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LARGE AND MODERATE DEVIATION PRINCIPLES FOR RECURSIVE KERNEL ESTIMATORS FOR SPATIAL DATA

SALIM BOUZEBDA* AND YOUSRI SLAOUI

Abstract. The main purpose of this paper is to establish large and moderate deviations principles for the recursive kernel estimators of a probability density function for spatial data defined by the stochastic approximation algorithm proposed by [9]. We show that the estimator constructed using the stepsize which minimize the variance of the class of the recursive estimators defined in [9] gives the same pointwise LDP and MDP as the nonrecursive kernel density estimator considered by [42]. We will prove moderate deviations and large deviations for statistic for testing symmetry, of interest by and of itself.

1. Introduction

Kernel nonparametric function estimation methods have been the subject of intense investigation by both statisticians and probabilists for many years and this has led to the development of a large variety of techniques. Although they are popular, they present only one of many approaches to the construction of good function estimators. These include, for example, nearest-neighbor, spline, neural network, and wavelet methods. In this article, we shall restrict attention to the some results concerning the kernel-type estimators of density based on spatial data. Spatial data, collected on measurement sites in a variety of fields and the statistical treatment, typically arise in various fields of research, including econometrics, epidemiology, environmental science, image analysis, oceanography, meteorology, geostatistics and many others. For good sources of references to research literature in this area along with statistical applications consult [22], [34], [36] and [14] and the references therein. In the context of nonparametric estimation for spacial data, the existing papier are mainly concerned with the estimation of a probability density and regression functions, due to lack of space we cite only some key references, among many others, [5], [13], [16], [42], [43] and [15]. In the works of [2] and [8], recursive versions of non-parametric density estimation for spatial data are investigated, we may refer also to [9, 10, 11] for related problems.

We start by giving some notation and definitions that are needed for the forthcoming sections. We consider a spatial process $\left(X_{\mathbf{i}} \in \mathbb{R}^d, \, \mathbf{i} \in \mathbb{N}^N\right)$ defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, where $d \geq 1$ and $N \geq 1$. We assume that the $X_{\mathbf{i}}$ have the same distribution for $\mathbf{i} \in \mathbb{N}^N$, with unknown probability density

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function $f(\cdot)$. As it is classically assumed in the literature, the process under study $(X_{\mathbf{i}})$ is observable over a region $\mathcal{D} \subset \mathbb{R}^N$. For convenience, we treat the observations sites as an array that is $\mathcal{I}_n = \{s_j,\, j = 1, \ldots, n\}$. In this paper we propose to estimate the probability density function $f(\cdot)$ based on $(X_{\mathbf{i}}, \mathbf{i} \in \mathcal{I}_n)$. In order to construct a stochastic algorithm for the estimation of the function $f(\cdot)$ at a point $\mathbf{x}$, we define an algorithm of search of the zero of the function $h : \mathbf{y} \to f(\mathbf{x}) - \mathbf{y}$. Following Robbins-Monro's procedure, this algorithm is defined by setting $f_0(\mathbf{x}) \in \mathbb{R}^d$, and, for all $n \geq 1$,

$$f_n(\mathbf{x}) = f_{n-1}(\mathbf{x}) + \gamma_{s_n} W_{s_n}(\mathbf{x}),$$

where $W_{s_n}(\mathbf{x})$ is an observation of the function h at the point $f_{n-1}(\mathbf{x})$ and (γ_{s_n}) is a sequence of positive real numbers that goes to zero. To define $W_{s_n}(\mathbf{x})$, they follow the approach of [32, 33], [44] and more recently [39] and introduced a kernel K (which is a function satisfying $\int_{\mathbb{R}^d} K(\mathbf{x})d\mathbf{x} = 1$) and a bandwidth (h_{s_n}) (which is a sequence of positive real numbers that goes to zero), and they set

$$W_{s_n}(\mathbf{x}) = h_{s_n}^{-d} K(h_{s_n}^{-1}[\mathbf{x} - X_{s_n}]) - f_{n-1}(\mathbf{x}).$$

The stochastic approximation algorithm introduced in [8, 9] which estimate recursively the density $f(\cdot)$ at the point $\mathbf{x}$ is defined by

$$f_n(\mathbf{x}) = (1 - \gamma_{s_n}) f_{n-1}(\mathbf{x}) + \gamma_{s_n} h_{s_n}^{-d} K\left(\frac{\mathbf{x} - X_{s_n}}{h_{s_n}}\right). \tag{1.1}$$

Recently, large and moderate deviations results have been proved for the recursive density estimators defined by stochastic approximation method in [37] in the non spatial case, for the averaged stochastic approximation method for the estimation of a regression function in [40] (for the non spatial case) and moderate deviations results for the stochastic approximation method for the estimation of a regression function in [41] (for the non spatial case). The purpose of this paper is to establish large and moderate deviations principles for the recursive density estimator for spatial data defined by the stochastic approximation algorithm (1.1).

Let us first recall that a $\mathbb{R}^m$-valued sequence $(Z_n)_{n\geq 1}$ satisfies a large deviations principle (LDP) with speed (ν_n) and good rate function I if :

(1) (ν_n) is a positive sequence such that $\lim_{n\to\infty} \nu_n = \infty$;
(2) $I : \mathbb{R}^m \to [0, \infty]$ has compact level sets;
(3) for every borel set $B \subset \mathbb{R}^m$,

$$\begin{aligned} -\inf_{x\in \overset{\circ}{B}} I(\mathbf{x}) &\leq \liminf_{n\to\infty} \nu_n^{-1} \log \mathbb{P}[Z_n \in B] \\ &\leq \limsup_{n\to\infty} \nu_n^{-1} \log \mathbb{P}[Z_n \in B] \leq -\inf_{x\in \overline{B}} I(\mathbf{x}), \end{aligned}$$

where $\overset{\circ}{B}$ and $\overline{B}$ denote the interior and the closure of B respectively. Moreover, let (v_n) be a nonrandom sequence that goes to infinity; if $(v_n Z_n)$ satisfies a LDP, then (Z_n) is said to satisfy a moderate deviations principle (MDP). For a background on the theory of large deviations, see [18] and references therein.

The first purpose of this paper is to establish pointwise LDP for the recursive kernel density estimators $f_n(\cdot)$ defined by the stochastic approximation algorithm (1.1). It turns out that the rate function depend on the choice of the stepsize (γ_{s_n}). We focus in the first part of this paper on the following two special cases:

(1) $(\gamma_{s_n}) = (n^{-1})$,

(2) $(\gamma_{s_n}) = \left(h_{s_n}^d \left(\sum_{k=1}^{n} h_{s_k}^d \right)^{-1} \right)$,

remark that the first stepsize belongs to the subclass of recursive kernel estimators which have a minimum $MISE$ and the second stepsize belongs to the subclass of recursive kernel estimators which have a minimum variance (see [9]).

We show that using the stepsize $(\gamma_{s_n}) = (n^{-1})$ and the bandwidth

$$(h_{s_n}) \equiv (cn^{-a})$$

with $c > 0$ and $a \in]0, 1/d[$, the sequence $(f_n(\mathbf{x}) - f(\mathbf{x}))$ satisfies a LDP with speed $(nh_{s_n}^d)$ and the rate function defined as follows:

$$\begin{cases} \text{if} \quad f(\mathbf{x}) \neq 0, \quad I_{a,\mathbf{x}} : t \to f(\mathbf{x}) I_a \left(1 + \frac{t}{f(\mathbf{x})} \right), \\ \text{if} \quad f(\mathbf{x}) = 0, \quad I_{a,\mathbf{x}}(0) = 0 \quad \text{and} \quad I_{a,\mathbf{x}}(t) = +\infty \quad \text{for} \quad t \neq 0, \end{cases} \tag{1.2}$$

where

$$I_a(t) = \sup_{u \in \mathbb{R}} \{ ut - \psi_a(u) \},$$

$$\psi_a(u) = \int_{[0,1] \times \mathbb{R}^d} s^{-ad} \left(e^{uK(\mathbf{z})} - 1 \right) ds d\mathbf{z},$$

which is the same rate function for the LDP of the [45] kernel estimator (see [27]) in the non spatial case. Moreover, we show that using the stepsize

$$(\gamma_{s_n}) = \left(h_{s_n}^d \left(\sum_{k=1}^{n} h_{s_k}^d \right)^{-1} \right)$$

and more general bandwidths defined as $h_{s_n} = h(s_n)$ for all n, where h is a regularly varing function with exponent $(-a)$, $a \in]0, 1/d[$. We prove that the sequence $(f_{s_n}(\mathbf{x}) - f(\mathbf{x}))$ satisfies a LDP with speed $(nh_{s_n}^d)$ and the following rate function:

$$\begin{cases} \text{if} \quad f(\mathbf{x}) \neq 0, \quad I_{\mathbf{x}} : t \to f(\mathbf{x}) I \left(1 + \frac{t}{f(\mathbf{x})} \right), \\ \text{if} \quad f(\mathbf{x}) = 0, \quad I_{\mathbf{x}}(0) = 0 \quad \text{and} \quad I_{\mathbf{x}}(t) = +\infty \quad \text{for} \quad t \neq 0, \end{cases} \tag{1.3}$$

where

$$I(\mathbf{t}) = \sup_{\mathbf{u} \in \mathbb{R}^d} \{ ut - \psi(\mathbf{u}) \},$$

$$\psi(\mathbf{u}) = \int_{\mathbb{R}^d} \left(e^{\mathbf{u}K(\mathbf{z})} - 1 \right) d\mathbf{z},$$

which is the same rate function for the LDP of the nonrecursive kernel density estimator considered by [42], see [27] in the non spatial case (Akaike-Parzen-Rosenblatt kernel density, [1], [35] and [30]).

