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STOCHASTIC 2-D NAVIER-STOKES EQUATION WITH ARTIFICIAL COMPRESSIBILITY

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ABSTRACT. In this paper we study the stochastic Navier-Stokes equation with artificial compressibility. The main results of this work are the existence and uniqueness theorem for strong solutions and the limit to incompressible flow. These results are obtained by utilizing a local monotonicity property of the sum of the Stokes operator and the nonlinearity.

1. Introduction

The stochastic Navier-Stokes equation is a well accepted model for atmospheric, aero and ocean dynamics. Chandrasekhar [5] and Novikov [14] first studied the Navier-Stokes equation with external random forces. After that several approaches have been poposed, from the classic paper by Bensoussan and Temam [4] to some more recent results, e.g., by Bensoussan [3], by Flandoli and Gatarek [8] and by Menaldi and Sritharan [13].

This paper is concerned with the existence and uniqueness of strong solutions for the Stochastic 2-D Navier-Stokes equation with artificial compressibility in bounded domains. The concept of artificial compressibility was first introduced by Chorin [6, 7] and Temam [17, 18], in order to overcome the computational difficulties connected with the incompressibility constraint. Using the classical Sobolev compactness embedding and exploiting the classical Lions [12] method of fractional derivatives, Temam in his papers [17, 18] and in his book [19](chapter 3) proved the existence, uniqueness and convergence of the deterministic Navier-Stokes equation with artificial compressibility in bounded domains.

In the rest of this section we formulate the abstract Navier-Stokes problem with artificial compressibility. We describe some standard well known results including the local monotonicity property of the Navier-Stokes operator. In Section 2 we establish certain new a priori estimates involving exponential weight for stochastic Navier-Stokes equation with artificial compressibility. These estimates play a fundamental role in the proof of existence and uniqueness of strong solutions proved in the second half of Section 2. The monotonicity argument used here is the generalization of the classical Minty-Browder method for dealing with local

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monotonicity. This method was first used by Menaldi and Sritharan [13] and for multiplicative noise by Sritharan and Sundar [16]. For similar ideas see also Barbu and Sritharan [1, 2]. In the last part of Section 2 we discuss the convergence of the corresponding perturbed problem. Here the use of local monotonicity avoids the classical method based on compactness and thus the results apply to unbounded domains and hence the existence and the uniqueness as well as convergence to incompressible flow are new even in the deterministic case.

2. Abstract Mathematical Framework and Local Monotonicity

Let $\mathcal{O} \subset \mathbb{R}^2$ be a bounded domain (for the sake of simplicity) with smooth boundary, $\mathbf{u}$ the velocity and p the pressure fields. The Navier-Stokes problem (with Newtonian constitutive relationship and *artificial compressible* medium) can be written as follows

$$\begin{cases} \partial_t \mathbf{u} - \nu \triangle \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \dfrac{1}{2}(\mathrm{Div}\, \mathbf{u})\mathbf{u} + \nabla p = \mathbf{f} \ \text{ in } \ \mathrm{L}^2(0,T;\mathbb{H}^{-1}(\mathcal{O})), \\ \varepsilon \partial_t p + \mathrm{Div}\, \mathbf{u} = 0 \ \text{ in } \ \mathrm{L}^2(0,T;\mathrm{L}^2(\mathcal{O})), \end{cases} \tag{2.1}$$

with the initial conditions

$$\mathbf{u}(0) = \mathbf{u}_0 \quad \text{in} \quad \mathbb{L}^2(\mathcal{O}) \qquad \text{and} \qquad p(0) = p_0 \quad \text{in} \quad \mathrm{L}^2(\mathcal{O}), \tag{2.2}$$

where $\varepsilon > 0$ is a vanishing parameter, $\mathbf{u}_0$ belong to $\mathbb{L}^2(\mathcal{O}) = \mathrm{L}^2(\mathcal{O}, \mathbb{R}^2)$, ν is the kinematic viscosity, p denotes pressure and is a scalar-valued function and the (force) field $\mathbf{f}$ is in $\mathrm{L}^2(0,T;\mathbb{L}^2(\mathcal{O}))$. A solution $(\mathbf{u}, p)$ should belongs to the space $\mathrm{L}^2(0,T;\mathbb{H}_0^1(\mathcal{O}) \times \mathrm{L}^2(\mathcal{O}))$, with $\mathbb{H}_0^1(\mathcal{O}) = \mathrm{H}_0^1(\mathcal{O}, \mathbb{R}^2)$ and $\mathbb{H}^{-1}(\mathcal{O})$ its dual space. The second equation in (2.1) is an artificial state equation of a slightly compressible medium and the extra term $(1/2)(\mathrm{Div}\, \mathbf{u})\mathbf{u}$ is a stabilization term to handle the nonlinearity. The standard spaces used here are as follows:
$\mathbb{H}_0^1(\mathcal{O})$ with the norm

$$\|\mathbf{v}\|_{\mathbb{H}_0^1} := \Big(\int_{\mathcal{O}} |\nabla \mathbf{v}|^2 dx \Big)^{1/2} = \|\mathbf{v}\|, \tag{2.3}$$

and $\mathbb{L}^2(\mathcal{O})$ with the norm

$$\|\mathbf{v}\|_{\mathbb{L}^2} := \Big(\int_{\mathcal{O}} |\mathbf{v}|^2 dx \Big)^{1/2} = |\mathbf{v}|. \tag{2.4}$$

Using the Gelfand triple $\mathbb{H}_0^1(\mathcal{O}) \subset \mathbb{L}^2(\mathcal{O}) \subset \mathbb{H}^{-1}(\mathcal{O})$ we may consider $\triangle$ or ∇ as a linear map from $\mathbb{H}_0^1(\mathcal{O})$ or $\mathrm{L}^2(\mathcal{O})$ into the dual of $\mathbb{H}_0^1(\mathcal{O})$ respectively. The inner product in the $\mathbb{L}^2$ or L^2 is denoted by $(\cdot, \cdot)$ and the induced duality by $\langle \cdot, \cdot \rangle$. Thus, for any $\mathbf{u} = (u_i)$, $\mathbf{v} = (v_i)$ and $\mathbf{w} = (w_i)$ in $\mathbb{H}_0^1(\mathcal{O})$ and p in $\mathrm{L}^2(\mathcal{O})$ we have

$$\langle -\nu \triangle \mathbf{u}, \mathbf{w} \rangle = \nu \sum_{i,j} \int_{\mathcal{O}} \partial_i u_j \, \partial_i w_j \mathrm{d}x, \tag{2.5}$$

$$\langle -\nabla p, \mathbf{w} \rangle = -\sum_i \int_{\mathcal{O}} \partial_i p \, w_i \mathrm{d}x = \int_{\mathcal{O}} p \, \partial_i w_i \mathrm{d}x = \langle p, \mathrm{Div}\, \mathbf{w} \rangle \tag{2.6}$$

and

$$\langle (\mathbf{u} \cdot \nabla)\mathbf{v}, \mathbf{w} \rangle = \sum_{i,j} \int_{\mathcal{O}} u_i \, \partial_i v_j \, w_j \mathrm{d}x. \tag{2.7}$$

It is clear that $\mathbf{u} \mapsto \operatorname{Div} \mathbf{u}$ is a linear continuous operator from $\mathbb{H}_0^1(\mathcal{O})$ into $\mathrm{L}^2(\mathcal{O})$. Next, an integration by parts and Hölder inequality yields

$$\langle (\mathbf{u} \cdot \nabla)\mathbf{v}, \mathbf{w} \rangle = -\langle (\operatorname{Div} \mathbf{u})\mathbf{w}, \mathbf{v} \rangle - \langle (\mathbf{u} \cdot \nabla)\mathbf{w}, \mathbf{v} \rangle, \tag{2.8}$$

$$|\langle (\mathbf{u} \cdot \nabla)\mathbf{v}, \mathbf{w} \rangle| \leq C \sum_{i,j} \|u_i w_j\|_{\mathrm{L}^2(\mathcal{O},\mathbb{R}^2)} \|\partial_i v_j\|_{\mathrm{L}^2(\mathcal{O},\mathbb{R}^2)}, \tag{2.9}$$

and in the right-hand-side we can use $\mathbb{L}^4$-norms to estimate the product $u_i v_j$.

Lemma 2.1. *For any real-valued smooth functions φ and ψ with compact support in $\mathbb{R}^2$, the following hold:*

$$\|\varphi\, \psi\|_{\mathrm{L}^2}^2 \leq \|\varphi\, \partial_1 \varphi\|_{\mathrm{L}^1} \|\psi\, \partial_2 \psi\|_{\mathrm{L}^1}, \tag{2.10}$$

$$\|\varphi\|_{L^4}^4 \leq 2\|\varphi\|_{\mathrm{L}^2}^2 \|\nabla \varphi\|_{\mathbb{L}^2}^2. \tag{2.11}$$

Proof. The results stated above are classical and well known [10]. □

As in Temam [19], chapter 3, the non-linear term is a trilinear continuous form on $\mathbb{H}_0^1(\mathcal{O}) \times \mathbb{H}_0^1(\mathcal{O}) \times \mathbb{H}_0^1(\mathcal{O})$

$$\hat{b}(\mathbf{u}, \mathbf{v}, \mathbf{w}) := \langle \hat{B}(\mathbf{u}, \mathbf{v}), \mathbf{w} \rangle := \frac{1}{2} \sum_{i,j} \int_{\mathcal{O}} [u_i\, \partial_i v_j\, w_j - u_i\, \partial_i w_j\, v_j] \mathrm{d}x, \tag{2.12}$$

where

$$\hat{B}(\mathbf{u}) = \hat{B}(\mathbf{u}, \mathbf{u}) = [(\mathbf{u} \cdot \nabla) + \frac{1}{2} \operatorname{Div} \mathbf{u}]\mathbf{u}. \tag{2.13}$$

We have the following lemmas.

