent: \(1\) \(f \leq g\), \(2\) There is a finite \(F \subseteq S \times T\) with \(f \leq F\).
There are finite subsets \( F, G \subseteq S \times T \) such that \( F \leq [y] \leq [G] \leq \mathfrak{S} \).

**Remark 11.** (Scott) \( f \in [S \to T] \), then \( f = \sup_{[y]} [f] \cdot t \iff f = f(x) \).

**Proof.** Use Lemma 6 and apply \( k \), recall Lemma 10.

**Proposition 11.** Let \( S, T \subseteq \mathfrak{S} \), \( f, g \in (S \to T) \). Then \( f \ll g \) implies \( k(f) \ll k(g) \).

**Proof.** If \( f \ll g \), then there is a finite set \( F \) with \( f \leq \sup_{[y]} [f] \cdot t \). (Prop. 9). If we show \( k(F) \ll k(g) \) we are done. Since \( k \) preserves sups (Prop. 1), it is no loss of generality to assume \( f = \sum_t [x] \cdot t \). By the Sublemma in Lemma 7 \( f \ll g \) then means \( t \ll g \). We claim that \( [x] \cdot t \ll g(t) \) for all \( t \). For this we must show that \( t \ll g \) implies \( t \ll k(g) \).

But \( t \ll g \) implies \( g(t) \leq \sup_{[y]} [g(y)] \cdot t \), and the hypothesis \( t \ll g \) then furnishes the claim.

**Corollary 14.** The left adjoint \( f \to \mathfrak{S} : [S \to T] \to [S \to T] \) of the corestriction of \( k \) is a \( \mathfrak{S} \)-embedding. Thus \( [S \to T] \) is a \( \mathfrak{S} \)-object of \( (S \to T) \).

**Proof.** As a left adjoint of a \( \mathfrak{S} \)-surjection, it is an \( \mathfrak{S} \)-preserving injections. Since \( x \) respects \( \ll \), its left adjoint is a \( \mathfrak{S} \)-morphism (see ATLAS).

We finally identify \( \mathfrak{S} \).

**Proposition 15.** Let \( f \in [S \to T] \). Then \( \mathfrak{S} \subseteq (S \to T) \) is defined by

\[
\mathfrak{S}(x) = \inf \{ f(t) : t \leq x \} = \inf \{ f(t) : t \ll x \}.
\]

**Proof.** Let \( \mathfrak{S} \subseteq (S \to T) \). We must show that \( \mathfrak{S} \subseteq \mathfrak{S} \iff \mathfrak{S} \subseteq k(g) \).

If \( f \leq k(g) \), then \( x \ll v \), then by assumption \( g(x) \leq f(v) \) which shows \( f \geq g \).

Conversely, assume \( f \geq g \). Take any \( x \ll v \), then \( g(u) \leq f(u) \leq f(x) \). So \( f \geq k(g) \).

**Proposition 16.** Let \( S, T \subseteq \mathfrak{S} \). Then the \( \mathfrak{S} \)-functor \( x : (S \to T) \to [S \to T] \) given by \( x(f) = k(f)^x \) is a \( \mathfrak{S} \)-retraction onto the set of all \( f \in [S \to T] \)

such that \( \mathfrak{S}(x) = \inf \{ f(t) : t \leq x \} \), \( \mathfrak{S}(x) = \inf \{ f(t) : t \ll x \} \).

Moreover, in \( r \in [S \to T] \).