Our second purpose in this paper is to provide pointwise MDP for the proposed density estimator for spatial data defined by the stochastic approximation algorithm (1.1). In this case, we consider more general stepsizes defined as

$\gamma_{s_n} = \gamma(s_n)$ for all n, where γ is a regularly function with exponent $(-\alpha)$, $\alpha \in]1/2, 1]$. Throughout this paper we will use the following notation:

$$\xi = \lim_{n\to+\infty} (n\gamma_{s_n})^{-1}. \tag{1.4}$$

For any positive sequence (v_{s_n}) satisfying

$$\lim_{n\to\infty} v_{s_n} = \infty \quad \text{and} \quad \lim_{n\to\infty} \frac{\gamma_{s_n} v_{s_n}^2}{h_{s_n}^d} = 0$$

and general bandwidths (h_{s_n}), we prove that the sequence

$$v_{s_n}(f_n(\mathbf{x}) - f(\mathbf{x}))$$

satisfies a LDP of speed $\left(h_{s_n}^d / \left(\gamma_{s_n} v_{s_n}^2\right)\right)$ and rate function $J_{a,\alpha,\mathbf{x}}$ defined by

$$\begin{cases} \text{if} \quad f(\mathbf{x}) \neq 0, \quad J_{a,\alpha,\mathbf{x}} : t \to \dfrac{t^2 (2 - (\alpha - ad)\xi)}{2f(\mathbf{x}) \displaystyle\int_{\mathbb{R}^d} K^2(\mathbf{z})\, d\mathbf{z}} \\ \text{if} \quad f(\mathbf{x}) = 0, \quad J_{a,\alpha,\mathbf{x}}(0) = 0 \quad \text{and} \quad J_{a,\alpha,\mathbf{x}}(t) = +\infty \quad \text{for} \quad t \neq 0. \end{cases} \tag{1.5}$$

Let us point out that using the stepsize

$$(\gamma_{s_n}) = \left(h_{s_n}^d \left(\sum_{k=1}^{d} h_{s_k}^d \right)^{-1} \right)$$

which minimize the variance of $f_n(\cdot)$, we obtain the same rate function for the pointwise LDP and MDP as the one obtained for the non recursive kernel density estimator. To our best knowledge, these problems were open up to present, and it gives the main motivation to our paper.

The layout of the article is as follows. In the forthcoming section, we will introduce our framework and give the main assumptions. Section 2.1 is devoted to the pointwise LDP for the density estimator defined in (1.1) and Section will be concerned with MDP results. In section 3, we investigate large and moderate deviations for the problem of symmetry of the density function $f(\cdot)$. To prevent from interrupting the flow of the presentation, all proofs are gathered in Section 5.

2. Assumptions and Main Results

We define the following class of regularly varying sequences.

Definition 2.1. Let $\gamma \in \mathbb{R}$ and $(v_{s_n})_{n\geq 1}$ be a nonrandom positive sequence. We say that $(v_{s_n}) \in \mathcal{GS}(\gamma)$ if

$$\lim_{n\to+\infty} n \left[1 - \frac{v_{s_{n-1}}}{v_{s_n}} \right] = \gamma. \tag{2.1}$$

Condition (2.1) was introduced (in the case when $s_n = n$) by [21] to define regularly varying sequences (see also [6]), and by [24] in the context of stochastic approximation algorithms. Typical sequences in $\mathcal{GS}(\gamma)$ are, for $b \in \mathbb{R}$, $n^\gamma (\log n)^b$, $n^\gamma (\log\log n)^b$, and so on.

2.1. Pointwise LDP for the density estimator defined by the stochastic approximation algorithm (1.1).

2.1.1. *Choices of* (γ_{s_n}) *minimizing the MISE of* $f_n(\cdot)$. It is clear that in order to minimize the MISE of $f_n(\cdot)$, the stepsize (γ_{s_n}) must be chosen in $\mathcal{GS}(-1)$ and should ensures that

$$\lim_{n\to\infty} n\gamma_{s_n} = 1.$$

A straightforward example of stepsize belonging to $\mathcal{GS}(-1)$ and satisfies

$$\lim_{n\to\infty} n\gamma_{s_n} = 1 \quad \text{is} \quad (\gamma_{s_n}) = \left(n^{-1}\right).$$

For this choice of stepsize, the estimator $f_n(\cdot)$ defined by (1.1) equals to the recursive kernel estimator introduced by [45] in the spatial case.

To establish pointwise LDP for $f_n(\cdot)$ in this special case, we need the following assumptions.

(L1) $K : \mathbb{R}^d \to \mathbb{R}$ is a continuous, bounded function satisfying

$$\int_{\mathbb{R}^d} K(z_1, \dots, z_d)\, dz_1 \dots dz_d := \int_{\mathbb{R}^d} K(\mathbf{z})\, d\mathbf{z} = 1,$$

$\int_{\mathbb{R}^d} \mathbf{z}K(\mathbf{z})\, d\mathbf{z} = 0$ and $\int_{\mathbb{R}^d} \|\mathbf{z}\|^2 K(\mathbf{z})\, d\mathbf{z} < \infty$.

(L2) (i) $(h_{s_n}) = (cn^{-a})$ with $a \in]0, 1/d[$ and $c > 0$.
(i) $(\gamma_{s_n}) = \left(n^{-1}\right)$.

(L3) (i) $f(\cdot)$ is bounded, twice differentiable, and, for all $i, j \in \{1, \dots, d\}$, $\partial^2 f(\cdot)/\partial x_i \partial x_j$ is bounded.
(ii) For any $i, j \in \{1, \dots, n\}$ such that $s_i \neq s_j$, the random vector $\left(X_{s_i}, X_{s_j}\right)$ has a density $f_{s_i,s_j}(\cdot)$ such that $\sup_{s_i\neq s_j} \left\|g_{s_i,s_j}\right\| < \infty$, where

$$g_{s_i,s_j}(\cdot) = f_{s_i,s_j}(\cdot) - f(\cdot) \otimes f(\cdot).$$

(L4) (i) The field $(X_{s_i})_{1\leq i\leq n}$ is α-mixing: there exists a function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\phi(t)$ goes to zero as t goes to infinity, such that for E, $F \subset \mathbb{R}^2$ with finite cardinals $Card(E)$, $Card(F)$

$$\begin{aligned}\alpha(\sigma(E), \sigma(F)) \quad &:= \sup_{A\in\sigma(E), B\in\sigma(F)} |\mathbb{P}(A \cap B) - \mathbb{P}(A)\mathbb{P}(B)| \\ &\leq \phi(dist(E, F))\, \psi(Card(E), Card(F)),\end{aligned}$$

where $\sigma(E) = \{X_\mathbf{i} : \mathbf{i} \in E\}$ and $\sigma(F) = \{X_\mathbf{i} : \mathbf{i} \in F\}$, $dist(E, F)$ is the Euclidean distance between E and F and $\psi(\cdot)$ is a positive symmetric function nondecreasing in each variable. The functions $\phi(\cdot)$ and $\psi(\cdot)$ are such that $\phi(i) \leq Ci^{-\theta}$ and

$$\psi(n, m) \leq C \min(m, n).$$

(ii) $\sum_{k=0}^{\infty} (k + 1)^2 \alpha_n^{\frac{\delta}{4+\delta}}(k) < c$ for some c, $\delta > 0$ and all n, where

$$\alpha_n(k) = \alpha_n(\mathbf{X}, k) = \sup_{A\in\mathcal{F}_{-\infty}^n, B\in\mathcal{F}_{n+k}^\infty} |\mathbb{P}(A \cap B) - \mathbb{P}(A)\mathbb{P}(B)|,$$

with $\mathbf{X} = \{X_{s_i}\}_{i=1}^n$, $\mathcal{F}_m^n$ denote the σ-algebra generated by $\{X_{s_i}\}_{i=m}^n$.