Lemma 2.2. *Let $\mathbf{u}$ and $\mathbf{w}$ be in the spaces $\mathbb{H}_0^1(\mathcal{O}, \mathbb{R}^2)$ and $\mathbb{L}^4(\mathcal{O}, \mathbb{R}^2)$ respectively. Then the following estimate holds:*

$$|\langle \hat{B}(\mathbf{u}), \mathbf{w} \rangle| \leq 2\|\mathbf{u}\|^{3/2}\ |\mathbf{u}|^{1/2}\ \|\mathbf{w}\|_{\mathbb{L}^4(\mathcal{O},\mathbb{R}^2)}. \tag{2.14}$$

Proof. We observe that

$$\hat{b}(\mathbf{u}, \mathbf{v}, \mathbf{w}) = \sum_{i,j=1}^{2} \int_{\mathcal{O}} u_i (D_i v_j) w_j\ dx\ +\ \frac{1}{2} \sum_{j=1}^{2} \int_{\mathcal{O}} (div\ \mathbf{u}) v_j w_j\ dx.$$

Using the Hölder inequality,

$$\begin{aligned} |\hat{b}(\mathbf{u}, \mathbf{v}, \mathbf{w})| \quad &\leq \quad \sum_{i,j=1}^{2} \|u_i\|_{\mathbb{L}^4(\mathcal{O},\mathbb{R}^2)}\ |D_i v_j|_{\mathbb{L}^2(\mathcal{O},\mathbb{R}^2)}\ \|w_j\|_{\mathbb{L}^4(\mathcal{O},\mathbb{R}^2)} \\ &+ \quad \frac{1}{2} \sum_{j=1}^{2}\ |div\ \mathbf{u}|_{\mathbb{L}^2(\mathcal{O},\mathbb{R}^2)}\ \|v_j\|_{\mathbb{L}^4(\mathcal{O},\mathbb{R}^2)}\ \|w_j\|_{\mathbb{L}^4(\mathcal{O},\mathbb{R}^2)}. \end{aligned}$$

In Frobenius norm divergence can be estimated by gradient. Hence

$$|\hat{b}(\mathbf{u}, \mathbf{u}, \mathbf{w})| \leq\ \frac{3}{2} \|\mathbf{u}\|_{\mathbb{L}^4(\mathcal{O},\mathbb{R}^2)}\ \|\mathbf{u}\|\ \|\mathbf{w}\|_{\mathbb{L}^4(\mathcal{O},\mathbb{R}^2)}.$$

Using the equation (2.11) in Lemma 2.1 we get

$$|\langle \hat{B}(\mathbf{u}), \mathbf{w} \rangle|\ =\ |\hat{b}(\mathbf{u}, \mathbf{u}, \mathbf{w})|\ \leq\ 2\|\mathbf{u}\|^{3/2}\ |\mathbf{u}|^{1/2}\ \|\mathbf{w}\|_{\mathbb{L}^4(\mathcal{O},\mathbb{R}^2)}.$$

□

Lemma 2.3. *Let* $\mathbf{u}$ *and* $\mathbf{v}$ *be in the space* $\mathbb{H}_0^1(\mathcal{O},\mathbb{R}^2)$. *Then the following estimate holds:*

$$|\langle \hat{B}(\mathbf{u}) - \hat{B}(\mathbf{v}), \mathbf{u}-\mathbf{v}\rangle| \le \frac{\nu}{2}\|\mathbf{u}-\mathbf{v}\|^2 + \frac{27}{2\nu^3}|\mathbf{u}-\mathbf{v}|^2\|\mathbf{v}\|^4_{\mathbb{L}^4(\mathcal{O},\mathbb{R}^2)}. \tag{2.15}$$

Proof. For any given $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{H}_0^1(\mathcal{O},\mathbb{R}^2)$, we have from (2.12)

$$\langle \hat{B}(\mathbf{u}), \mathbf{u}\rangle = 0, \tag{2.16}$$

and

$$\langle \hat{B}(\mathbf{u},\mathbf{v}), \mathbf{v}\rangle = \langle (\mathbf{u}\cdot\nabla)\mathbf{v}, \mathbf{v}\rangle + \frac{1}{2}\langle (\operatorname{Div}\mathbf{u})\mathbf{v}, \mathbf{v}\rangle = 0. \tag{2.17}$$

Then using (2.16) we obtain

$$\begin{aligned}\langle \hat{B}(\mathbf{u}) - \hat{B}(\mathbf{v}), \mathbf{u}-\mathbf{v}\rangle &= \langle \hat{B}(\mathbf{u}),\mathbf{u}\rangle - \langle \hat{B}(\mathbf{u}),\mathbf{v}\rangle + \langle \hat{B}(\mathbf{v}),\mathbf{v}\rangle - \langle \hat{B}(\mathbf{v}),\mathbf{u}\rangle \\ &= -\langle \hat{B}(\mathbf{u}),\mathbf{v}\rangle - \langle \hat{B}(\mathbf{v}),\mathbf{u}\rangle. \end{aligned}\tag{2.18}$$

Now

$$\begin{aligned}\langle \hat{B}(\mathbf{u}-\mathbf{v}), \mathbf{v}\rangle &= \langle ((\mathbf{u}-\mathbf{v})\cdot\nabla)(\mathbf{u}-\mathbf{v}) + \frac{1}{2}(\operatorname{Div}(\mathbf{u}-\mathbf{v}))(\mathbf{u}-\mathbf{v}), \mathbf{v}\rangle \\ &= \langle \hat{B}(\mathbf{u}),\mathbf{v}\rangle + \langle \hat{B}(\mathbf{v}),\mathbf{v}\rangle - \langle (\mathbf{u}\cdot\nabla)\mathbf{v} + \frac{1}{2}(\operatorname{Div}\mathbf{u})\mathbf{v}, \mathbf{v}\rangle \\ &\quad - \langle (\mathbf{v}\cdot\nabla)\mathbf{u} + \frac{1}{2}(\operatorname{Div}\mathbf{v})\mathbf{u}, \mathbf{v}\rangle.\end{aligned}$$

Hence using (2.16) and (2.17) we have

$$\begin{aligned}\langle \hat{B}(\mathbf{u}-\mathbf{v}), \mathbf{v}\rangle &= \langle \hat{B}(\mathbf{u}),\mathbf{v}\rangle - \langle (\mathbf{v}\cdot\nabla)\mathbf{u} + \frac{1}{2}(\operatorname{Div}\mathbf{v})\mathbf{u}, \mathbf{v}\rangle \\ &= \langle \hat{B}(\mathbf{u}),\mathbf{v}\rangle + \langle \hat{B}(\mathbf{v}),\mathbf{u}\rangle - \langle (\mathbf{v}\cdot\nabla)\mathbf{u} + \frac{1}{2}(\operatorname{Div}\mathbf{v})\mathbf{u}, \mathbf{v}\rangle \\ &\quad - \langle (\mathbf{v}\cdot\nabla)\mathbf{v} + \frac{1}{2}(\operatorname{Div}\mathbf{v})\mathbf{v}, \mathbf{u}\rangle \\ &= \langle \hat{B}(\mathbf{u}),\mathbf{v}\rangle + \langle \hat{B}(\mathbf{v}),\mathbf{u}\rangle \\ &\quad - \langle (\mathbf{v}\cdot\nabla)(\mathbf{u}+\mathbf{v}) + \frac{1}{2}(\operatorname{Div}\mathbf{v})(\mathbf{u}+\mathbf{v}), \mathbf{u}+\mathbf{v}\rangle \\ &\quad + \langle (\mathbf{v}\cdot\nabla)\mathbf{u} + \frac{1}{2}(\operatorname{Div}\mathbf{v})\mathbf{u}, \mathbf{u}\rangle \\ &\quad + \langle (\mathbf{v}\cdot\nabla)\mathbf{v} + \frac{1}{2}(\operatorname{Div}\mathbf{v})\mathbf{v}, \mathbf{v}\rangle.\end{aligned}$$

With the help of (2.16) and (2.17) last three terms of the right hand side vanish.Thus

$$\langle \hat{B}(\mathbf{u}-\mathbf{v}), \mathbf{v}\rangle = \langle \hat{B}(\mathbf{u}),\mathbf{v}\rangle + \langle \hat{B}(\mathbf{v}),\mathbf{u}\rangle. \tag{2.19}$$

Thus (2.18) and (2.19) yield

$$\langle \hat{B}(\mathbf{u}) - \hat{B}(\mathbf{v}), \mathbf{u}-\mathbf{v}\rangle = -\langle \hat{B}(\mathbf{u}-\mathbf{v}), \mathbf{v}\rangle, \tag{2.20}$$

which by Lemma 2.2 gives the estimate

$$|\langle \hat{B}(\mathbf{u}) - \hat{B}(\mathbf{v}), \mathbf{u}-\mathbf{v}\rangle| \le 2\|\mathbf{u}-\mathbf{v}\|^{3/2}|\mathbf{u}-\mathbf{v}|^{1/2}\|\mathbf{v}\|_{\mathbb{L}^4(\mathcal{O},\mathbb{R}^2)},$$

where $\|\cdot\|$ and $|\cdot|$ denotes the norm in $\mathbb{H}_0^1(\mathcal{O})$ and $\mathbb{L}^2(\mathcal{O})$ respectively.
Now using the fact that for any two real numbers a, b and any p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$,

$$ab \leq \frac{|a|^p}{p} + \frac{|b|^q}{q},$$

we obtain the estimate (2.15). □

Notice that $\mathbb{H}_0^1(\mathcal{O})$ is continuously included in $\mathbb{L}^4(\mathcal{O}) = \mathrm{L}^4(\mathcal{O}, \mathbb{R}^2)$ and $\mathbf{u} \mapsto (\nabla \cdot \mathbf{u})\mathbf{u}$ is a (nonlinear) continuous mapping from $\mathbb{H}_0^1(\mathcal{O})$ into its dual $\mathbb{H}^{-1}(\mathcal{O})$. Hence the nonlinear operator $\hat{B}(\cdot)$ can be considered as a map from $\mathbb{H}_0^1(\mathcal{O})$ into the space $\mathbb{H}^{-1}(\mathcal{O}) \cap \mathbb{L}^{4/3}(\mathcal{O}, \mathbb{R}^2)$. Then combination of previous lemmas yield the following local monotonicity property.