**Proof.** It follows from the preceding that \( r \) is a \( \mathfrak{S} \)-retraction onto a sub-

object of \( (S \to T) \) which is isomorphic to \( k \in [S \to T] \). Remains to identify the image of \( r \). If \( \mathfrak{S} \subseteq \mathfrak{S} \), then \( f = g \) for some \( g \subseteq [S \to T] \). Then \( \inf \mathfrak{S}(x) \)

\[
\mathfrak{S}(x) = \inf \{ f(t) : t \leq x \} = \inf \{ f(t) : t \ll x \} = \inf \{ f(t) : t \ll x \}.
\]

Conversely, suppose that \( r \) satisfies \( (k) \), we claim \( r = k(f)^x \), and since \( f \leq k(g)^x \) (by adjunction theory) we have to show that \( k(f)^x \leq f(x) \) for any \( x \). Let \( a \ll k(f)^x (x) \). Since \( a \subseteq \mathfrak{S} \) and \( f(x) = \inf \{ f(t) : t \leq x \} \), we take an arbitrary \( y \) with \( a \leq y \), and we must show that \( a \leq f(y) \). From \( a \ll k(f)^x (x) \) we now have \( a \subseteq k(f)^x (y) \) so \( f(y) \). So there is an \( a \subseteq y \) with \( a \subseteq f(x) \). But \( f(x) \leq f(y) \) implying the claim.

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We note that the elements of $[S \rightarrow T]$ in $(S \rightarrow T)$ are characterized in a dual fashion by what have been done before:

**Corollary 17.** Let $S, T \in \mathcal{CL}$. Then a map $f \in (S \rightarrow T)$ is in $[S \rightarrow T] \subseteq (S \rightarrow T)$ iff

\[(f \succeq f) \Rightarrow \exists \xi \in T, \forall s \in S \quad f(s) = \xi(s) = \sup \{s \in S \mid f(x) \prec f(s)\} \text{ for all } s \in S.
\]

Proof. We recall $\sup(f \circ g) = k(f)(g)$. Thus $(f \succeq f)$ is equivalent to $k(f) \circ k = f$, and since $k$ is a retraction, this is equivalent to $f \circ k = k \circ f$ (Prop. 4). \]

One may consider Corollary 17 as a characterization of the Scott continuous maps among the monotone maps $S \rightarrow T$. Recall that $\mathcal{K}(L)$ is the set of compact elements of $L$, i.e., elements characterized by $c \preceq c$.

We make a few comments on the compact elements in $(S \rightarrow T)$ and $[S \rightarrow T]$.

**Proposition 18.** Let $S, T \in \mathcal{CL}$ and $f \in (S \rightarrow T)$. Then the following are equivalent:

1. $f \in \mathcal{K}(S \rightarrow T)$
2. There is a finite set $F \subseteq S \times T$ such that $(s, t) \in F$ implies $t \in \mathcal{K}(T)$.

Proof. (1) $\Rightarrow$ (2): Suppose $f \preceq f$. Then by Proposition 9, there is a finite set $G \subseteq S \times T$ such that $f \preceq (G) \preceq f$, whence $f \preceq (G) \preceq (G)$. We define

\[F = \{(s, t(s)) : \max \{t': (s', t') \in G \text{ for some } t' \text{ and } t(s) = \max \{t'' : (s'', t'') \in F \text{ with } t'' \leq s'' \leq s\}\}.\]

**Sublemma:** $(F) = (G)$.

Proof of the Sublemma. Note $(G) = \sup \{t(s) : (s, t) \in F\}. \quad \square$

First, we observe that for $(s, t) \in F$, we have $(F)(s) = t$. From $(F) = (G) \circ f \preceq f$, we have $(F) \preceq (G) \circ f$, hence $t = (F)(s) \preceq (G)(s) - t$, i.e., $t \in \mathcal{K}(T)$.

$(2) \Rightarrow (1)$: If $t \in \mathcal{K}(T)$, then $t \preceq t$, whence $(s) \preceq (s)$, and so $(s) \preceq (s)$. Since $\mathcal{K}(S \rightarrow T)$ is a sup-semilattice with $\mathcal{K}(S \rightarrow T)$, (2) follows. \]
Recall that the category $\mathcal{Z}$ of compact zero dimensional semilattices and continuous semilattice morphisms is isomorphic to the category of (complete) algebraic lattices and maps preserving all infs and sup of subdirect components. (See [5, Duality]).