Assumption (**L1**) on the kernel is widely used in the recursive and the nonrecursive framework for the functional estimation. Assumption (**L2**) on the stepsize and the bandwidth was used in the recursive framework for the estimation of the density function (see [25] and [37, 38]) and for the estimation of the distribution function (see [39]). Assumption (**L3**) on the density of X was used in the nonrecursive framework for the estimation of the density function (see [35] and [30]) and in the recursive framework (see [25] and [37, 38, 39]). Assumption (**L4**) **i**) are classical in nonparametric estimation in the spatial literature (see [2]). However, assumption (**L4**) **ii**) was considered in [20] to establish a general central limit theorem for strong mixing sequences, see [35].

The following Theorem gives the pointwise LDP for $f_n(\cdot)$ in this case.

Theorem 2.2 (Pointwise LDP for Wolverton and Wagner estimator)**.** *Let Assumptions* (**L1**)-(**L4**) *hold and assume that* $f(\cdot)$ *is continuous at* $\mathbf{x}$*. Then, the sequence* $(f_n(\mathbf{x}) - f(\mathbf{x}))$ *satisfies a LDP with speed* $\left(nh_{s_n}^d\right)$ *and rate function defined by (1.2).*

2.1.2. *Choices of* (γ_{s_n}) *minimizing the variance of* $f_n(\cdot)$. In order to minimize the asymptotic variance of $f_n(\cdot)$, the stepsize (γ_{s_n}) must be chosen in $\mathcal{GS}(-1)$ and must satisfy

$$\lim_{n\to\infty} n\gamma_{s_n} = 1 - ad$$

A straightforward example of stepsize belonging to $\mathcal{GS}(-1)$ and such that

$$\lim_{n\to\infty} n\gamma_{s_n} = 1 - ad \quad \text{is} \quad (\gamma_{s_n}) = \left((1-ad)\, n^{-1}\right),$$

a second example of stepsize satisfying these two conditions is

$$(\gamma_{s_n}) = \left(h_{s_n}^d \left(\sum_{k=1}^{n} h_{s_k}^d \right)^{-1} \right).$$

For this choice of stepsize, the estimator $f_n(\cdot)$ defined by (1.1) gives in the non spatial case the estimator considered by [17] and [19]. To establish pointwise LDP for $f_n(\cdot)$ in this case, we assume that.

(**L2′**) (i) $(h_{s_n}) \in \mathcal{GS}(-a)$ with $a \in]0, 1/d[$.
(ii) $(\gamma_{s_n}) = \left(h_{s_n}^d \left(\sum_{k=1}^n h_{s_k}^d\right)^{-1}\right)$.

The following Theorem gives the pointwise LDP for $f_n(\cdot)$ in this case.

Theorem 2.3 (Pointwise LDP for Deheuvels estimator)**.** *Let the assumptions* (**L1**)*,* (**L2′**)-(**L4**) *be fulfilled and assume that* $f(\cdot)$ *is continuous at* $\mathbf{x}$*. Then, the sequence* $(f_n(\mathbf{x}) - f(\mathbf{x}))$ *satisfies a LDP with speed* $\left(nh_{s_n}^d\right)$ *and rate function defined by (1.3).*

2.2. Pointwise MDP for the density estimator defined by the stochastic approximation algorithm (1.1). Let (v_n) be a positive sequence, we assume that

(**M1**) $K : \mathbb{R}^d \to \mathbb{R}$ is a continuous, bounded function satisfying

$$\int_{\mathbb{R}^d} K(z_1, \ldots, z_d)\, dz_1 \ldots dz_d := \int_{\mathbb{R}^d} K(\mathbf{z})\, d\mathbf{z} = 1,$$

$\int_{\mathbb{R}^d} \mathbf{z} K(\mathbf{z})\, d\mathbf{z} = 0$ and $\int_{\mathbb{R}^d} \|\mathbf{z}\|^2 K(\mathbf{z})\, d\mathbf{z} < \infty$.

(**M2**) (i) $(\gamma_{s_n}) \in \mathcal{GS}(-\alpha)$ with $\alpha \in]1/2, 1]$.
(ii) $(h_{s_n}) \in \mathcal{GS}(-a)$ with $a \in]0, \alpha/d[$.
(iii) $\lim_{n\to\infty} (n\gamma_{s_n}) \in]\min\{2a, (\alpha - ad)/2\}, \infty]$.

(**M3**) (i) $f(\cdot)$ is bounded, twice differentiable, and, for all $i, j \in \{1, \ldots, d\}$, $\partial^2 f(\cdot)/\partial x_i \partial x_j$ is bounded.

(ii) For any $i, j \in \{1, \ldots, n\}$ such that $s_i \neq s_j$, the random vector $\left(X_{s_i}, X_{s_j}\right)$ has a density $f_{s_i,s_j}(\cdot)$ such that

$$\sup_{s_i \neq s_j} \left\| g_{s_i,s_j} \right\| < \infty.$$

(M4) (i) The field $(X_{s_i})_{1 \leq i \leq n}$ is α-mixing.

(ii) $\sum_{k=0}^{\infty} (k+1)^2 \alpha_n^{\frac{\delta}{4+\delta}}(k) < c$ for some c, $\delta > 0$ and all n.

(M5) $\lim_{n\to\infty} v_{s_n} = \infty$ and $\lim_{n\to\infty} \gamma_{s_n} v_{s_n}^2 / h_{s_n}^d = 0$.

The following Theorem gives the pointwise MDP for $f_n(\cdot)$.

Theorem 2.4 (Pointwise MDP for the recursive estimator defined by (1.1))**.** *Let Assumptions* (**M1**)-(**M5**) *hold and assume that $f(\cdot)$ is continuous at $\mathbf{x}$. Then, the sequence $(f_n(\mathbf{x}) - f(\mathbf{x}))$ satisfies a MDP with speed $\left(h_{s_n}^d / \left(\gamma_{s_n} v_{s_n}^2\right)\right)$ and rate function $J_{a,\alpha,\mathbf{x}}$ defined in (1.5).*

3. Application to Testing Symmetry

In this section, we study large and moderate deviations for an important problem of symmetry testing for statistics for spatial data based on the kernel density estimator $f_n(\cdot)$ given in (1.1). More precisely, we investigate the test of symmetry of the density $f(\cdot)$ at a given $\mathbf{x}$ (i.e., to test if $f(-\mathbf{x}) = f(\mathbf{x})$) by using the statistic $|f_n(\mathbf{x}) - f_n(-\mathbf{x})|$. We may refer for more details on testing symmetry to [29], [23], [4], [7] and more recently [12]. Testing symmetry has not been studied in a systematical way until present and the results obtained here are believed to be novel in the spatial data framework. Let us define

$$\begin{aligned} f_n(\mathbf{x}) - f_n(-\mathbf{x}) &= (1-\gamma_{s_n})\left[f_{n-1}(\mathbf{x}) - f_{n-1}(-\mathbf{x})\right] \\ &\quad + \gamma_{s_n} h_{s_n}^{-d} \left[K\left(\frac{\mathbf{x} - X_{s_n}}{h_{s_n}}\right) - K\left(\frac{-\mathbf{x} - X_{s_n}}{h_{s_n}}\right)\right]. \end{aligned} \tag{3.1}$$

Let $\mathbf{J}_{a,\alpha,\mathbf{x}}$ be the rate function defined by

$$\begin{cases} \text{if} \quad f(\mathbf{x}) \neq 0, \quad \mathbf{J}_{a,\alpha,\mathbf{x}} : t \to \dfrac{t^2\left(2 - (\alpha - ad)\,\xi\right)}{4 f(\mathbf{x}) \displaystyle\int_{\mathbb{R}^d} K^2(\mathbf{z})\, d\mathbf{z}} \\ \text{if} \quad f(\mathbf{x}) = 0, \quad \mathbf{J}_{a,\alpha,\mathbf{x}}(0) = 0 \quad \text{and} \quad \mathbf{J}_{a,\alpha,\mathbf{x}}(t) = +\infty \quad \text{for} \quad t \neq 0. \end{cases} \tag{3.2}$$

The following Theorem gives the pointwise LDP for $f_n(\cdot)$ in this case.

Theorem 3.1 (Pointwise LDP for Wolverton and Wagner estimator)**.** *Let Assumptions* (**L1**)-(**L4**) *hold and assume that $f(\cdot)$ is continuous at $\mathbf{x}$. Then, the sequence $(f_n(\mathbf{x}) - f_n(-\mathbf{x}))$ satisfies a LDP with speed $\left(nh_{s_n}^d\right)$ and rate function defined by (1.2).*

The following Theorem gives the pointwise LDP for $f_n(\cdot)$ in this case.

Theorem 3.2 (Pointwise LDP for Deheuvels estimator)**.** *Let the assumptions* (**L1**), (**L2′**)-(**L4**) *be fulfilled and assume that $f(\cdot)$ is continuous at $\mathbf{x}$. Then, the sequence $(f_n(\mathbf{x}) - f_n(-\mathbf{x}))$ satisfies a LDP with speed $\left(nh_{s_n}^d\right)$ and rate function defined by (1.3).*

The following Theorem gives the pointwise MDP for $f_n(\cdot)$.