Lemma 2.4. *For a given $r > 0$, let us denote by $\mathbb{B}_r$ the (closed) $\mathbb{L}^4$-ball in $\mathbb{H}_0^1$*

$$\mathbb{B}_r = \{\mathbf{v} \in \mathrm{H}_0^1(\mathcal{O}, \mathbb{R}^2) \ ; \ \|\mathbf{v}\|_{\mathrm{L}^4(\mathcal{O}, \mathbb{R}^2)} \leq r\}, \tag{2.21}$$

then the nonlinear operator $\mathbf{u} \mapsto A\mathbf{u} + \hat{B}(\mathbf{u}) := -\nu\triangle\mathbf{u} + [(\mathbf{u} \cdot \nabla) + (1/2)\,\mathrm{Div}\,\mathbf{u}]\mathbf{u}$ is monotone in the convex ball $\mathbb{B}_r$ i.e.,

$$\langle A\mathbf{w}, \mathbf{w}\rangle + \langle \hat{B}(\mathbf{u}) - \hat{B}(\mathbf{v}), \mathbf{w}\rangle + \frac{27r^4}{2\nu^3}\,|\mathbf{w}|^2 \geq \frac{\nu}{2}\,\|\mathbf{w}\|^2, \tag{2.22}$$

$\forall \mathbf{u} \in \mathrm{H}_0^1(\mathcal{O}, \mathbb{R}^2)$, $\mathbf{v} \in \mathbb{B}_r$ *and* $\mathbf{w} = \mathbf{u} - \mathbf{v}$.

Proof. First, it is clear that

$$\langle A\mathbf{w}, \mathbf{w}\rangle = \nu\|\mathbf{w}\|^2,$$

and the equation(2.15) yields

$$\langle \hat{B}(\mathbf{u}) - \hat{B}(\mathbf{v}), \mathbf{w}\rangle \geq -\frac{\nu}{2}\,\|\mathbf{w}\|^2 - \frac{27r^4}{2\nu^3}\,|\mathbf{w}|^2.$$

Summing these equations up we get the desired result (2.22). □

3. Stochastic 2-D Navier-Stokes Equation with Artificial Compressibility

Let us consider the Navier-Stokes equation subject to a random (Gaussian) term i.e., the forcing field $\mathbf{f}$ has a mean value still denoted by $\mathbf{f}$ and a noise denoted by $\dot{\mathbf{G}}$. We can write (to simplify notation we use time-invariant forces) $\mathbf{f}(t) = \mathbf{f}(x,t)$ and the noise process $\dot{\mathbf{G}}(t) = \dot{\mathbf{G}}(x,t)$ as a series $d\mathbf{G}_k = \sum_k \mathbf{g}_k(x,t)dw_k(t)$, where $\mathbf{g} = (\mathbf{g}_1, \mathbf{g}_2, \cdots)$ and $w = (w_1, w_2, \ldots)$ are regarded as ℓ^2-valued functions in x and t respectively. The stochastic noise process represented by $\mathbf{g}(t)dw(t) = \sum_k \mathbf{g}_k(x,t)dw_k(t,\omega)$ is normal distributed in $\mathbb{H}$ with a trace-class co-variance operator denoted by $\mathbf{g}^2 = \mathbf{g}^2(t)$ and given by

$$\begin{cases} (\mathbf{g}^2(t)\mathbf{u}, \mathbf{v}) = \displaystyle\sum_k (\mathbf{g}_k(t), \mathbf{u})\,(\mathbf{g}_k(t), \mathbf{v}), \\ \mathrm{Tr}(\mathbf{g}^2(t)) = \displaystyle\sum_k |\mathbf{g}_k(t)|^2 < \infty. \end{cases} \tag{3.1}$$

We interpret the stochastic Navier-Stokes equations as an Itô stochastic equations in variational form

$$\begin{cases} \mathrm{d}(\mathbf{u}(t),\mathbf{v}) + \langle -\nu\triangle\mathbf{u}(t) + [(\mathbf{u}(t)\cdot\nabla) + \frac{1}{2}\operatorname{Div}\mathbf{u}(t)]\mathbf{u}(t) \\ \qquad + \nabla p(t),\mathbf{v}\rangle\,\mathrm{d}t = (\mathbf{f},\mathbf{v})\,\mathrm{d}t + \sum_k (\mathbf{g}_k,\mathbf{v})\,\mathrm{d}w_k(t), \\ \langle \varepsilon\dot{p}(t) + \operatorname{Div}\mathbf{u}(t), q\rangle = 0, \end{cases} \tag{3.2}$$

in $(0,T)$, with the initial condition

$$(\mathbf{u}(0),\mathbf{v}) = (\mathbf{u}_0,\mathbf{v}) \quad \text{and} \quad (p(0),q) = (p_0,q), \tag{3.3}$$

for any $\mathbf{v}$ in the space $\mathbb{H}_0^1(\mathcal{O})$ and any q in $\mathrm{L}^2(\mathcal{O})$.

A finite-dimensional (Galerkin) approximation of the stochastic Navier-Stokes equation can be defined as follows. Let $\{\mathbf{e}_1,\mathbf{e}_2,\ldots\}$ be a complete orthonormal system (i.e., a basis) in the Hilbert space $\mathbb{L}^2(\mathcal{O})$ belonging to the space $\mathbb{H}_0^1(\mathcal{O})$ (and $\mathbb{L}^4(\mathcal{O})$). Denote by $\mathbb{L}_n^2(\mathcal{O})$ the n-dimensional subspace of $\mathbb{L}^2(\mathcal{O})$ and $\mathbb{H}_0^1(\mathcal{O})$ of all linear combinations of the first n elements $\{\mathbf{e}_1,\mathbf{e}_2,\ldots,\mathbf{e}_n\}$. Also denote by $\mathrm{L}_n^2(\mathcal{O}) := \nabla\cdot\mathbb{L}_n^2(\mathcal{O})$ the image of ∇.

Consider the following stochastic ODE in $\mathbb{R}^n$

$$\begin{cases} \mathrm{d}(\mathbf{u}^n(t),\mathbf{v}) + \langle -\nu\triangle\mathbf{u}^n(t) + [(\mathbf{u}^n(t)\cdot\nabla) + \frac{1}{2}\operatorname{Div}\mathbf{u}^n(t)]\mathbf{u}^n(t) \\ \qquad + \nabla p^n(t),\mathbf{v}\rangle\mathrm{d}t = (\mathbf{f},\mathbf{v})\mathrm{d}t + \sum_k (\mathbf{g}_k,\mathbf{v})\mathrm{d}w_k(t), \\ \langle \varepsilon\dot{p}^n(t) + \operatorname{Div}\mathbf{u}^n(t), q\rangle = 0, \end{cases} \tag{3.4}$$

in $(0,T)$, with the initial condition

$$(\mathbf{u}(0),\mathbf{v}) = (\mathbf{u}_0,\mathbf{v}), \tag{3.5}$$

for any $\mathbf{v}$ in the space $\mathbb{L}_n^2(\mathcal{O})$ and q in $\mathrm{L}_n^2(\mathcal{O})$. The coefficients involved are locally Lipschitz and we need some *a priori* estimate to show global existence of a solution $\mathbf{u}^n(t)$ as an adapted process in the space $C^0(0,T,\mathbb{L}_n^2(\mathcal{O}))$.

Proposition 3.1 (energy estimate)**.** *Under the above mathematical setting let*

$$\mathbf{f}\in\mathrm{L}^2(0,T;\mathbb{L}^2(\mathcal{O})),\ \mathbf{g}\in\mathrm{L}^2(0,T;\ell_2(\mathbb{L}^2(\mathcal{O}))),\ \mathbf{u}_0\in\mathbb{L}^2(\mathcal{O}),\ p_0\in\mathrm{L}^2(\mathcal{O}). \tag{3.6}$$

Let $\mathbf{u}^n(t)$ be an adapted process in $C^0(0,T,\mathbb{H}_n)$ which solves the stochastic ODE (3.4). Then we have the energy equality

$$\begin{cases} \mathrm{d}\big[|\mathbf{u}^n(t)|^2 + \varepsilon|p^n(t)|^2\big] + 2\nu\,|\nabla\mathbf{u}^n(t)|^2\mathrm{d}t \\ \qquad = \big[2\,(\mathbf{f}(t),\mathbf{u}^n(t)) + \mathrm{Tr}(\mathbf{g}^2(t))\big]\mathrm{d}t + 2\sum_k(\mathbf{g}_k(t),\mathbf{u}^n(t))\mathrm{d}w_k(t), \end{cases} \tag{3.7}$$

which yields the following estimate for any $\delta>0$

$$\begin{cases} E\big\{|\mathbf{u}^n(t)|^2 + \varepsilon|p^n(t)|^2\big\}e^{-\delta t} + 2\,\nu\int_0^T E\big\{|\nabla\mathbf{u}^n(t)|^2\big\}e^{-\delta t}\mathrm{d}t \\ \qquad \le |\mathbf{u}(0)|^2 + \varepsilon|p(0)|^2 + \int_0^T \big[\frac{1}{\delta}|\mathbf{f}(t)|^2 + \mathrm{Tr}(\mathbf{g}^2(t))\big]e^{-\delta t}\mathrm{d}t, \end{cases} \tag{3.8}$$

for any $0 \le t \le T$. *Moreover, if we suppose*

$$\mathbf{f} \in \mathrm{L}^{\mathrm{p}}(0,T;\mathbb{L}^2(\mathcal{O})),\ \mathbf{g} \in \mathrm{L}^{\mathrm{p}}(0,T;\ell_2(\mathbb{L}^2(\mathcal{O}))) \tag{3.9}$$

then we also have

$$\begin{cases} E\big\{ \sup\limits_{0\le t\le T} \big[|\mathbf{u}^n(t)|^{\mathrm{p}} + \varepsilon|p^n(t)|^{\mathrm{p}}\big]e^{-\delta t} \\ \qquad + \mathrm{p}\,\nu \displaystyle\int_0^T |\nabla \mathbf{u}^n(t)|^2 |\mathbf{u}^n(t)|^{\mathrm{p}-2} e^{-\delta t}\mathrm{d}t\big\} \le |\mathbf{u}(0)|^{\mathrm{p}} \\ \qquad + \varepsilon|p(0)|^{\mathrm{p}} + C_{\delta,\mathrm{p},T} \displaystyle\int_0^T \big[|\mathbf{f}(t)|^{\mathrm{p}} + \mathrm{Tr}(\mathbf{g}^2(t))^{\mathrm{p}/2}\big]e^{-\delta t}\mathrm{d}t, \end{cases} \tag{3.10}$$

for some constant $C_{\delta,\mathrm{p},T}$ *depending only on* $\delta > 0$, $\varepsilon > 0$, $1 \le \mathrm{p} < \infty$ *and* $T > 0$.

