Seminar on Continuity in Semilattices

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SCS 8: On the Theorem of Lawson's that all Compact Locally Connected Finite Dimensional Semilattices are CL

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SEMINAR ON CONTINUITY IN SEMILATTICES (SCS)

NAME(S)	Hofmann and Mislove	DATE	M 6	D 28	Y 76
TOPIC	On the Theorem of Lawson's that all compact locally connected finite dimensional semilattices are CL				
REFERENCE	Memo from Hofmann of 3-23-76 p.3, bottom "On peripherality in CL-theory", uncirculated correspondence between Mislove and Hofmann				

In the March memo mentioned above it was proposed to link the topological concept of peripherality with the lattice theoretical concept of "faciality". We pursue this to reprove a slight generalisation of Lawson's theorem.

For the definition of peripheral points we refer to the literature, notably to

[LM] Lawson, J.D., and B. Madison, Peripheral and inner points, Fund. Math. 69 (1970), 253-266.

We use the following facts which suffice for our discussion.

LEMMA A. Let $(s,x) \mapsto sx: S \times X \longrightarrow X$ be a continuous function between topological spaces, where X is compact. Suppose that there is an element $1 \in S$ with $1x = x$ for all $x \in X$ and a non-peripheral element $p \in X$. Then there is an open neighborhood U of 1 in S such that $p \in sX$ for all s in the component U_o of 1 in U. (See [LM], p.262, Theorem 3.4). □

LEMMA B. The non-peripheral points of a finite dimensional topological locally compact space are dense. (Dimension is cohomological dimension. The Lemma is proved in LM, but it was around since about 68.) □

For a compact semilattice S we write $x <\cdot\, y$ iff $y \in \operatorname{int} \uparrow x$. We observe that $x <\cdot\, y$ implies $x \ll y$,we do not know anything on the converse yet (see Memo Carruth 5-28). A later memo reporting on some activities in Darmstadt will show that the interpolation property is crucial in the analysis of these relations.

LEMMA 1 . Let S be a compact semilattice. Suppose that for all open neighborhoods U of 1 the component U_o of 1 in U has non-empty interior. (This is certainly the case if S is locally connected at 1.) If p is

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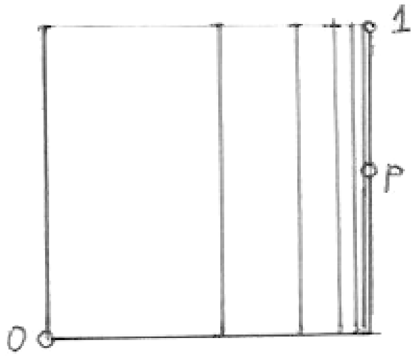
a non-peripheral point of S, then $p \ll 1$, and so also $p \ll 1$.

Proof. Apply the Lemma A with S=X and find $U_o \subseteq \uparrow p$, hence $\text{int} \uparrow p \neq \emptyset$ and so $1 \in \text{int} \uparrow p$. □

If one wishes to put it the other way around: If p is on a face (i.e. $p \ll 1$ does not hold) then p is peripheral(given the other hypotheses).

The following example shows that without some local connectivity at 1 the result must fail (and this cancels a conjecture in the memo of 3-23)

EXAMPLE 2. The following is a CL-subobject of the square. The point p is facial and non-peripheral.



□

Here is a technical definition

DEFINITION 3. Let S be a topological semilattice, say, locally compact. A point s is called <u>hyperinternal</u> iff $s = \sup\{x \in \downarrow s : x \text{ is non-peripheral in } \downarrow s\}$. □

PROPOSITION 4. Let S be a compact semigroup. Suppose that $\downarrow s$ is finite dimensional at s (i.e there is a finite dimensional open neighborhood U of s in $\downarrow s$). Then s is hyperinternal.

Proof. Let V be any open neighborhood of s in U. Then V contains a non-peripheral point by LEMMA B. Such a point is also non-peripheral in $\downarrow s$ (see [LM]). The assertion follows. □

Here is another technical definition.

DEFINITION 5. Let S be a locally compact semilattice. A point s is called <u>approximately</u> <u>hyperinternal</u> (shortly AHI) iff $s = \sup\{x \in \downarrow s : x \text{ is hyperinternal}\}$. □

This thing occurs:

EXAMPLE 6. Every point in I^X, X any set, is AHI.

Proof. Indeed if $s \in I^X$, then $s = \sup \Downarrow s$ (where $\Downarrow s = \{x : x \ll s\}$) since $I^X \in$ <u>CL</u>. If $x \ll s$, then $\downarrow x$ is finite dimensional, hence hyperinternal by Proposition 4. □

Of course, any S which is embedded into I^X under preservation of $\ll$ retains this property. On the other hand, let X be the compact 2-cell and $S = \Gamma(X)$

the semilattice of compact subsets under $\cup$. We believe that virtually no point of S other than zero = X is hyperinternal. For $A \in \Gamma(X)$ we have $\downarrow A = \{B = \bar{B} \subseteq X : A \subseteq B\}$; this looks very much like a Hilbert cube; one would have to ask the LSU infinite dimensional topologists (Dick Schori or Dick Anderson).

Abide with another technical definition:

DEFINITION 7. Let us say that S is <u>locally connected below</u> s if for every open neighborhood U of s in $\downarrow s$, the component U_o of s in U has inner points of $\downarrow s$. □

THEOREM 8. <u>Let</u> $S \in$ <u>CS</u> (i.e. S is a compact topological semilattice. <u>Suppose that the following conditions are satisfied</u>:

(I) <u>Every non-zero point of</u> S <u>is AHI</u>.

(II) S <u>is locally connected below</u> s <u>for every hyperinternal point</u> s.

<u>Then</u> $S \in$ <u>CL</u> (i.e. S is a continuous lattice).

Proof. Let $s \in S$. Let $x \le s$ be hyperinternal and $p \le x$ a non-peripheral point of $\downarrow x$. Then ~~pxxxxxxxxxsxaxneighborhoodxofxx~~ $p \ll_{\downarrow x} x$ by Lemma 17 and (II). Since S is lower continuous (i.e. satisfies $t \sup D = \sup tD$ for $t \in S$ and every up-directed set $D \subseteq S$) we conclude $p \ll x$, hence $p \ll s$. From (I) we conclude $s = \sup \downarrow s$. Hence $S \in$ CL. □

COROLLARY 9. <u>Let</u> $S \in$ <u>CS</u>. <u>Suppose that every point in</u> S <u>is the sup of points</u> x <u>so that</u> $\downarrow x$ <u>is finite dimensional in</u> x. <u>If all</u> $\downarrow s$ <u>are locally connected at</u> s, <u>then</u> $S \in$ <u>CL</u>.

COROLLARY 10. <u>Every locally connected (locally) finite dimensional</u> $S \in$ <u>CS is in CL</u>.

Reminder: I^X is locally connected; if X is infinite, then I^X contains only peripheral points. (Apply the Third Fundamental Theorem to get one peripheral point (the identity); then use homogeneity.) We leave an algebraic parallel as an exercise:

PROPOSITION 11. Let $S \in$ <u>CS</u>. Suppose that every point in S is the sup of points x so that $\downarrow x$ has finite breadth (near x). Then $S \in$ <u>CL</u>. □

We also recall in the general context the existence of Lawson's study on the relation of breadth and dimention in <u>CL</u>.