Theorem 3.3 (Pointwise MDP for a test of symmetry based on recursive kernel estimators of a probability density function for spatial data defined by (3.1))**.** *Let Assumptions* (**M1**)-(**M5**) *hold and assume that* $f(\cdot)$ *is continuous at* $\mathbf{x}$*. Then, the sequence* $(f_n(\mathbf{x}) - f_n(-\mathbf{x}))$ *satisfies a MDP with speed* $\left(h_{s_n}^d / \left(\gamma_{s_n} v_{s_n}^2\right)\right)$ *and rate function* $\mathbf{J}_{a,\alpha,\mathbf{x}}$ *defined in (3.2).*

Large deviations results are useful and efficient tools to study the asymptotic efficiency of tests, in particular to obtain the Bahadur exact slope for comparison of statistics. This problem has been deeply investigated; we refer to [3] and the book of [28] for an accessible introduction to this topic. We will not investigate the efficiency problem in the present paper. We plan to make an extension of the current paper by considering other symmetries context.

4. Discussions

The advantage of recursive estimators on their nonrecursive version is that their update, from a sample of size n to one of size $n+1$, require less computations. This property can be generalized, one can check that it follows from (1.1) that for all $n_1 \in [0, n-1]$,

$$\begin{aligned} f_n(\mathbf{x}) &= \prod_{j=n_1+1}^{n} \left(1-\gamma_{s_j}\right) f_{n_1}(\mathbf{x}) \\ &\quad + \sum_{k=n_1}^{n-1} \left[\prod_{j=k+1}^{n} \left(1-\gamma_{s_j}\right)\right] \frac{\gamma_{s_k}}{h_{s_k}^d} K\left(\frac{\mathbf{x}-X_{s_k}}{h_{s_k}}\right) + \frac{\gamma_{s_n}}{h_{s_n}^d} K\left(\frac{\mathbf{x}-X_{s_n}}{h_{s_n}}\right) \\ &= \alpha_1 f_{n_1}(\mathbf{x}) + \sum_{k=n_1}^{n-1} \beta_k \frac{\gamma_{s_k}}{h_{s_k}^d} K\left(\frac{\mathbf{x}-X_{s_k}}{h_k}\right) + \frac{\gamma_{s_n}}{h_{s_n}^d} K\left(\frac{\mathbf{x}-X_{s_n}}{h_{s_n}}\right), \end{aligned}$$

where

$$\alpha_1 = \prod_{j=n_1+1}^{n} \left(1-\gamma_{s_j}\right) \text{ and } \beta_k = \prod_{j=k+1}^{n} \left(1-\gamma_{s_j}\right).$$

We suppose that we receive a first sample of size $n_1 = \lfloor n/2 \rfloor$ (the lower integer part of $n/2$) an then, we suppose that we receive an additional sample of size $n - n_1$. It is clear, that we can use a data-driven bandwidth to construct an optimal bandwidth based on the first sample of size n_1 and separately an optimal bandwidth based on the second sample of size $n - n_1$, and then the proposed estimator can be viewed as a linear combination of two estimators, which improve considerably the computational cost.

5. Proofs

This section is devoted to the detailed proofs of our results. The previously displayed notation continue to be used in the sequel. Through this section we use the following notation:

$$\Pi_n = \prod_{j=1}^{n}\left(1-\gamma_{s_j}\right),$$
$$Z_{s_n}(\mathbf{x}) = h_{s_n}^{-d}Y_{s_n},$$
$$Y_{s_n} = K\left(\frac{\mathbf{x}-X_{s_n}}{h_{s_n}}\right), \quad (5.1)$$
$$T_{s_n} = Y_{s_n}(\mathbf{x}) - Y_{s_n}(-\mathbf{x}). \quad (5.2)$$

In order to prove the results we require the following technical lemma. The proof of this lemma is quite similar to the proof of Lemma 2 of [25].

Lemma 5.1. *Let* $(v_n) \in \mathcal{GS}(v^*)$, $(\gamma_{s_n}) \in \mathcal{GS}(-\alpha)$, *and* $m > 0$ *such that* $m - v^*\xi > 0$ *where* ξ *is defined in (1.4). We have*

$$\lim_{n\to+\infty} v_n\Pi_n^m \sum_{k=1}^{n}\Pi_k^{-m}\frac{\gamma_{s_k}}{v_k} = \frac{1}{m-v^*\xi}.$$

Moreover, for all positive sequence (α_{s_n}) *such that* $\lim_{n\to+\infty}\alpha_{s_n}=0$, *and for all* $\delta\in\mathbb{R}$,

$$\lim_{n\to+\infty} v_n\Pi_n^m\left[\sum_{k=1}^{n}\Pi_k^{-m}\frac{\gamma_{s_k}}{v_k}\alpha_{s_k}+\delta\right]=0.$$

First, it follows from (1.1), that

$$\begin{aligned}
&f_n(\mathbf{x}) - f(\mathbf{x})\\
&= \left(1-\gamma_{s_n}\right)\left(f_{n-1}(\mathbf{x})-f(\mathbf{x})\right)+\gamma_{s_n}\left(Z_n(\mathbf{x})-f(\mathbf{x})\right)\\
&= \sum_{k=1}^{n-1}\left[\prod_{j=k+1}^{n}\left(1-\gamma_{s_j}\right)\right]\gamma_{s_k}\left(Z_{s_k}(\mathbf{x})-f(\mathbf{x})\right)\\
&\quad+\gamma_{s_n}\left(Z_{s_n}(\mathbf{x})-f(\mathbf{x})\right)\\
&\quad+\left[\prod_{j=1}^{n}\left(1-\gamma_{s_j}\right)\right]\left(f_0(\mathbf{x})-f(\mathbf{x})\right)\\
&= \Pi_n\sum_{k=1}^{n}\Pi_k^{-1}\gamma_{s_k}\left(Z_{s_k}(\mathbf{x})-f(\mathbf{x})\right)+\Pi_n\left(f_0(\mathbf{x})-f(\mathbf{x})\right).
\end{aligned}$$

Then, we readily infer that

$$\begin{aligned}
&\mathbb{E}\left[f_n(\mathbf{x})\right]-f(\mathbf{x})\\
&= \Pi_n\sum_{k=1}^{n}\Pi_k^{-1}\gamma_{s_k}\left(\mathbb{E}\left[Z_{s_k}(\mathbf{x})\right]-f(\mathbf{x})\right)+\Pi_n\left(f_0(\mathbf{x})-f(\mathbf{x})\right).
\end{aligned}$$

Hence, it follows that

$$\begin{aligned}
f_n(\mathbf{x})-\mathbb{E}\left[f_n(\mathbf{x})\right] &= \Pi_n\sum_{k=1}^{n}\Pi_k^{-1}\gamma_{s_k}\left(Z_{s_k}(\mathbf{x})-\mathbb{E}\left[Z_{s_k}(\mathbf{x})\right]\right)\\
&= \Pi_n\sum_{k=1}^{n}\Pi_k^{-1}\gamma_{s_k}h_{s_k}^{-d}\left(Y_{s_k}-\mathbb{E}\left[Y_{s_k}\right]\right).
\end{aligned}$$

Now, we let (Ψ_n) and (B_n) be the sequences defined as follows:

$$\begin{aligned}\Psi_n(\mathbf{x}) &= \Pi_n \sum_{k=1}^{n} \Pi_k^{-1} \gamma_{s_k} h_{s_k}^{-d} \left(Y_{s_k} - \mathbb{E}\left[Y_{s_k}\right]\right)\\ B_n(\mathbf{x}) &= \mathbb{E}\left[f_n(\mathbf{x})\right] - f(\mathbf{x}).\end{aligned}$$

It is easy to see that

$$f_n(\mathbf{x}) - f(\mathbf{x}) = \Psi_n(\mathbf{x}) + B_n(\mathbf{x}). \tag{5.3}$$

We then deduce that, Theorems 2.2, 2.3 and 2.4 are consequences of (5.3) and the pointwise LDP and MDP for (Ψ_n), which is given in the following propositions.

Proposition 5.2 (Pointwise LDP and MDP for (Ψ_n))**.** *(1) Under the assumptions* (**L1**)-(**L4**)*, the sequence* $(f_n(\mathbf{x}) - \mathbb{E}(f_n(\mathbf{x})))$ *satisfies a LDP with speed* $\left(nh_{s_n}^d\right)$ *and rate function* $I_{a,\mathbf{x}}$*.*

(2) Under the assumptions (**L1**), (**L2′**)-(**L4**), *the sequence* $(f_n(\mathbf{x}) - \mathbb{E}(f_n(\mathbf{x})))$ *satisfies a LDP with speed* $\left(nh_{s_n}^d\right)$ *and rate function* $I_{\mathbf{x}}$*.*

(3) Under the assumptions (**M1**)-(**M5**)*, the sequence* $(v_n \Psi_n(\mathbf{x}))$ *satisfies a LDP with speed* $\left(h_{s_n}^d / \left(\gamma_{s_n} v_n^2\right)\right)$ *and rate function* $J_{a,\alpha,\mathbf{x}}$*.*

The proof of the following proposition is given in [9].