Proof. From (3.4) we notice that,

$$\begin{aligned} d(\mathbf{u}^n(t),\mathbf{u}^n(t)) &+ \langle -\nu\triangle \mathbf{u}^n(t), \mathbf{u}^n(t)\rangle \mathrm{d}t + \langle (\mathbf{u}^n(t)\cdot\nabla)\mathbf{u}^n(t), \mathbf{u}^n(t)\rangle \mathrm{d}t \\ &+ \frac{1}{2}\langle \mathbf{u}^n(t)\,\mathrm{Div}\,\mathbf{u}^n(t), \mathbf{u}^n(t)\rangle \mathrm{d}t + \langle \nabla p^n(t), \mathbf{u}^n(t)\rangle \mathrm{d}t \\ &= (\mathbf{f}(t), \mathbf{u}^n(t))\mathrm{d}t + \sum_k (\mathbf{g}_k(t), \mathbf{u}^n(t))\mathrm{d}w_k(t). \end{aligned} \tag{3.11}$$

It is clear that

$$\langle -\nu\triangle \mathbf{u}^n(t), \mathbf{u}^n(t)\rangle = \nu|\nabla \mathbf{u}^n(t)|^2,$$

and the equation (2.8) yields

$$\langle (\mathbf{u}^n(t)\cdot\nabla)\mathbf{u}^n(t), \mathbf{u}^n(t)\rangle = -\langle \mathbf{u}^n(t)\,\mathrm{Div}\,\mathbf{u}^n(t), \mathbf{u}^n(t)\rangle - \langle (\mathbf{u}^n(t)\cdot\nabla)\mathbf{u}^n(t), \mathbf{u}^n(t)\rangle.$$

Hence

$$\langle (\mathbf{u}^n(t)\cdot\nabla)\mathbf{u}^n(t), \mathbf{u}^n(t)\rangle + \frac{1}{2}\langle \mathbf{u}^n(t)\,\mathrm{Div}\,\mathbf{u}^n(t), \mathbf{u}^n(t)\rangle = 0. \tag{3.12}$$

Using the equation $\varepsilon\dot{p}^n(t) + \mathrm{Div}\,\mathbf{u}^n(t) = 0$ we get from (2.6)

$$\langle \nabla p^n(t), \mathbf{u}^n(t)\rangle = -\langle p^n(t), \mathrm{Div}\,\mathbf{u}^n(t)\rangle = -\langle p^n(t), -\varepsilon\dot{p}^n(t)\rangle = \frac{\varepsilon}{2}\ \frac{\mathrm{d}}{\mathrm{d}t}|p^n(t)|^2. \tag{3.13}$$

Combining all the above results one can get from (3.11)

$$\begin{aligned} \frac{1}{2}\ \mathrm{d}|\mathbf{u}^n(t)|^2 &+ \nu|\nabla \mathbf{u}^n(t)|^2\mathrm{d}t + \frac{\varepsilon}{2}\ \mathrm{d}|p^n(t)|^2 \\ &= (\mathbf{f}(t), \mathbf{u}^n(t))\mathrm{d}t + \frac{1}{2}\ \mathrm{Tr}(\mathbf{g}^2(t))\mathrm{d}t + \sum_k (\mathbf{g}_k(t), \mathbf{u}^n(t))\mathrm{d}w_k(t). \end{aligned}$$

Rearranging the terms we get the desired energy equality (3.7).

Next, we calculate the stochastic differential of the process

$$\mathrm{F}(t) := \big[|\mathbf{u}^n(t)|^2 + \varepsilon|p^n(t)|^2\big]e^{-\delta t}$$

to get

$$\mathrm{dF}(t) = e^{-\delta t}\mathrm{d}\big[|\mathbf{u}^n(t)|^2 + \varepsilon|p^n(t)|^2\big] - \delta\big[|\mathbf{u}^n(t)|^2 + \varepsilon|p^n(t)|^2\big]e^{-\delta t}\mathrm{d}t.$$

Using the energy equality (3.7) we have

$$\begin{aligned} \mathrm{dF}(t) = &-2\nu|\nabla\mathbf{u}^n(t)|^2 e^{-\delta t}\mathrm{d}t + 2\,(\mathbf{f}(t),\mathbf{u}^n(t))e^{-\delta t}\mathrm{d}t + \mathrm{Tr}(\mathbf{g}^2(t))e^{-\delta t}\mathrm{d}t \\ &+ 2\sum_k(\mathbf{g}_k(t),\mathbf{u}^n(t))e^{-\delta t}\mathrm{d}w_k(t) - \delta\big[|\mathbf{u}^n(t)|^2+\varepsilon|p^n(t)|^2\big]e^{-\delta t}\mathrm{d}t. \end{aligned} \tag{3.14}$$

Now using the inequality

$$2ab \le \delta a^2 + \frac{1}{\delta}b^2$$

on $2\,(\mathbf{f}(t),\mathbf{u}^n(t))$ we have

$$2\,(\mathbf{f}(t),\mathbf{u}^n(t)) \le \delta|\mathbf{u}^n(t)|^2 + \frac{1}{\delta}|\mathbf{f}(t)|^2 \le \delta\big[|\mathbf{u}^n(t)|^2+\varepsilon|p^n(t)|^2\big] + \frac{1}{\delta}|\mathbf{f}(t)|^2.$$

Then (3.14) yields

$$\begin{aligned} \mathrm{dF}(t) = &-2\nu|\nabla\mathbf{u}^n(t)|^2 e^{-\delta t}\mathrm{d}t + \delta\big[|\mathbf{u}^n(t)|^2+\varepsilon|p^n(t)|^2\big]e^{-\delta t}\mathrm{d}t \\ &+ \frac{1}{\delta}|\mathbf{f}(t)|^2 e^{-\delta t}\mathrm{d}t + \mathrm{Tr}(\mathbf{g}^2(t))e^{-\delta t}\mathrm{d}t + 2\sum_k(\mathbf{g}_k(t),\mathbf{u}^n(t))e^{-\delta t}\mathrm{d}w_k(t) \\ &- \delta\big[|\mathbf{u}^n(t)|^2+\varepsilon|p^n(t)|^2\big]e^{-\delta t}\mathrm{d}t. \end{aligned}$$

Rearranging the terms we have

$$\begin{aligned} &\mathrm{d}\big[\big\{|\mathbf{u}^n(t)|^2+\varepsilon|p^n(t)|^2\big\}e^{-\delta t}\big] + 2\nu|\nabla\mathbf{u}^n(t)|^2 e^{-\delta t}\mathrm{d}t \\ &\quad\le \frac{1}{\delta}|\mathbf{f}(t)|^2 e^{-\delta t}\mathrm{d}t + \mathrm{Tr}(\mathbf{g}^2(t))e^{-\delta t}\mathrm{d}t + 2\sum_k(\mathbf{g}_k(t),\mathbf{u}^n(t))e^{-\delta t}\mathrm{d}w_k(t). \end{aligned}$$

Next we integrate in $[0,T]$ to get

$$\begin{aligned} &\big[|\mathbf{u}^n(t)|^2+\varepsilon|p^n(t)|^2\big]e^{-\delta t} + 2\nu\int_0^T|\nabla\mathbf{u}^n(t)|^2 e^{-\delta t}\mathrm{d}t \\ &\quad\le |\mathbf{u}(0)|^2+\varepsilon|p(0)|^2 + \int_0^T\big[\frac{1}{\delta}|\mathbf{f}(t)|^2+\mathrm{Tr}(\mathbf{g}^2(t))\big]e^{-\delta t}\mathrm{d}t \\ &\qquad + 2\sum_k\int_0^T(\mathbf{g}_k(t),\mathbf{u}^n(t))e^{-\delta t}\mathrm{d}w_k(t). \end{aligned}$$

Finally taking mathematical expectation and keeping in mind the expectation of a stochastic integral is zero, we have the desired result (3.8).
Similarly, consider

$$\mathrm{G}(t) := \big[|\mathbf{u}^n(t)|^{\mathrm{p}}+\varepsilon|p^n(t)|^{\mathrm{p}}\big]e^{-\delta t}$$

and use Itô calculus. Here we check that its stochastic differential satisfies

$$\begin{aligned} \mathrm{dG}(t) = &-\delta\big[|\mathbf{u}^n(t)|^{\mathrm{p}}+\varepsilon|p^n(t)|^{\mathrm{p}}\big]e^{-\delta t}\mathrm{d}t \\ &+ \frac{\mathrm{p}}{2}|\mathbf{u}^n(t)|^{\mathrm{p}-2}\big\{\mathrm{d}\big[|\mathbf{u}^n(t)|^2+\varepsilon|p^n(t)|^2\big]\big\}e^{-\delta t} \\ &+ \frac{\mathrm{p}(\mathrm{p}-1)}{8}|\mathbf{u}^n(t)|^{\mathrm{p}-4}\big\{\mathrm{d}\big[|\mathbf{u}^n(t)|^2+\varepsilon|p^n(t)|^2\big]\big\}^2 e^{-\delta t}. \end{aligned}$$

Using the energy equality (3.7) we get

$$\begin{aligned}
&\mathrm{dG}(t) + \delta\big[|\mathbf{u}^n(t)|^{\mathrm{p}} + \varepsilon|p^n(t)|^{\mathrm{p}}\big]e^{-\delta t}\mathrm{d}t \\
&\quad = \frac{\mathrm{p}}{2}|\mathbf{u}^n(t)|^{\mathrm{p}-2}\Big\{-2\nu\,\|\mathbf{u}^n(t)\|^2\mathrm{d}t + \big[2\,(\mathbf{f}(t),\mathbf{u}^n(t)) + \mathrm{Tr}(\mathbf{g}^2(t))\big]\mathrm{d}t \\
&\qquad + 2\sum_k(\mathbf{g}_k(t),\mathbf{u}^n(t))\mathrm{d}w_k(t)\Big\}e^{-\delta t} \\
&\qquad + \frac{\mathrm{p}(\mathrm{p}-1)}{8}|\mathbf{u}^n(t)|^{\mathrm{p}-4}\big[4\sum_k(\mathbf{g}_k(t),\mathbf{u}^n(t))^2\mathrm{d}t\big]e^{-\delta t}.
\end{aligned}$$