**Proposition 20.** Let $S, T \in \mathcal{CL}$. Then the following statements are equivalent:

1. $T \subseteq S$.
2. $T \Rightarrow S$.
3. $(S \Rightarrow T) \subseteq S$.
4. $[S \Rightarrow T] \subseteq S$.

**Proof.** (1)$\Rightarrow$(2) is clear (Consider the $\mathcal{CL}$-topologies). (3)$\Rightarrow$(4) By Prop. 13, $k$ preserves infs, and after Prop. 13, $k$ preserves compact elements.

(4)$\Rightarrow$(1): $T$ is a $\mathcal{CL}$-retract of $[S \Rightarrow T]$ under the map $f \mapsto f(0)$.]

Scott raised the question whether $\ker(S)$ was in $\mathcal{CL}$ for any $\mathcal{CL}$-object. We wish to comment on this question. The inclusion map $\ker(S) \rightarrow [S \Rightarrow S]$ preserves infs, so that $\ker S$ is a complete lattice in its own right, and we know from Lemma 1 that $\ker(S) \in \mathcal{CL}$ iff

(6) For all $f \in \ker S$ we have $f = \sup \{g \in \ker S : g \ll [S \Rightarrow S] \}$.
LEMMA 21. \( I_{\gamma}^x = I_{\alpha}^x \cdot I_{\gamma}^x \).

Proof. 1) \( I_{\gamma}^x \cdot I_{\gamma}^x = I_{\gamma}^x \cdot I_{\gamma}^x \).
2) \( I_{\gamma}^x \cdot I_{\gamma}^x = I_{\gamma}^x \cdot I_{\gamma}^x \).

LEMMA 22. Let \( F \subseteq S \times S \) be finite. Then there is some \( H \subseteq S \times S \) finite such that \( [F][G] = [H] \).

Proof. We have \( [F][G] = (\sup_{(a,b) \subseteq F} [a]^u) \cdot (\sup_{(u,v) \subseteq G} [u]_v) = \sup \{ [a]^u : (a,b) \subseteq F, (u,v) \subseteq G \} \) for \( H = \{ (u, [a]^u(v)) : (u,v) \subseteq F \} \) by LEMMA 21.

PROPOSITION 23. Let \( S \subseteq G \) and \( F \subseteq S \times S \) finite. Then there is a natural number \( n = n(F) \) and a finite set \( F' = F'(F) \subseteq S \times S \) such that

(i) \( [F']^n = [F'] \)
(ii) \( [F']^2 = [F'] \).

Proof. Let \( X \) be the finite set \( [F](S) \). For each \( x \in X \) we have \( y = [F](x) \), and by the finiteness of \( X \) for each \( x \) there is a natural number \( n(x) \) such that \( [F']^n(x) = [F'](x) \). Let us define \( n(x) = \max \{ n(x) : x \in X \} \) and take \( s \in S \). Then \( x = [F](s) \in X \) where \( [F']^n(x) = [F']^n(x) = [F']^n(x) \cdot n(x) + 1 = [F']^n(x) \cdot n(x) \).

By Lemma 22 (and induction) there is some finite set \( F' \subseteq S \times S \) such that \( [F']^n = [F'] \). Then (i) (and (ii)) is a consequence of (ii).

COROLLARY 24. Let \( S \subseteq G \). For \( f \in \ker(S) \) the following statements are equivalent:

1. \( f = \sup \{ g \in \ker(S) : g \ll f \} = \sup \{ f \cap \ker(S) \} \).
2. \( f = \sup \{ [F] \subseteq \ker(S) : F \subseteq S \times S \) finite, \( [F] \ll f \} \).

Proof. Trivially (2) \Rightarrow (1). In order to prove (1) \Rightarrow (2), let \( g \ll f \).

By Corollary 11, there is a finite \( G \subseteq S \times S \) such that \( g \ll [G] \ll f \).