Proposition 5.3 (Pointwise convergence rate of (B_n))**.** *Let the assumptions* (**M1**)-(**M3**) *be satisfied. We assume that, for all* $i, j \in \{1, \ldots, d\}$, $\partial^2 f(\cdot)/\partial x_i \partial x_j$ *is continuous at* $\mathbf{x}$*. Then, we have*

(1) If $a \leq \alpha/(d+4)$,

$$B_n(\mathbf{x}) = O\left(h_{s_n}^2\right).$$

(2) If $a > \alpha/(d+4)$,

$$B_n(\mathbf{x}) = o\left(\sqrt{\gamma_{s_n} h_{s_n}^{-d}}\right).$$

Set $\mathbf{x} \in \mathbb{R}^d$; since the assumptions of Theorems 2.2 and 2.3 gives that

$$\lim_{n \to \infty} B_n(\mathbf{x}) = 0,$$

Theorem 2.2 (respectively Theorem 2.3) is a consequence of the application of the first Part (respectively of the second Part) of Proposition 5.2. Moreover, under the assumptions of Theorem 2.4, the application of Proposition 5.3,

$$\lim_{n \to \infty} v_n B_n(\mathbf{x}) = 0;$$

Theorem 2.4 thus follows from the application of third Part of Proposition 5.2. Let us now state a preliminary lemma, which is the key of the proof of Proposition 5.2. For any $u \in \mathbb{R}$, Set

$$\begin{aligned}\Lambda_{n,\mathbf{x}}(u) &= \frac{\gamma_{s_n} v_n^2}{h_{s_n}^d} \log \mathbb{E}\left[\exp\left(u \frac{h_{s_n}^d}{\gamma_{s_n} v_{s_n}} \Psi_n(\mathbf{x})\right)\right],\\ \Lambda_{\mathbf{x}}^{L,1}(u) &= f(\mathbf{x})\left(\psi_a(u) - u\right),\\ \Lambda_{\mathbf{x}}^{L,2}(u) &= f(\mathbf{x})\left(\psi(u) - u\right),\\ \Lambda_{\mathbf{x}}^{M}(u) &= \frac{u^2}{2\left(2 - (\alpha - ad)\xi\right)} f(\mathbf{x}) \int_{\mathbb{R}^d} K^2(\mathbf{z})\, d\mathbf{z}.\end{aligned}$$

Lemma 5.4. *[Convergence of $\Lambda_{n,\mathbf{x}}$] If $f(\cdot)$ is continuous at $\mathbf{x}$, then for all $u \in \mathbb{R}$*

$$\lim_{n\to\infty} \Lambda_{n,\mathbf{x}}(u) = \Lambda_{\mathbf{x}}(u), \tag{5.4}$$

where

$$\Lambda_{\mathbf{x}}(u) = \begin{cases} \Lambda_{\mathbf{x}}^{L,1}(u) & \text{when} \quad v_n \equiv 1, & (\mathbf{L1})\text{-}(\mathbf{L4}) \text{ hold,} \\ \Lambda_{\mathbf{x}}^{L,2}(u) & \text{when} \quad v_n \equiv 1, & (\mathbf{L1}), (\mathbf{L2'})\text{-}(\mathbf{L4}) \text{ hold,} \\ \Lambda_{\mathbf{x}}^{M}(u) & \text{when} \quad v_n \to \infty, & (\mathbf{M1})\text{-}(\mathbf{M5}) \text{ hold.} \end{cases}$$

Our proofs are now organized as follows. We first proof Lemma 5.4 and after give the proof of Proposition 5.2.

Proof of Lemma 5.4. Set $u \in \mathbb{R}$, $u_n = u/v_n$ and $a_{s_n} = h_{s_n}^d \gamma_{s_n}^{-1}$. We have:

$$\begin{aligned} \Lambda_{n,\mathbf{x}}(u) &= \frac{v_n^2}{a_{s_n}} \log \mathbb{E}\left[\exp\left(u_n a_{s_n} \Psi_n(\mathbf{x})\right)\right] \\ &= \frac{v_n^2}{a_{s_n}} \log \mathbb{E}\left[\exp\left(u_n a_{s_n} \Pi_n \sum_{k=1}^n \Pi_k^{-1} a_{s_k}^{-1}\left(Y_{s_k} - \mathbb{E}\left[Y_{s_k}\right]\right)\right)\right] \\ &= \frac{v_n^2}{a_{s_n}} \sum_{k=1}^n \log \mathbb{E}\left[\exp\left(u_n \frac{a_{s_n}\Pi_n}{a_{s_k}\Pi_k} Y_{s_k}\right)\right] \\ &\quad - u v_n \Pi_n \sum_{k=1}^n \Pi_k^{-1} a_{s_k}^{-1} \mathbb{E}\left[Y_{s_k}\right]. \end{aligned}$$

By Taylor expansion, there exists $c_{k,n}$ between 1 and $\mathbb{E}\left[\exp\left(u_n \frac{a_{s_n}\Pi_n}{a_{s_k}\Pi_k} Y_{s_k}\right)\right]$ such that

$$\begin{aligned} &\log \mathbb{E}\left[\exp\left(u_n \frac{a_{s_n}\Pi_n}{a_{s_k}\Pi_{s_k}} Y_{s_k}\right)\right] \\ &\quad = \mathbb{E}\left[\exp\left(u_n \frac{a_{s_n}\Pi_n}{a_{s_k}\Pi_k} Y_{s_k}\right) - 1\right] \\ &\qquad - \frac{1}{2c_{k,n}^2}\left(\mathbb{E}\left[\exp\left(u_n \frac{a_{s_n}\Pi_n}{a_{s_k}\Pi_k} Y_{s_k}\right) - 1\right]\right)^2, \end{aligned}$$

and $\Lambda_{n,\mathbf{x}}$ can be rewritten as

$$\begin{aligned} \Lambda_{n,\mathbf{x}}(u) &= \frac{v_n^2}{a_{s_n}} \sum_{k=1}^n \mathbb{E}\left[\exp\left(u_n \frac{a_{s_n}\Pi_n}{a_{s_k}\Pi_k} Y_{s_k}\right) - 1\right] \\ &\quad - \frac{v_n^2}{2a_{s_n}} \sum_{k=1}^n \frac{1}{c_{k,n}^2}\left(\mathbb{E}\left[\exp\left(u_n \frac{a_{s_n}\Pi_n}{a_{s_k}\Pi_k} Y_{s_k}\right) - 1\right]\right)^2 \\ &\quad - u v_n \Pi_n \sum_{k=1}^n \Pi_k^{-1} a_{s_k}^{-1} \mathbb{E}\left[Y_{s_k}\right]. \end{aligned} \tag{5.5}$$

First case: $v_n \to \infty$. A Taylor's expansion implies the existence of $c'_{k,n}$ between 0 and $u_n \frac{a_{s_n}\Pi_n}{a_{s_k}\Pi_k} Y_{s_k}$ such that

$$\begin{aligned} &\mathbb{E}\left[\exp\left(u_n \frac{a_{s_n}\Pi_n}{a_{s_k}\Pi_k} Y_{s_k}\right) - 1\right] \\ &= u_n \frac{a_{s_n}\Pi_n}{a_{s_k}\Pi_k}\mathbb{E}\left[Y_{s_k}\right] + \frac{1}{2}\left(u_n \frac{a_{s_n}\Pi_n}{a_{s_k}\Pi_k}\right)^2 \mathbb{E}\left[Y_{s_k}^2\right] \\ &\quad + \frac{1}{6}\left(u_n \frac{a_{s_n}\Pi_n}{a_{s_k}\Pi_k}\right)^3 \mathbb{E}\left[Y_{s_k}^3 e^{c'_{k,n}}\right]. \end{aligned}$$