Simplification and rearrangement of the terms in the above equation yields

$$\begin{aligned}
&\mathrm{dG}(t) + \nu\mathrm{p}\|\mathbf{u}^n(t)\|^2|\mathbf{u}^n(t)|^{\mathrm{p}-2}e^{-\delta t}\mathrm{d}t + \delta\big[|\mathbf{u}^n(t)|^{\mathrm{p}} + \varepsilon|p^n(t)|^{\mathrm{p}}\big]e^{-\delta t}\mathrm{d}t \\
&\quad = |\mathbf{u}^n(t)|^{\mathrm{p}-2}\big[\mathrm{p}\,(\mathbf{f}(t),\mathbf{u}^n(t)) + \frac{\mathrm{p}^2}{2}\,\mathrm{Tr}(\mathbf{g}^2(t))\big]e^{-\delta t}\mathrm{d}t \\
&\qquad + \mathrm{p}\sum_k(\mathbf{g}_k(t),\mathbf{u}^n(t))|\mathbf{u}^n(t)|^{\mathrm{p}-2}e^{-\delta t}\mathrm{d}w_k(t).
\end{aligned} \tag{3.15}$$

Now in the first and the second terms on the right hand side of the above equation we apply the following elementary inequality

$$\lambda\mathrm{ab} \le \frac{(\alpha\mathrm{a})^{\mathrm{p}}}{\mathrm{p}} + \frac{(\beta\mathrm{b})^{\mathrm{q}}}{\mathrm{q}},$$

where $\frac{1}{\mathrm{p}} + \frac{1}{\mathrm{q}} = 1,\ \lambda = \alpha\beta > 0,\ \mathrm{ab} > 0$. Then choosing

$$\lambda = \mathrm{p}, \quad \alpha = \frac{\mathrm{p}}{(\frac{\delta\mathrm{q}}{2})^{\frac{1}{\mathrm{q}}}}, \quad \beta = \Big(\frac{\delta\mathrm{q}}{2}\Big)^{\frac{1}{\mathrm{q}}},$$

we get

$$\begin{aligned}
\mathrm{p}\,(\mathbf{f}(t),\mathbf{u}^n(t))|\mathbf{u}^n(t)|^{\mathrm{p}-2} &\le \alpha^{\mathrm{p}}\frac{|\mathbf{f}(t)|^{\mathrm{p}}}{\mathrm{p}} + \beta^{\mathrm{q}}\frac{|\mathbf{u}^n(t)|^{(\mathrm{p}-1)\mathrm{q}}}{\mathrm{q}} \\
&= C_{\delta,\mathrm{p}}\ |\mathbf{f}(t)|^{\mathrm{p}} + \frac{\delta}{2}|\mathbf{u}^n(t)|^{\mathrm{p}},
\end{aligned}$$

where the constant $C_{\delta,\mathrm{p}} > 0$ depends only on $\delta > 0$ and $1 \le \mathrm{p} < \infty$. Similarly with proper choices of α and β, we can prove that

$$\frac{\mathrm{p}^2}{2}\,\mathrm{Tr}(\mathbf{g}^2(t))|\mathbf{u}^n(t)|^{\mathrm{p}-2} \le C_{\delta,\mathrm{p}}\ \mathrm{Tr}(\mathbf{g}^2(t))^{\frac{\mathrm{p}}{2}} + \frac{\delta}{2}|\mathbf{u}^n(t)|^{\mathrm{p}}.$$

Then (3.15) yields

$$\begin{aligned}
&\mathrm{dG}(t) + \nu\mathrm{p}\|\mathbf{u}^n(t)\|^2|\mathbf{u}^n(t)|^{\mathrm{p}-2}e^{-\delta t}\mathrm{d}t \\
&\quad \le C_{\delta,\mathrm{p}}\big[|\mathbf{f}(t)|^{\mathrm{p}} + \mathrm{Tr}(\mathbf{g}^2(t))^{\frac{\mathrm{p}}{2}}\big]e^{-\delta t}\mathrm{d}t \\
&\qquad + \mathrm{p}\sum_k(\mathbf{g}_k(t),\mathbf{u}^n(t))|\mathbf{u}^n(t)|^{\mathrm{p}-2}e^{-\delta t}\mathrm{d}w_k(t).
\end{aligned} \tag{3.16}$$

Integrating the stochastic differential (3.16),then taking the sup norm in $[0,T]$ and finally taking the mathematical expectation we have

$$
\begin{aligned}
E\Big\{ \sup_{0\le t\le T} \big[|\mathbf{u}^n(t)|^{\mathrm{p}} + \varepsilon |p^n(t)|^{\mathrm{p}}\big] e^{-\delta t} \\
&+ \mathrm{p}\,\nu \int_0^T |\nabla \mathbf{u}^n(t)|^2 |\mathbf{u}^n(t)|^{\mathrm{p}-2} e^{-\delta t} \mathrm{d}t \Big\} \le |\mathbf{u}(0)|^{\mathrm{p}} \\
&+ \varepsilon |p(0)|^{\mathrm{p}} + C_{\delta,\mathrm{p},T} \int_0^T \big[|\mathbf{f}(t)|^{\mathrm{p}} + \mathrm{Tr}(\mathbf{g}^2(t))^{\mathrm{p}/2}\big] e^{-\delta t} \mathrm{d}t \\
&+ \mathrm{p}\ E\Big\{ \sup_{0\le t\le T} \Big| \int_0^t \sum_k (\mathbf{g}_k(s), \mathbf{u}^n(s)) |\mathbf{u}^n(s)|^{\mathrm{p}-2} e^{-\delta s} \mathrm{d}w_k(s) \Big| \Big\}. \qquad (3.17)
\end{aligned}
$$

By means of martingale inequality, we deduce

$$
\begin{aligned}
E\Big\{ \sup_{0\le t\le T} \Big| \int_0^t \sum_k (\mathbf{g}_k(s), \mathbf{u}^n(s)) |\mathbf{u}^n(s)|^{\mathrm{p}-2} e^{-\delta s} \mathrm{d}w_k(s) \Big| \Big\} \\
&\le C\ E\Big\{ \Big(\int_0^T \sum_k \big[(\mathbf{g}_k(t), \mathbf{u}^n(t)) |\mathbf{u}^n(t)|^{\mathrm{p}-2} e^{-\delta t}\big]^2 \mathrm{d}t \Big)^{1/2} \Big\} \\
&\le C\ E\Big\{ \Big(\int_0^T \mathrm{Tr}(\mathbf{g}^2(t)) |\mathbf{u}^n(t)|^{2\mathrm{p}-2} e^{-2\delta t} \mathrm{d}t \Big)^{1/2} \Big\} \\
&\le C\ E\Big\{ \sup_{0\le t\le T} (|\mathbf{u}^n(t)|^{\mathrm{p}-1} e^{-\delta t/\mathrm{p}'}) \Big(\int_0^T \mathrm{Tr}(\mathbf{g}^2(t)) e^{-2\delta t/\mathrm{p}} \mathrm{d}t \Big)^{1/2} \Big\} \\
&\le \frac{\delta}{2}\ E\Big\{ \sup_{0\le t\le T} (|\mathbf{u}^n(t)|^{\mathrm{p}} e^{-\delta t}) \Big\} + C_{\delta,\mathrm{p},T}\ E\Big\{ \int_0^T \mathrm{Tr}(\mathbf{g}^2(t))^{\mathrm{p}/2} e^{-\delta t} \mathrm{d}t \Big\},
\end{aligned}
\qquad (3.18)
$$

where the constant $C_{\delta,\mathrm{p},T}$ depends only on $\delta > 0$, $1 \le \mathrm{p} < \infty$ and $T > 0$. Using (3.18) in (3.17) we get the desired estimate (3.10). □

Now we deal with the existence and uniqueness of the SPDE and its finite-dimensional approximation.

Proposition 3.2 (uniqueness)**.** *Let* $\mathbf{u}$ *be a solution of the stochastic Navier-Stokes equation (SPDE)* (3.2) *with the regularity*

$$
\begin{cases}
\mathbf{u} \in \mathrm{L}^2(\Omega; C^0(0,T;\mathbb{L}^2(\mathcal{O})) \cap \mathrm{L}^2(0,T;\mathbb{H}^1_0(\mathcal{O}))), \\
\mathbf{u} \in \mathbb{L}^4(\Omega \times \mathcal{O} \times (0,T)), \quad p \in \mathrm{L}^2(\Omega \times \mathcal{O} \times (0,T)),
\end{cases}
\qquad (3.19)
$$

and let the data $\mathbf{f}$, $\mathbf{g}$, $\mathbf{u}_0$ *and* p_0 *satisfy the condition*

$$
\begin{cases}
\mathbf{f} \in \mathrm{L}^2(0,T;\mathbb{H}^{-1}(\mathcal{O})), \quad \mathbf{g} \in \mathrm{L}^2(0,T;\ell_2(\mathbb{L}^2(\mathcal{O}))), \\
\mathbf{u}_0 \in \mathbb{L}^2(\mathcal{O}), \quad p_0 \in \mathrm{L}^2(\mathcal{O}).
\end{cases}
\qquad (3.20)
$$

If $\mathbf{v}$ *in* $\mathrm{L}^2(\Omega; C^0(0,T,\mathbb{L}^2(\mathcal{O}))\cap \mathrm{L}^2(0,T,\mathbb{H}^1_0(\mathcal{O})))$ *is another solution of the stochastic Navier-Stokes equation* (3.2)*, then*

$$\begin{cases} \Big[|\mathbf{u}(t)-\mathbf{v}(t)|^2+\varepsilon|p(t)-q(t)|^2\Big]\ \exp\Big[-\dfrac{27}{\nu^3}\displaystyle\int_0^t \|\mathbf{u}(s)\|^4_{\mathbb{L}^4(\mathcal{O})}\mathrm{d}s\Big]\le \\ \qquad\qquad\le |\mathbf{u}(0)-\mathbf{v}(0)|^2+\varepsilon|p(0)-q(0)|^2, \end{cases} \tag{3.21}$$

with probability 1 *for any* $0\le t\le T$ *and* $\varepsilon>0$.