Let \( F = G' \) according to Prop. 23. Then \( g \ll [G]^n = [F] \ll [G] \ll f \), and \( [F] \subseteq \ker(S) \) by Prop. 23.
The following is an observation which is dual to one made by Scott.

PROPOSITION 25. Let \( S \subseteq \text{Cl} \) and \( f \in \ker S \). Set \( f' = \sup_{g \in \ker S} g \leq f \).

For each \( g \leq f \), we have \( g[S] \subseteq \text{Cl} \). Let \( \varphi : \text{Cl} \to \text{Cl} \) be the corestriction of \( g \). Now suppose \( g \leq h \ll f \). Then \( g[S] \subseteq h[S] \); let \( \varphi : h[S] \to g[S] \) be its left adjoint, which is in \( \text{Cl} \). Sim. For each \( g \leq f' \), there is a commutative diagram:

\[
\begin{array}{ccc}
S & \xrightarrow{f'} & f'(S) \\
\downarrow{\varphi} & \nearrow{\gamma_g} & \\
g(S) & \xrightarrow{\gamma_f} & f'' \text{ corest } f
\end{array}
\]

Thus there is a natural morphism \( \varphi : f'(S) \to \lim_{g \leq h \ll f} (g[S], \varphi : g[S]) \).

This map is an isomorphism.

Proof. The relation \( g[S] \subseteq h[S] \) is proved by Scott (p.121). Hence the inverse system \( (g[S], \varphi : g[S]) \) is well defined as is its \( \xi \) limit in \( \text{Cl} \). All maps \( \varphi : g[S] \) are surjective, hence so are all \( \gamma_g \). It follows that \( \gamma_g \) is surjective. We must show that \( \gamma_g \) separates points. Suppose that \( a, b \in S \) are such that \( f(a) \neq f(b) \). Then find an \( a \ll f \) with \( a \neq f(a) \) but \( a \ll f(b) \). Going back to the definition of \( f' \) as the sup of the up-directed set of all \( g \) with \( g \ll f \) we find a \( g \ll f \) in \( \ker S \) such that \( a \ll g(t) \). Then \( g(a) \neq f(a) \) implies \( f(a) \neq f(t) \). Thus \( g(a) \neq g(t) \), which implies \( \gamma_g f(a) \neq \gamma_g f(t) \).

COROLLARY 26. If the conditions of Corollary 24 are all satisfied, then \( f(S) \neq 2 \).

Proof. We apply Prop. 25 to condition (2) in Coroll. 24 and conclude that \( f(S) \neq 2 \).

LEMMA 27. Let \( S \subseteq \ker S \) such that \( f(S) \neq 2 \). Then

\[
f = \sup \{ [^\circ] : o \in K(f(S)) \},
\]

where \( \hat{k} \).

Proof. \( f = \sup \{ [^\circ] : o \in K(f(S)) \} \). But \( [^\circ] \) subst. if \( \circ \leq 2 \) and \( \circ \) otherwise \( \hat{g} \), since \( \circ \subseteq K(f(S)) \subseteq K(S) \), because the inclusion \( f(S) \subseteq \text{Cl} \) respects \( \subseteq \) as a \( \text{Cl} \) map (Lem. A). But \( \hat{f} \) the corestriction \( f : S \to f(S) \) is left adjoint to the inclusion, whence \( \circ \subseteq 2 \) iff \( \circ \subseteq f(2) \), so that \( [^\circ] = [^\circ] \circ f = \hat{f} \circ [^\circ] \).

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But on any $2$-object $f$ we clearly have $1_f = \sup\{[c]_f : c \in K(S)\}$, and $\beta$ preserves arbitrary sups. Thus $\sup\{\beta f(a) : c \in K(f(S))\}$ = $\beta(\sup\{f(a) : c \in K(f(S))\})$. 

A compact semigroup is dimensionally stable if its topological dimension dominates that of all of its quotients.