Therefore, we obtain

$$\begin{aligned} &\Lambda_{n,\mathbf{x}}(u) \\ &= \frac{1}{2}u^2 a_{s_n}\Pi_n^2 \sum_{k=1}^n \Pi_k^{-2} a_{s_k}^{-2}\mathbb{E}\left[Y_{s_k}^2\right] \\ &\quad + \frac{1}{6}u^2 u_n a_{s_n}^2 \Pi_n^3 \sum_{k=1}^n \Pi_k^{-3} a_{s_k}^{-3}\mathbb{E}\left[Y_{s_k}^3 e^{c'_{k,n}}\right] \\ &\quad - \frac{v_n^2}{2a_{s_n}} \sum_{k=1}^n \frac{1}{c_{k,n}^2}\left(\mathbb{E}\left[\exp\left(u_n \frac{a_{s_n}\Pi_n}{a_{s_k}\Pi_k} Y_{s_k}\right) - 1\right]\right)^2 \\ &= \frac{1}{2}f(\mathbf{x})u^2 a_{s_n}\Pi_n^2 \sum_{k=1}^n \Pi_k^{-2} a_{s_k}^{-1}\gamma_{s_k}\int_{\mathbb{R}^d} K^2(\mathbf{z})\, d\mathbf{z} + R_{n,\mathbf{x}}^{(1)}(u) \\ &\quad + R_{n,\mathbf{x}}^{(2)}(u), \end{aligned} \tag{5.6}$$

with

$$\begin{aligned} R_{n,\mathbf{x}}^{(1)}(u) &= \frac{1}{2}u^2 a_{s_n}\Pi_n^2 \sum_{k=1}^n \Pi_k^{-2} a_{s_k}^{-1}\gamma_{s_k}\int_{\mathbb{R}^d} K^2(\mathbf{z})\left[f(\mathbf{x}-\mathbf{z}h_{s_k}) - f(\mathbf{x})\right] d\mathbf{z} \\ R_{n,\mathbf{x}}^{(2)}(u) &= \frac{1}{6}\frac{u^3}{v_n} a_{s_n}^2 \Pi_n^3 \sum_{k=1}^n \Pi_k^{-3} a_{s_k}^{-3}\mathbb{E}\left[Y_{s_k}^3 e^{c'_{k,n}}\right] \\ &\quad - \frac{v_n^2}{2a_{s_n}} \sum_{k=1}^n \frac{1}{c_{k,n}^2}\left(\mathbb{E}\left[\exp\left(u_n \frac{a_{s_n}\Pi_n}{a_{s_k}\Pi_k} Y_{s_k}\right) - 1\right]\right)^2. \end{aligned}$$

Since $f(\cdot)$ is continuous, we have

$$\lim_{k\to\infty} \left|f(\mathbf{x}-\mathbf{z}h_{s_k}) - f(\mathbf{x})\right| = 0.$$

Thus a straightforward application of Lebesgue dominated convergence theorem in connection with condition (**M1**) implies that

$$\lim_{k\to\infty} \int_{\mathbb{R}^d} K^2(\mathbf{z}) \left|f(\mathbf{x}-\mathbf{z}h_{s_k}) - f(\mathbf{x})\right| d\mathbf{z} = 0.$$

Since $(a_{s_n}) \in \mathcal{GS}(\alpha - ad)$, and

$$\lim_{n\to\infty} (n\gamma_{s_n}) > (\alpha - ad)/2.$$

By an application of Lemma 5.1 we can therefore write

$$a_{s_n}\Pi_n^2\sum_{k=1}^{n}\Pi_k^{-2}a_{s_k}^{-1}\gamma_{s_k}=\frac{1}{\left(2-\left(\alpha-ad\right)\xi\right)}+o\left(1\right). \tag{5.7}$$

From this we see that

$$\lim_{n\to\infty}\left|R_{n,\mathbf{x}}^{(1)}\left(u\right)\right|=0.$$

Moreover, in view of (5.1), we have $|Y_{s_k}|\leq\|K\|_\infty$, then

$$\begin{aligned} c'_{k,n} &\leq \left|u_n\frac{a_{s_n}\Pi_n}{a_{s_k}\Pi_k}Y_{s_k}\right| \\ &\leq \left|u_n\right|\left\|K\right\|_\infty. \end{aligned} \tag{5.8}$$

Since, we have

$$\mathbb{E}\left|Y_{s_k}\right|^3\leq h_{s_k}^d\left\|f\right\|_\infty\int_{\mathbb{R}^d}\left|K^3\left(\mathbf{z}\right)\right|d\mathbf{z}.$$

It follows from, Lemma 5.1 and (5.8), that, there exists a positive constant c_1 such that, for n large enough,

$$\begin{aligned} &\left|\frac{u^3}{v_n}a_{s_n}^2\Pi_n^3\sum_{k=1}^{n}\Pi_k^{-3}a_{s_k}^{-3}\mathbb{E}\left[Y_{s_k}^3e^{c'_{k,n}}\right]\right| \\ &\qquad\leq c_1e^{|u_n|\|K\|_\infty}\frac{u^3}{v_n}\left\|f\right\|_\infty\int_{\mathbb{R}^d}\left|K^3\left(\mathbf{z}\right)\right|d\mathbf{z}, \end{aligned} \tag{5.9}$$

which goes to 0 as $n\to\infty$. An application of Lemma 5.1 ensures that

$$\begin{aligned} &\left|\frac{v_n^2}{2a_{s_n}}\sum_{k=1}^{n}\frac{1}{c_{k,n}^2}\left(\mathbb{E}\left[\exp\left(u_n\frac{a_{s_n}\Pi_n}{a_{s_k}\Pi_k}Y_{s_k}\right)-1\right]\right)^2\right| \\ &\leq\frac{v_n^2}{2a_{s_n}}\sum_{k=1}^{n}\left(\mathbb{E}\left[\exp\left(u_n\frac{a_{s_n}\Pi_n}{a_{s_k}\Pi_k}Y_{s_k}\right)-1\right]\right)^2 \\ &=\frac{u^2}{2}\left\|f\right\|_\infty^2a_{s_n}\Pi_n^2\sum_{k=1}^{n}\Pi_k^{-2}a_{s_k}^{-1}\gamma_{s_k}h_{s_k}^d \\ &\quad+o\left(a_{s_n}\Pi_n^2\sum_{k=1}^{n}\Pi_k^{-2}a_{s_k}^{-1}\gamma_{s_k}h_{s_k}^d\right) \\ &=o\left(1\right). \end{aligned} \tag{5.10}$$

The combination of (5.9) and (5.10) ensures that

$$\lim_{n\to\infty}\left|R_{n,\mathbf{x}}^{(2)}\left(u\right)\right|=0.$$

Then, it follows from (5.6) and (5.7), that

$$\lim_{n\to\infty}\Lambda_{n,\mathbf{x}}\left(u\right)=\Lambda_{\mathbf{x}}^{M}\left(u\right).$$

Second case: $(v_n) \equiv 1$. We obtain from (5.5) that

$$\begin{aligned}
&\Lambda_{n,\mathbf{x}}(u) \\
&= \frac{1}{a_{s_n}}\sum_{k=1}^{n}\mathbb{E}\left[\exp\left(u\frac{a_{s_n}\Pi_n}{a_{s_k}\Pi_k}Y_{s_k}\right)-1\right] \\
&\quad -\frac{1}{2a_{s_n}}\sum_{k=1}^{n}\frac{1}{c_{k,n}^2}\left(\mathbb{E}\left[\exp\left(u\frac{a_{s_n}\Pi_n}{a_{s_k}\Pi_k}Y_{s_k}\right)-1\right]\right)^2 \\
&\quad -u\Pi_n\sum_{k=1}^{n}\Pi_k^{-1}a_{s_k}^{-1}\mathbb{E}\left[Y_{s_k}\right] \\
&= \frac{1}{a_{s_n}}\sum_{k=1}^{n}h_{s_k}^d\int_{\mathbb{R}^d}\left[\exp\left(u\frac{a_{s_n}\Pi_n}{a_{s_k}\Pi_k}K(\mathbf{z})\right)-1\right]f(\mathbf{x})d\mathbf{z} \\
&\quad -u\Pi_n\sum_{k=1}^{n}\Pi_k^{-1}\gamma_{s_k}\int_{\mathbb{R}^d}K(\mathbf{z})f(\mathbf{x})d\mathbf{z} \\
&\quad -R_{n,\mathbf{x}}^{(3)}(u)+R_{n,\mathbf{x}}^{(4)}(u) \\
&= f(\mathbf{x})\frac{1}{a_{s_n}}\sum_{k=1}^{n}h_{s_k}^d\left[\int_{\mathbb{R}^d}\left(\exp\left(uV_{n,k}K(\mathbf{z})\right)-1\right)-uV_{n,k}K(\mathbf{z})\right]d\mathbf{z} \\
&\quad -R_{n,\mathbf{x}}^{(3)}(u)+R_{n,\mathbf{x}}^{(4)}(u), \qquad (5.11)
\end{aligned}$$

with

$$\begin{aligned}
V_{n,k} &= \frac{a_{s_n}\Pi_n}{a_{s_k}\Pi_k} \\
R_{n,\mathbf{x}}^{(3)}(u) &= \frac{1}{2a_{s_n}}\sum_{k=1}^{n}\frac{1}{c_{k,n}^2}\left(\mathbb{E}\left[\exp\left(u\frac{a_{s_n}\Pi_n}{a_{s_k}\Pi_k}Y_{s_k}\right)-1\right]\right)^2 \\
R_{n,\mathbf{x}}^{(4)}(u) &= \frac{1}{a_{s_n}}\sum_{k=1}^{n}h_{s_k}^d\int_{\mathbb{R}^d}\left[\exp\left(u\frac{a_{s_n}\Pi_n}{a_{s_k}\Pi_k}K(\mathbf{z})\right)-1\right] \\
&\quad\times\left[f\left(\mathbf{x}-\mathbf{z}h_{s_k}\right)-f(\mathbf{x})\right]d\mathbf{z} \\
&\quad -u\Pi_n\sum_{k=1}^{n}\Pi_k^{-1}\gamma_{s_k}\int_{\mathbb{R}^d}K(\mathbf{z})\left[f\left(\mathbf{x}-\mathbf{z}h_{s_k}\right)-f(\mathbf{x})\right]d\mathbf{z}.
\end{aligned}$$