Proof. Indeed if $\mathbf{u}$ and $\mathbf{v}$ are two solutions then $\mathbf{w}=\mathbf{v}-\mathbf{u}$ solves the deterministic equation

$$\partial_t\mathbf{w}(t)-\nu\triangle\mathbf{w}(t)+\nabla(q(t)-p(t))=\hat{B}(\mathbf{u})-\hat{B}(\mathbf{v})\quad \text{in } \mathbb{L}^2(0,T;\mathbb{H}^{-1}(\mathcal{O})),$$

with $\hat{B}(\mathbf{u})=[(\mathbf{u}\cdot\nabla)+\frac{1}{2}\operatorname{Div}\mathbf{u}]\mathbf{u}$. Notice that actually p and q are better processes, they belong to the space $\mathrm{L}^2(\Omega;C^0(0,T;\mathbb{L}^2(\mathcal{O})))$.

Next, setting

$$r(t):=\frac{27}{\nu^3}\int_0^t \|\mathbf{u}(\mathbf{s})\|^4_{\mathbb{L}^4(\mathcal{O})}\mathrm{d}s$$

we have

$$\begin{aligned} \mathrm{d}\langle\mathbf{w}(t),2e^{-r(t)}\mathbf{w}(t)\rangle-\nu\langle\triangle\mathbf{w}(t),2e^{-r(t)}\mathbf{w}(t)\rangle\mathrm{d}t& \\ +\langle\nabla(q(t)-p(t)),2e^{-r(t)}\mathbf{w}(t)\rangle\mathrm{d}t& \\ =\langle\hat{B}(\mathbf{u})-\hat{B}(\mathbf{v}),2e^{-r(t)}\mathbf{w}(t)\rangle\mathrm{d}t&. \end{aligned}$$

Using (2.6) we get

$$\begin{aligned} e^{-r(t)}\mathrm{d}(|\mathbf{w}(t)|^2)+2\nu e^{-r(t)}\|\mathbf{w}(t)\|^2\mathrm{d}t+2e^{-r(t)}\langle p(t)-q(t),\operatorname{Div}\mathbf{w}(t)\rangle\mathrm{d}t& \\ =2e^{-r(t)}\langle\hat{B}(\mathbf{u})-\hat{B}(\mathbf{v}),\mathbf{w}(t)\rangle\mathrm{d}t&. \end{aligned} \tag{3.22}$$

Since $\varepsilon\partial_t(q(t)-p(t))+\operatorname{Div}\mathbf{w}(t)=0$, Lemma 2.3 and (3.22) yield

$$\begin{aligned} &e^{-r(t)}\mathrm{d}\Big[|\mathbf{w}(t)|^2+\varepsilon|q(t)-p(t)|^2\Big] \\ &\quad=-2\nu e^{-r(t)}\|\mathbf{w}(t)\|^2\mathrm{d}t-2e^{-r(t)}\langle\hat{B}(\mathbf{v})-\hat{B}(\mathbf{u}),\mathbf{w}(t)\rangle\mathrm{d}t \\ &\quad\le-2\nu e^{-r(t)}\|\mathbf{w}(t)\|^2\mathrm{d}t+2e^{-r(t)}\Big[\frac{\nu}{2}\|\mathbf{w}(t)\|^2+\frac{27}{2\nu^3}|\mathbf{w}(t)|^2\|\mathbf{u}(\mathbf{t})\|^4_{\mathbb{L}^4(\mathcal{O})}\Big]\mathrm{d}t \\ &\quad=-\nu e^{-r(t)}\|\mathbf{w}(t)\|^2\mathrm{d}t+\dot{r}(t)e^{-r(t)}|\mathbf{w}(t)|^2\mathrm{d}t \\ &\quad\le-\nu e^{-r(t)}\|\mathbf{w}(t)\|^2\mathrm{d}t+\dot{r}(t)e^{-r(t)}\Big[|\mathbf{w}(t)|^2+\varepsilon|q(t)-p(t)|^2\Big]\mathrm{d}t. \end{aligned}$$

Hence

$$\mathrm{d}\Big[e^{-r(t)}\Big\{|\mathbf{w}(t)|^2+\varepsilon|q(t)-p(t)|^2\Big\}\Big]\le 0.$$

Hence, integrating in t, we deduce (3.21), with probability 1. □

Each solution $\mathbf{u}$ in the space $\mathrm{L}^2(\Omega;\mathrm{L}^\infty(0,T;\mathbb{H}^{-1}(\mathcal{O}))\cap\mathrm{L}^2(0,T;\mathbb{H}^1_0(\mathcal{O})))$ of the stochastic Navier-Stokes equation actually belongs to a better space, namely the space $\mathrm{L}^2(\Omega;C^0(0,T;\mathbb{L}^2(\mathcal{O}))\cap\mathbb{L}^4(\mathcal{O}\times(0,T)))$ in 2-D, $\mathcal{O}\subset\mathbb{R}^2$. Thus in 2-D, the uniqueness holds in the space $\mathrm{L}^2(\Omega;\mathrm{L}^2(0,T;\mathbb{H}^{-1}(\mathcal{O})))$.

If a given adapted process $\mathbf{u}$ in $\mathrm{L}^2(\Omega;\mathrm{L}^\infty(0,T;\mathbb{L}^2(\mathcal{O}))\cap\mathrm{L}^2(0,T;\mathbb{H}_0^1(\mathcal{O})))$ satisfies

$$\mathrm{d}(\mathbf{u}(t),\mathbf{v}) = \langle \mathbf{F}(t),\mathbf{v}\rangle \mathrm{d}t + (\mathbf{g}(t),\mathbf{v})\mathrm{d}w(t), \tag{3.23}$$

for any function $\mathbf{v}$ in $\mathbb{H}_0^1(\mathcal{O})$ and some functions $\mathbf{F}$ in $\mathrm{L}^2(0,T;\mathbb{H}^{-1}(\mathcal{O}))$ and $\mathbf{g}$ in $\mathrm{L}^2(0,T;\ell_2(\mathbb{L}^2(\mathcal{O})))$, then we can find a version of $\mathbf{u}$ (which is still denoted by $\mathbf{u}$) in $\mathrm{L}^2(\Omega;C^0(0,T;\mathbb{L}^2(\mathcal{O})))$ satisfying the energy equality

$$\mathrm{d}|\mathbf{u}(t)|^2 = \big[2\langle \mathbf{F}(t),\mathbf{u}(t)\rangle + \mathrm{Tr}(\mathbf{g}^{\mathbf{2}}(t)\big]\mathrm{d}t + 2(\mathbf{g}(t),\mathbf{u}(t))\mathrm{d}w(t) \tag{3.24}$$

see e.g. Gyongy and Krylov [9], Pardoux [15].

Definition 3.3. (*Strong Solution*) A strong solution $\mathbf{u}$ is defined on a given probability space $(\Omega,\ \Sigma,\ \Sigma_t,\ \mathcal{M})$ as a $\mathrm{L}^2(\Omega;\mathrm{L}^\infty(0,T;\mathbb{L}^2(\mathcal{O}))\cap\mathrm{L}^2(0,T;\mathbb{H}_0^1(\mathcal{O}))\cap C^0(0,T;\mathbb{L}^2(\mathcal{O})))$ valued function which satisfies the stochastic Navier-Stokes equation (3.2) in the weak sense and also the energy inequality

$$E\Big\{\sup_{0\le t\le T}\big[|\mathbf{u}(t)|^{\mathrm{p}} + \varepsilon|p(t)|^{\mathrm{p}}\big]e^{-\delta t} + \mathrm{p}\,\nu\int_0^T |\nabla\mathbf{u}(t)|^2|\mathbf{u}(t)|^{\mathrm{p}-2}e^{-\delta t}\mathrm{d}t\Big\}$$
$$\le |\mathbf{u}(0)|^{\mathrm{p}} + \varepsilon|p(0)|^{\mathrm{p}} + C_{\delta,\mathrm{p},T}\int_0^T \big[|\mathbf{f}(t)|^{\mathrm{p}} + \mathrm{Tr}(\mathbf{g}^2(t))^{\mathrm{p}/2}\big]e^{-\delta t}\mathrm{d}t,$$

where the constant $C_{\delta,\mathrm{p},T}$ depends only on $\delta>0$, $\varepsilon>0$, $1\le \mathrm{p}<\infty$ and $T>0$.

Proposition 3.4 (2-D existence)**.** *Let* $\mathbf{f}$, $\mathbf{g}$ *and* $\mathbf{u}_0$ *be such that*

$$\begin{cases} \mathbf{f}\in \mathrm{L}^{\mathrm{p}}(0,T;\mathbb{H}^{-1}(\mathcal{O})), \quad \mathbf{g}\in\mathrm{L}^{\mathrm{p}}(0,T;\ell_2(\mathbb{L}^2(\mathcal{O}))),\\ \mathbf{u}_0\in\mathbb{L}^2(\mathcal{O}), \quad p_0\in\mathrm{L}^2(\mathcal{O}), \end{cases} \tag{3.25}$$

for some $\mathrm{p}\ge 4$. *Then there is adapted processes* $\mathbf{u}(t,x,\omega)$ *and* $p(t,x,\omega)$ *with the regularity*

$$\begin{cases} \mathbf{u}\in\mathrm{L}^{\mathrm{p}}(\Omega;C^0(0,T;\mathbb{L}^2(\mathcal{O})))\cap\mathrm{L}^2(\Omega;\mathrm{L}^2(0,T;\mathbb{H}_0^1(\mathcal{O}))),\\ p,\ \dot{p}\in\mathrm{L}^2(\Omega;\mathrm{L}^2(0,T;\mathrm{L}^2(\mathcal{O}))) \end{cases} \tag{3.26}$$

satisfying the stochastic Navier-Stokes equation (3.2) *and the a priori bound* (3.10) *for every* $\varepsilon>0$.