THEOREM I. Let $S \leq CL$. Then the following statements are equivalent:

1. $\ker S \leq CL$ (Scott's parlance: $S$ is a continuous lattice).

2. $S$ is a dimensionally stable compact zero-dimensional semilattice.

3. $S$ is a (complete) algebraic lattice such that the set $K(S)$ of compact elements does not contain any non-degenerate order dense chain.

4. $\ker S < 2^0$ (ker $S$ is a complete algebraic lattice).

Proof. (1) is equivalent to (3) preceding Lemma 21. Corollary 11, Proposition 25 and Lemma 27 then show that (1) if

$1'$ For all $f \in \ker S$ the image $f(S)$ is closed.

Evidently, (2) implies $(1')$. Conversely suppose that $S$ is not dimensionally stable. Then there is a $CL$-sequence $\mathcal{S} : S \rightarrow 1 = \{0, 1\}$. Let $d : I \rightarrow S$ be its right adjoint and set $f(d) : S \rightarrow S$, $[f] = g$ d. Then $ \beta f = \beta g$, and since $\mathcal{S} \in \text{Cont}$, then $\mathcal{S} \in \text{Cont}$. Thus $\mathcal{S} \in \ker S$, but $f(S) = 1$, so

$(1')$ is violated.

The equivalence of (2) and (3) is not entirely trivial; it was proved in DIMENSION RAISING (Hofmann, Mislove, Snelbecker, Math. Z. 135 (1971) 1–16).

$4' \Rightarrow 1'$ is trivial. If (1) holds then for each $f \in \ker S$ we have $f = \sup\{[c]_f : c \in K(f(S))\}$ and all $[c]_f$ are compact in $\ker S$.

In another note (to emanate from Darmstadt) it is shown that many of the set-theoretical results (1) establish a few axioms describing basic properties of $\leq$ and occurring in a letter from Scott to Hofmann of 3–30–70, pp. 7–8 (being attributed to Mike Smyth) is order isomorphic to $\ker PL$ where $PL$ is the $\leq$-object of lattice ideals of $L$ considered in ATLAS. The questions whether the totality of $\leq$-relations on a complete lattice $L$ is a continuous lattice is then answered in the following.

COROLLARY 23. Let $L$ be a complete lattice. Then $\ker PL \leq CL$ iff $L$ does not contain any non-degenerate order dense chains.

Proof. We have $KPL = (L, \vee)$, where $(L, \vee)$ is the discrete sup semilattice underlying $L$. Since the assertion then follows from THEOREM I.
One might ask the question which \( \mathbb{Z} \)-objects can occur as \( \ker S \).

**Proposition 29.** Let \( S \in \mathbb{Z} \) be dimensionally stable. Then \( \ker S \) is dimensionally stable.

**Proof.** We must show that no chain \( C \) in \( K(\ker S) \) can be order dense.

If \( f \in K(\ker S) \), then \( f \preceq \bar{f} \) and so by Lemma 27 there are elements \( f_1, \ldots, f_n \) such that \( f \leq f_1 \lor \ldots \lor f_n \vartriangleleft f \). Hence

\[
f = \sup_3 \left[ \frac{\left[ f_i \right]}{n} \right].
\]

Thus \( f(S) \) is finite. If now \( C \) is a chain in \( K(\ker S) \) then \( f \leq g \leq h \) in \( C \) is equivalent to \( f(S) \leq g(S) \leq h(S) \).

Since \( h(S) \) is finite, there are only finitely many \( g \) with \( f \preceq g \preceq h \). This shows that \( C \) cannot be order dense. \( \Box \)

We note in conclusion that \( \ker(S) \) is isomorphic to the lattice of \( \mathbb{Z}^{\text{op}} \)-subobjects of \( S \) and thus isomorphic to \( \text{Cong}(S)^{\text{op}} \), the opposite of the lattice of closed (\( \mathbb{Z}^{\text{op}} \)) congruences of \( S \). Thus \( \text{Sim}(S) \) is always an object.