Moreover, it follows from (5.10), that

$$\lim_{n\to\infty}\left|R_{n,\mathbf{x}}^{(3)}(u)\right|=0.$$

Now, since

$$\left|e^t-1\right|\leq|t|\,e^{|t|},$$

one can see that

$$\begin{aligned}
&\left|R_{n,\mathbf{x}}^{(4)}(u)\right| \\
&\leq \frac{1}{a_{s_n}}\sum_{k=1}^{n}h_{s_k}^{d}\int_{\mathbb{R}^d}\left|\left[\exp\left(u\frac{a_{s_n}\Pi_n}{a_{s_k}\Pi_k}K(\mathbf{z})\right)-1\right]\left[f(\mathbf{x}-\mathbf{z}h_{s_k})-f(\mathbf{x})\right]\right|d\mathbf{z} \\
&\quad +|u|\,\Pi_n\sum_{k=1}^{n}\Pi_k^{-1}\gamma_{s_k}\int_{\mathbb{R}^d}|K(\mathbf{z})|\,|f(\mathbf{x}-\mathbf{z}h_{s_k})-f(\mathbf{x})|\,d\mathbf{z} \\
&\leq |u|\,e^{|u|\|K\|_\infty}\Pi_n\sum_{k=1}^{n}\Pi_k^{-1}\gamma_{s_k}\int_{\mathbb{R}^d}|K(\mathbf{z})|\,|f(\mathbf{x}-\mathbf{z}h_{s_k})-f(\mathbf{x})|\,d\mathbf{z} \\
&\quad +|u|\,\Pi_n\sum_{k=1}^{n}\Pi_k^{-1}\gamma_{s_k}\int_{\mathbb{R}^d}|K(\mathbf{z})|\,|f(\mathbf{x}-\mathbf{z}h_{s_k})-f(\mathbf{x})|\,d\mathbf{z} \\
&\leq |u|\left(e^{|u|\|K\|_\infty}+1\right)\Pi_n\sum_{k=1}^{n}\Pi_k^{-1}\gamma_{s_k}\int_{\mathbb{R}^d}|K(\mathbf{z})|\,|f(\mathbf{x}-\mathbf{z}h_{s_k})-f(\mathbf{x})|\,d\mathbf{z}.
\end{aligned}$$

Since, Lemma 5.1 ensures that, the sequence $\left(\Pi_n\sum_{k=1}^{n}\Pi_k^{-1}\gamma_{s_k}\right)$ is bounded, then, the dominated convergence theorem ensures that

$$\lim_{n\to\infty}R_{n,\mathbf{x}}^{(4)}(u)=0.$$

Then, it follows from (5.11), that

$$\begin{aligned}
\lim_{n\to\infty}\Lambda_{n,\mathbf{x}}(u) &= \lim_{n\to\infty}f(\mathbf{x})\frac{\gamma_{s_n}}{h_{s_n}^{d}}\sum_{k=1}^{n}h_{s_k}^{d} \\
&\quad \int_{\mathbb{R}^d}\left[\left(\exp\left(uV_{n,k}K(\mathbf{z})\right)-1\right)-uV_{n,k}K(\mathbf{z})\right]d\mathbf{z}. \qquad (5.12)
\end{aligned}$$

In the case when $(v_n)\equiv 1$, (**L1**)-(**L4**) hold.
Noting that

$$\frac{\Pi_n}{\Pi_k}=\prod_{j=k+1}^{n}\left(1-\gamma_{s_j}\right)=\frac{k}{n},$$

we then see that

$$V_{n,k}=\frac{a_{s_n}\Pi_n}{a_{s_k}\Pi_k}=\left(\frac{k}{n}\right)^{ad}.$$

Consequently, it follows from (5.12) and from some calculus that

$$\begin{aligned}
&\lim_{n\to\infty}\Lambda_{n,\mathbf{x}}(u) \\
&= f(\mathbf{x})\int_{\mathbb{R}^d}\left[\int_0^1 t^{-ad}\left(\exp\left(ut^{ad}K(\mathbf{z})\right)-1\right)-ut^{ad}K(\mathbf{z})\right]dtd\mathbf{z} \\
&= \Lambda_{\mathbf{x}}^{L,1}(u).
\end{aligned}$$

In the case when $(v_n) \equiv 1$, (**L1**), $(L2')$-(**L4**) hold.
We have

$$\begin{aligned}
\frac{\Pi_n}{\Pi_k} &= \prod_{j=k+1}^{n}\left(1-\gamma_{s_j}\right) = \prod_{j=k+1}^{n}\left(1-\frac{h_{s_j}^d}{\sum_{l=1}^{j} h_{s_l}^d}\right) \\
&= \prod_{j=k+1}^{n}\left(\frac{\sum_{l=1}^{j-1} h_{s_l}^d}{\sum_{l=1}^{j} h_{s_l}^d}\right) = \frac{\sum_{l=1}^{k} h_{s_l}^d}{\sum_{l=1}^{n} h_{s_l}^d} \\
&= \frac{\sum_{l=1}^{k} h_{s_l}^d}{h_{s_k}^d}\frac{h_{s_k}^d}{h_{s_n}^d}\frac{h_{s_n}^d}{\sum_{l=1}^{n} h_{s_l}^d} = \frac{\gamma_{s_n}}{\gamma_{s_k}}\frac{h_{s_k}^d}{h_{s_n}^d}.
\end{aligned}$$

From this, we infer that

$$V_{n,k} = 1.$$

Consequently, it follows from (5.23) that

$$\begin{aligned}
\lim_{n\to\infty} \Lambda_{n,\mathbf{x}}(u) &= f(\mathbf{x})\int_{\mathbb{R}^d}\left[\left(\exp\left(uK(\mathbf{z})\right)-1\right)-uK(\mathbf{z})\right]d\mathbf{z}. \\
&= \Lambda_{\mathbf{x}}^{L,2}(u).
\end{aligned}$$

Therefore Lemma 5.4 is proved. □

Proof of Proposition 5.2. To prove Proposition 5.2, we apply similar result as the one given by Proposition 1 in [26] in the non spatial case, Lemma 5.4 and the following result (see [31]).

Lemma 5.5. *Let (Z_n) be a sequence of real random variables, (ν_n) a positive sequence satisfying*

$$\lim_{n\to\infty} \nu_n = +\infty,$$

and suppose that there exists some convex non-negative function Γ defined on $\mathbb{R}$ such that

$$\forall u \in \mathbb{R},\ \lim_{n\to\infty}\frac{1}{\nu_n}\log\mathbb{E}\left[\exp\left(u\nu_n Z_n\right)\right] = \Gamma(u).$$

If the Legendre function Γ^* of Γ is a strictly convex function, then the sequence (Z_n) satisfies a LDP of speed (ν_n) and good rate fonction Γ^*.

In our framework, when $v_n \equiv 1$ and $\gamma_{s_n} = n^{-1}$, we take

$$Z_n = f_n(\mathbf{x}) - \mathbb{E}\left(f_n(\mathbf{x})\right),\quad \nu_n = nh_{s_n}^d,$$

with $h_{s_n} = cn^{-a}$, where $a \in \left]0, 1/d\right[$ and

$$\Gamma = \Lambda_{\mathbf{x}}^{L,1}.$$

In this case, the Legendre transform of $\Gamma = \Lambda_{\mathbf{x}}^{L,1}$ is the rate function

$$I_{a,\mathbf{x}} : t \to f(\mathbf{x}) I_a \left(\frac{t}{f(\mathbf{x})} + 1 \right),$$

which is strictly convex by Proposition 1 in [26]. Furthermore, when $v_n \equiv 1$ and

$$\gamma_{s_n} = h_{s_n}^d \left(\sum_{k=1}^n h_{s_k}^d \right)^{-1},$$

we take

$$Z_n = f_n(\mathbf{x}) - \mathbb{E}\left(f_n(\mathbf{x})\right), \nu_n = n h_{s_n}^d,$$

with $h_{s_n} \in \mathcal{GS}(-a)$ where $a \in]0, 1/d[$ and

$$\Gamma = \Lambda_{\mathbf{x}}^{L,2}.$$

In this case, the Legendre transform of $\Gamma = \Lambda_{\mathbf{x}}^{L,2}$ is the rate function

$$I_{\mathbf{x}} : t \to f(\mathbf{x}) I \left(\frac{t}{f(\mathbf{x})} + 1 \right),$$

which is strictly convex by Proposition 1 in [27]. Otherwise, when, $v_n \to \infty$, we take

$$Z_n = v_n \left(f_n(\mathbf{x}) - \mathbb{E}\left(f_n(\mathbf{x})\right)\right), \nu_n = h_{s_n}^d / \left(\gamma_{s_n} v_n^2\right)$$

and

$$\Gamma = \Lambda_{\mathbf{x}}^M,$$

Γ^* is then the quadratic rate function $J_{a,\alpha,\mathbf{x}}$ defined in (1.5) and thus Proposition 5.2 follows. □