Proof. Denoting

$$F(\mathbf{u}) := A\mathbf{u} + \hat{B}(\mathbf{u}) - \mathbf{f} := -\nu\triangle\mathbf{u} + [(\mathbf{u}\cdot\nabla) + (1/2)\,\mathrm{Div}\,\mathbf{u}]\mathbf{u} - \mathbf{f},$$

we have

$$\mathrm{d}\mathbf{u}^n(t) + F(\mathbf{u}^n(t))\mathrm{d}t + \nabla p^n(t)\mathrm{d}t = \mathbf{g}(t)\mathrm{d}w(t).$$

Then using the a priori estimate (3.10), it follows from the Banach-Alaoglu theorem that along a subsequence, the Galerkin approximations $\{\mathbf{u}^n\}$ have the following limits:

$$\mathbf{u}^n \longrightarrow \mathbf{u} \quad \text{weakly star in } \mathrm{L}^{\mathrm{p}}(\Omega;\mathrm{L}^\infty(0,T;\mathbb{L}^2(\mathcal{O})))\cap\mathrm{L}^2(\Omega;\mathrm{L}^2(0,T;\mathbb{H}_0^1(\mathcal{O}))),$$
$$p^n \longrightarrow p \text{ weakly in } \mathrm{L}^2(\Omega;\mathrm{L}^2(0,T;\mathbb{L}^2(\mathcal{O}))),$$
$$F(\mathbf{u}^n) \longrightarrow F_0 \quad \text{weakly in } \mathrm{L}^2(\Omega;\mathrm{L}^2(0,T;\mathbb{H}^{-1}(\mathcal{O}))),$$

where $\mathbf{u}$ has the Itô differential

$$\mathrm{d}\mathbf{u}(t) + F_0(t)\mathrm{d}t + \nabla p(t)\mathrm{d}t = \mathbf{g}(t)\mathrm{d}w(t) \qquad \text{in } \mathrm{L}^2(\Omega;\mathrm{L}^2(0,T;\mathbb{H}^{-1}(\mathcal{O})))$$

and the energy equality holds, i.e.,

$$\mathrm{d}\big[|\mathbf{u}(t)|^2+\varepsilon|p(t)|^2\big]+2\langle F_0(t),\mathbf{u}(t)\rangle\mathrm{d}t=\mathrm{Tr}(\mathbf{g}^2(t))\mathrm{d}t+2(\mathbf{g}(t),\mathbf{u}(t))\mathrm{d}w(t).$$

Now, for any adapted process $\mathbf{v}(t,x,\omega)$ in $\mathrm{L}^\infty((0,T)\times\Omega;\mathbb{L}^2(\mathcal{O}))$, we define

$$r(t,\omega):=\frac{27}{\nu^3}\int_0^t\|\mathbf{v}(\mathbf{s},.,\omega)\|^4_{\mathbb{L}^4(\mathcal{O})}\mathrm{d}s$$

as an adapted, continuous (and bounded in ω) real-valued process in $[0,T]$. Then from the energy equality

$$\begin{aligned}\mathrm{d}\Big[e^{-r(t)}\big\{|\mathbf{u}^n(t)|^2+\varepsilon|p^n(t)|^2\big\}\Big]&+e^{-r(t)}\langle 2F(\mathbf{u}^n(t))+\dot r(t)\mathbf{u}^n(t),\mathbf{u}^n(t)\rangle\mathrm{d}t\\&+\varepsilon|p^n(t)|^2\dot r(t)e^{-r(t)}\mathrm{d}t\\&\quad=\mathrm{Tr}(\mathbf{g}^2(t))e^{-r(t)}\mathrm{d}t+2(\mathbf{g}(t),\mathbf{u}^n(t))e^{-r(t)}\mathrm{d}w(t).\end{aligned}$$

Integrating between $0\le t\le T$ and taking the mathematical expectation we have

$$\begin{aligned}&E\Big[e^{-r(T)}\big\{|\mathbf{u}^n(T)|^2+\varepsilon|p^n(T)|^2\big\}-|\mathbf{u}^n(0)|^2-\varepsilon|p^n(0)|^2\Big]\\&\quad+E\Big[\int_0^T e^{-r(t)}\langle 2F(\mathbf{u}^n(t))+\dot r(t)\mathbf{u}^n(t),\mathbf{u}^n(t)\rangle\mathrm{d}t\Big]\\&\quad+E\Big[\varepsilon\int_0^T|p^n(t)|^2\dot r(t)e^{-r(t)}\mathrm{d}t\Big]\\&\quad=E\Big[\int_0^T\mathrm{Tr}(\mathbf{g}^2(t))e^{-r(t)}\mathrm{d}t\Big].\end{aligned}$$

Considering the fact that the initial conditions $\mathbf{u}^n(0)$ and $p^n(0)$ converge to $\mathbf{u}(0)$ and $p(0)$ respectively in $\mathbb{L}^2$, and the lower-semi-continuity of the $\mathbb{L}^2$-norm, we deduce

$$\begin{aligned}&\liminf_n E\Big[-\int_0^T e^{-r(t)}\langle 2F(\mathbf{u}^n(t))+\dot r(t)\mathbf{u}^n(t),\mathbf{u}^n(t)\rangle\mathrm{d}t\Big]\\&\quad=\liminf_n E\Big[e^{-r(T)}\big\{|\mathbf{u}^n(T)|^2+\varepsilon|p^n(T)|^2\big\}-|\mathbf{u}^n(0)|^2-\varepsilon|p^n(0)|^2\\&\qquad\qquad+\varepsilon\int_0^T|p^n(t)|^2\dot r(t)e^{-r(t)}\mathrm{d}t-\int_0^T\mathrm{Tr}(\mathbf{g}^2(t))e^{-r(t)}\mathrm{d}t\Big]\\&\quad\ge E\Big[e^{-r(T)}\big\{|\mathbf{u}(T)|^2+\varepsilon|p(T)|^2\big\}-|\mathbf{u}(0)|^2-\varepsilon|p(0)|^2\\&\qquad\qquad+\varepsilon\int_0^T|p(t)|^2\dot r(t)e^{-r(t)}\mathrm{d}t-\int_0^T\mathrm{Tr}(\mathbf{g}^2(t))e^{-r(t)}\mathrm{d}t\Big]\\&\quad=E\Big[-\int_0^T e^{-r(t)}\langle 2F_0(t)+\dot r(t)\mathbf{u}(t),\mathbf{u}(t)\rangle\mathrm{d}t\Big].\end{aligned}$$

Next, by monotonicity on $\mathbb{L}^4$-balls, i.e. by Lemma 2.4, we have

$$\begin{aligned}2E\Big[\int_0^T e^{-r(t)}\langle F(\mathbf{u}^n(t))-F(\mathbf{v}(t)),\mathbf{u}^n(t)-\mathbf{v}(t)\rangle\mathrm{d}t\Big]&\\+E\Big[e^{-r(t)}\dot r(t)|\mathbf{u}^n(t)-\mathbf{v}(t)|^2\mathrm{d}t\Big]&\ \ge 0.\end{aligned}$$

Rearranging the terms we find

$$E\Big[\int_0^T e^{-r(t)}\langle 2F(\mathbf{v}(t))+\dot r(t)\mathbf{v}(t),\mathbf{v}(t)-\mathbf{u}^n(t)\rangle \mathrm{d}t\Big]$$
$$\geq E\Big[\int_0^T e^{-r(t)}\langle 2F(\mathbf{u}^n(t))+\dot r(t)\mathbf{u}^n(t),\mathbf{v}(t)-\mathbf{u}^n(t)\rangle \mathrm{d}t\Big].$$

Taking limit in n , we get

$$E\Big[\int_0^T e^{-r(t)}\langle 2F(\mathbf{v}(t))+\dot r(t)\mathbf{v}(t),\mathbf{v}(t)-\mathbf{u}(t)\rangle \mathrm{d}t\Big]$$
$$\geq E\Big[\int_0^T e^{-r(t)}\langle 2F_0(t)+\dot r(t)\mathbf{u}(t),\mathbf{v}(t)-\mathbf{u}(t)\rangle \mathrm{d}t\Big].$$

Now we take $\mathbf{v}:=\mathbf{u}+\lambda\mathbf{w}$ with $\lambda>0$ and $\mathbf{w}$ is an adapted process in the space

$$\mathrm{L}^4(\Omega;\mathrm{L}^\infty(0,T;\mathbb{L}^2(\mathcal{O})))\cap\mathrm{L}^2(\Omega;\mathrm{L}^2(0,T;\mathbb{H}_0^1(\mathcal{O}))).$$

Then we have

$$\lambda E\Big[\int_0^T e^{-r(t)}\langle 2F\big(\mathbf{u}(t)+\lambda\mathbf{w}(t)\big)-2F_0(t),\mathbf{w}(t)\rangle \mathrm{d}t\Big]$$
$$+\lambda^2 E\Big[\int_0^T e^{-r(t)}\dot r(t)|\mathbf{w}(t)|^2\mathrm{d}t\Big]\ \geq 0.$$

Dividing by λ on both sides of the inequality above, and letting λ go to 0, one obtains

$$E\Big[\int_0^T e^{-r(t)}\langle F(\mathbf{u}(t))-F_0(t),\mathbf{w}(t)\rangle \mathrm{d}t\Big]\ \geq 0.$$

Since $\mathbf{w}$ is arbitrary, we conclude that $F_0(t)=F(\mathbf{u}(t))$. Thus the existence of a strong solution of the stochastic Navier-Stokes equation (3.2) has been proved. □

4. Convergence as $\varepsilon\to 0$

We will now study the asymptotic limit of $\varepsilon\to 0$. Let us consider the family of perturbed systems(depending on the positive parameter ε),

$$\partial_t\mathbf{u}^\varepsilon-\nu\triangle\mathbf{u}^\varepsilon+\sum_{i=1}^2 u_i^\varepsilon D_i\mathbf{u}^\varepsilon+\frac{1}{2}(\mathrm{Div}\,\mathbf{u}^\varepsilon)\mathbf{u}^\varepsilon+\nabla p^\varepsilon=\mathbf{f}+\mathbf{g}(t)\mathrm{d}w(t) \quad \text{in } \mathrm{L}^2(0,T;\mathbb{H}^{-1}(\mathcal{O})), \tag{4.1}$$

$$\varepsilon\partial_t p^\varepsilon+\mathrm{Div}\,\mathbf{u}^\varepsilon=0 \quad \text{in } \mathrm{L}^2(0,T;\mathrm{L}^2(\mathcal{O})), \tag{4.2}$$

with the initial conditions

$$\mathbf{u}^\varepsilon(0)=\mathbf{u}_0 \quad \text{in} \quad \mathbb{L}^2(\mathcal{O}) \qquad \text{and} \qquad p^\varepsilon(0)=p_0 \quad \text{in} \quad \mathrm{L}^2(\mathcal{O}). \tag{4.3}$$

This is a method to overcome the computational difficulties connected with the constraint "$\mathrm{Div}\,\mathbf{u}=0$". The equations (4.1)-(4.2) are easier to approximate than the original stochastic Navier-Stokes equation as the constraint "$\mathrm{Div}\,\mathbf{u}=0$" has been replaced by the evolution equation (4.2).

Here we will show how the solutions of the perturbed problems converge to the solutions of the incompressible stochastic Navier-Stokes equation as $\varepsilon \to 0$. The idea of the proof is similar to the deterministic case presented in Temam [19].

Let

$$\mathbf{f} \in \mathrm{L}^2(0,T;\mathbb{L}^2(\mathcal{O})),\ \mathbf{g} \in \mathrm{L}^2(0,T;\ell_2(\mathbb{L}^2(\mathcal{O}))),\ \mathbf{u}_0 \in \mathbb{L}^2(\mathcal{O}),\ p_0 \in \mathrm{L}^2(\mathcal{O}).$$

Then we can write the above mentioned perturbed systems as Itô Stochastic equations in variational form

$$\begin{cases} \mathrm{d}(\mathbf{u}^\varepsilon(t),\mathbf{v}) + \langle -\nu\triangle \mathbf{u}^\varepsilon(t),\mathbf{v}\rangle\,\mathrm{d}t + \hat{B}(\mathbf{u}^\varepsilon,\mathbf{u}^\varepsilon,\mathbf{v})\mathrm{d}t \\ \qquad\qquad + \langle \nabla p^\varepsilon(t),\mathbf{v}\rangle\,\mathrm{d}t = (\mathbf{f},\mathbf{v})\,\mathrm{d}t + (\mathbf{g}(t),\mathbf{v})\,\mathrm{d}w(t), \\ \langle \varepsilon\dot{p^\varepsilon}(t) + \mathrm{Div}\,\mathbf{u}^\varepsilon(t), q\rangle = 0, \end{cases} \tag{4.4}$$

in $(0,T)$, with the initial conditions

$$(\mathbf{u}^\varepsilon(0),\mathbf{v}) = (\mathbf{u}_0,\mathbf{v}) \quad \text{and} \quad (p^\varepsilon(0),q) = (p_0,q), \tag{4.5}$$

for any $\mathbf{v}$ in the space $\mathbb{H}^1_0(\mathcal{O})$ and any q in $\mathrm{L}^2(\Omega;\mathrm{L}^2(0,T;\mathbb{L}^2(\mathcal{O})))$.

Proposition 4.1. *As $\varepsilon \to 0$, the solutions $\{\mathbf{u}^\varepsilon, p^\varepsilon\}$ of the equations* (4.4)-(4.5) *converge to the solution $\mathbf{u}$ of the incompressible stochastic Navier-Stokes equation.*

Proof. First we should point out that the solutions $\{\mathbf{u}^\varepsilon, p^\varepsilon\}$ of the equations (4.4)-(4.5) satisfy the monotonicity property in Lemma 2.4, the energy equality (3.7) and the a priori estimates (3.8) and (3.10). By virtue of these a priori estimates and using the Banach-Alaoglu theorem, along a subsequence the approximations $\{\mathbf{u}^\varepsilon, p^\varepsilon\}$ have the following limits:

$$\begin{aligned} \mathbf{u}^\varepsilon \longrightarrow \mathbf{u} \quad &\text{weakly star in } \mathrm{L}^p(\Omega;\mathrm{L}^\infty(0,T;\mathbb{L}^2(\mathcal{O}))), \\ &\text{weakly in } \mathrm{L}^2(\Omega;\mathrm{L}^2(0,T;\mathbb{H}^1_0(\mathcal{O}))), \end{aligned} \tag{4.6}$$

$$\sqrt{\varepsilon}p^\varepsilon \longrightarrow \chi \text{ weakly in } \mathrm{L}^2(\Omega;\mathrm{L}^2(0,T;\mathbb{L}^2(\mathcal{O}))). \tag{4.7}$$

Let us denote

$$F(\mathbf{u}^\varepsilon) := A\mathbf{u}^\varepsilon + \hat{B}(\mathbf{u}^\varepsilon) - \mathbf{f} := -\nu\triangle\mathbf{u}^\varepsilon + [(\mathbf{u}^\varepsilon\cdot\nabla) + (1/2)\,\mathrm{Div}\,\mathbf{u}^\varepsilon]\mathbf{u}^\varepsilon - \mathbf{f},$$

and

$$\tilde{F}(\mathbf{u}) := A\mathbf{u} + B(\mathbf{u}) - \mathbf{f} := -\nu\triangle\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} - \mathbf{f}.$$

Let ϕ be a C^∞ scalar function on $[0,T]$ with $\phi(T)=0$. Multiplying the equation (4.4) by $\phi(t)$, integrating in t and taking mathematical expectation, we obtain

$$\begin{aligned} (\mathbf{u}^\varepsilon(T),\mathbf{v})\phi(T) - (\mathbf{u}^\varepsilon(0),\mathbf{v})\phi(0) - E\Big[\int_0^T (\mathbf{u}^\varepsilon(t),\mathbf{v}\phi'(t))\mathrm{d}t\Big] \\ + E\Big[\int_0^T \big(F(\mathbf{u}^\varepsilon(t)),\mathbf{v}\phi(t)\big)\mathrm{d}t\Big] + E\Big[\int_0^T \langle\nabla p^\varepsilon(t),\mathbf{v}\phi(t)\rangle\,\mathrm{d}t\Big] \\ = E\Big[\int_0^T (\mathbf{g}(t),\mathbf{v}\phi(t))\,\mathrm{d}w(t)\Big], \quad \text{for all } \mathbf{v} \text{ in } \mathbb{H}^1_0(\mathcal{O}). \end{aligned} \tag{4.8}$$

Now passing to the limit in (4.7) we have in the sense of distribution,

$$E\big[\sqrt{\varepsilon}\big(\frac{\mathrm{d}p^\varepsilon}{\mathrm{d}t},q\big)\big] \longrightarrow E\big[\big(\frac{\mathrm{d}\chi}{\mathrm{d}t},q\big)\big]. \tag{4.9}$$

Hence in the same sense

$$E\big[\varepsilon\big(\frac{\mathrm{d}p^\varepsilon}{\mathrm{d}t},q\big)\big] \longrightarrow 0.$$

Then passing to the limit in the following equation

$$E\big[\langle\varepsilon\dot{p^\varepsilon}(t)+\mathrm{Div}\,\mathbf{u}^\varepsilon(t),q\rangle\big]=0,$$

we get

$$E\big[\langle\mathrm{Div}\,\mathbf{u},q\rangle\big]=0,\quad \forall q\in\mathrm{L}^2(\Omega;\mathrm{L}^2(0,T;\mathbb{L}^2(\mathcal{O}))),$$

which implies that $\mathrm{Div}\,\mathbf{u}=0$, almost everywhere and almost surely.
Hence $\mathbf{u}\in\mathrm{L}^{\mathrm{p}}(\Omega;\mathrm{L}^\infty(0,T;\mathbb{L}^2(\mathcal{O})))\cap\mathrm{L}^2(\Omega;\mathrm{L}^2(0,T;\mathbb{H}^1_0(\mathcal{O})))$.
Now it is clear that

$$\langle\nabla p^\varepsilon,\mathbf{v}\phi(t)\rangle=\langle p^\varepsilon,\phi(t)\,\mathrm{Div}\,\mathbf{v}\rangle=0,\quad \text{almost surely.} \tag{4.10}$$

Now using the same Minty-Browder monotonicity argument used in Proposition 3.4 we can show that in limit

$$F(\mathbf{u}^\varepsilon)\longrightarrow\tilde{F}(\mathbf{u})\quad \text{weakly in } \mathrm{L}^2(\Omega;\mathrm{L}^2(0,T;\mathbb{H}^{-1}(\mathcal{O}))). \tag{4.11}$$

Using the results (4.10) and (4.11), in limit we have from equation (4.8)

$$-E\Big[\int_0^T(\mathbf{u}(t),\mathbf{v}\phi'(t))\mathrm{d}t\Big]+E\Big[\int_0^T\big(\tilde{F}(\mathbf{u}(t)),\mathbf{v}\phi(t)\big)\mathrm{d}t\Big]=(\mathbf{u}_0,\mathbf{v})\phi(0),$$
$$\text{for all } \mathbf{v} \text{ in } \mathbb{H}^1_0(\mathcal{O}).$$

This proves that $\mathbf{u}$ is a solution of the incompressible stochastic Navier-Stokes equation. □

Remark 4.2. If one considers the multiplicative noise $\sigma(t,\mathbf{u})$ of the type considered in the hypotheses $(A.1-A.3)$ in Sritharan and Sundar [16], then under these conditions same a priori estimates (3.8)-(3.10) hold. Thus $\sigma^\varepsilon(.,\mathbf{u}^\varepsilon)\to S$ weakly in $\mathrm{L}^2(\Omega;\mathrm{L}^2(0,T;\mathrm{L}_Q))$, where L_Q denote the space of linear operators S such that $SQ^{1/2}$ is a Hilbert-Schmidt operator from L^2 to L^2 and the norm on the space L_Q is defined by $|S|^2_{\mathrm{L}_Q}=\mathrm{Tr}(SQS^\star)$ and Q is a trace class operator. Hence with the help of Minty-Browder monotonicity arguments the existence and uniqueness proofs go through and we can also establish the limit to incompressible flow.

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