Proofs of the results concerning the symmetry test. In the proofs of this part, we will use similar arguments as those in the proofs of the preceding theorems that we include with all details in odder to make our presentation more self-contained. By using similar arguments to obtain (5.3), one can see that we have

$$\begin{aligned} & f_n(\mathbf{x}) - f_n(-\mathbf{x}) - \mathbb{E}\left[f_n(\mathbf{x}) - f_n(-\mathbf{x})\right] \\ &= \Pi_n \sum_{k=1}^n \Pi_k^{-1} \gamma_{s_k} h_{s_k}^{-d} \left(T_{s_k} - \mathbb{E}\left[T_{s_k}\right]\right). \end{aligned}$$

Now, we let $\left(\widetilde{\Psi}_n\right)$ and $\left(\widetilde{B}_n\right)$ be the sequences defined as follows:

$$\begin{aligned} \widetilde{\Psi}_n(\mathbf{x}) &= \Pi_n \sum_{k=1}^n \Pi_k^{-1} \gamma_{s_k} h_{s_k}^{-d} \left(T_{s_k} - \mathbb{E}\left[T_{s_k}\right]\right) \\ \widetilde{B}_n(\mathbf{x}) &= \mathbb{E}\left[f_n(\mathbf{x})\right] - \mathbb{E}\left[f_n(-\mathbf{x})\right]. \end{aligned}$$

It is easy to see that

$$f_n(\mathbf{x}) - f_n(-\mathbf{x}) = \widetilde{\Psi}_n(\mathbf{x}) + \widetilde{B}_n(\mathbf{x}). \tag{5.13}$$

We then deduce that, Theorems 3.1, 3.2 and 3.3 are consequences of (5.13) and the pointwise LDP and MDP for $\left(\widetilde{\Psi}_n\right)$, which is given in the following propositions.

Proposition 5.6 (Pointwise LDP and MDP for $\left(\widetilde{\Psi}_n\right)$). *(1) Under the assumptions* (**L1**)-(**L4**)*, the sequence*

$$\left(f_n(\mathbf{x}) - f_n(-\mathbf{x}) - \mathbb{E}\left[f_n(\mathbf{x}) - f_n(-\mathbf{x})\right]\right)$$

satisfies a LDP with speed $\left(nh_{s_n}^d\right)$ *and rate function* $I_{a,\mathbf{x}}$.

(2) Under the assumptions (**L1**), (**L2′**)-(**L4**)*, the sequence*

$$\left(f_n(\mathbf{x}) - f_n(-\mathbf{x}) - \mathbb{E}\left[f_n(\mathbf{x}) - f_n(-\mathbf{x})\right]\right)$$

satisfies a LDP with speed $\left(nh_{s_n}^d\right)$ *and rate function* $I_{\mathbf{x}}$.

(3) Under the assumptions (**M1**)-(**M5**)*, the sequence*

$$\left(v_n\widetilde{\Psi}_n(\mathbf{x})\right)$$

satisfies a LDP with speed $\left(h_{s_n}^d / \left(\gamma_{s_n} v_n^2\right)\right)$ *and rate function* $J_{a,\alpha,\mathbf{x}}$.

Set $\mathbf{x} \in \mathbb{R}^d$; since the assumptions of Theorems 3.1 and 3.2 gives that

$$\lim_{n\to\infty} \widetilde{B}_n(\mathbf{x}) = 0,$$

Theorem 2.2 (respectively Theorem 3.2) is a consequence of the application of the first Part (respectively of the second Part) of Proposition 5.6. Moreover, under the assumptions of Theorem 3.3, the application of Proposition 5.3,

$$\lim_{n\to\infty} v_n\widetilde{B}_n(\mathbf{x}) = 0;$$

Theorem 2.4 thus follows from the application of third Part of Proposition 5.2. Let us now state a preliminary lemma, which is the key of the proof of Proposition 5.2. For any $u \in \mathbb{R}$, Set

$$\begin{aligned}
\widetilde{\Lambda}_{n,\mathbf{x}}(u) &= \frac{\gamma_{s_n} v_n^2}{h_{s_n}^d} \log \mathbb{E}\left[\exp\left(u\frac{h_{s_n}^d}{\gamma_{s_n} v_{s_n}}\widetilde{\Psi}_n(\mathbf{x})\right)\right],\\
\Lambda_{\mathbf{x}}^{L,1}(u) &= f(\mathbf{x})\left(\psi_a(u) - u\right),\\
\Lambda_{\mathbf{x}}^{L,2}(u) &= f(\mathbf{x})\left(\psi(u) - u\right),\\
\widetilde{\Lambda}_{\mathbf{x}}^{M}(u) &= \frac{u^2}{\left(2 - (\alpha - ad)\xi\right)} f(\mathbf{x}) \int_{\mathbb{R}^d} K^2(\mathbf{z})\, d\mathbf{z}.
\end{aligned}$$

Lemma 5.7. (Convergence of $\widetilde{\Lambda}_{n,\mathbf{x}}$) *If* $f(\cdot)$ *is continuous at* $\mathbf{x}$*, then for all* $u \in \mathbb{R}$

$$\lim_{n\to\infty} \widetilde{\Lambda}_{n,\mathbf{x}}(u) = \widetilde{\Lambda}_{\mathbf{x}}(u), \tag{5.14}$$

where

$$\widetilde{\Lambda}_{\mathbf{x}}(u) = \begin{cases} \Lambda_{\mathbf{x}}^{L,1}(u) & \text{when} \quad v_n \equiv 1, & (\mathbf{L1})\text{-}(\mathbf{L4}) \text{ hold.}\\ \Lambda_{\mathbf{x}}^{L,2}(u) & \text{when} \quad v_n \equiv 1, & (\mathbf{L1}), (\mathbf{L2'})\text{-}(\mathbf{L4}) \text{ hold.}\\ \Lambda_{\mathbf{x}}^{M}(u) & \text{when} \quad v_n \to \infty, & (\mathbf{M1})\text{-}(\mathbf{M5}) \text{ hold.} \end{cases}$$

Our proofs are now organized as follows: Lemma 5.7 is proved in the forthcoming Section and after we give the proof of Proposition 5.2.

Proof of Lemma 5.7. Set $u \in \mathbb{R}$, $u_n = u/v_n$ and $a_{s_n} = h_{s_n}^d \gamma_{s_n}^{-1}$. We have:

$$\begin{aligned}
\widetilde{\Lambda}_{n,\mathbf{x}}(u) &= \frac{v_n^2}{a_{s_n}} \log \mathbb{E}\left[\exp\left(u_n a_{s_n} \widetilde{\Psi}_n(\mathbf{x})\right)\right] \\
&= \frac{v_n^2}{a_{s_n}} \log \mathbb{E}\left[\exp\left(u_n a_{s_n} \Pi_n \sum_{k=1}^n \Pi_k^{-1} a_{s_k}^{-1} \left(T_{s_k} - \mathbb{E}\left[T_{s_k}\right]\right)\right)\right] \\
&= \frac{v_n^2}{a_{s_n}} \sum_{k=1}^n \log \mathbb{E}\left[\exp\left(u_n \frac{a_{s_n} \Pi_n}{a_{s_k} \Pi_k} T_{s_k}\right)\right] \\
&\quad - u v_n \Pi_n \sum_{k=1}^n \Pi_k^{-1} a_{s_k}^{-1} \mathbb{E}\left[T_{s_k}\right].
\end{aligned}$$

By Taylor expansion, there exists $c_{k,n}$ between 1 and $\mathbb{E}\left[\exp\left(u_n \frac{a_{s_n}\Pi_n}{a_{s_k}\Pi_k} T_{s_k}\right)\right]$ such that

$$\begin{aligned}
&\log \mathbb{E}\left[\exp\left(u_n \frac{a_{s_n} \Pi_n}{a_{s_k} \Pi_{s_k}} T_{s_k}\right)\right] \\
&= \mathbb{E}\left[\exp\left(u_n \frac{a_{s_n} \Pi_n}{a_{s_k} \Pi_k} T_{s_k}\right) - 1\right] \\
&\quad - \frac{1}{2c_{k,n}^2} \left(\mathbb{E}\left[\exp\left(u_n \frac{a_{s_n} \Pi_n}{a_{s_k} \Pi_k} T_{s_k}\right) - 1\right]\right)^2,
\end{aligned}$$

and $\widetilde{\Lambda}_{n,\mathbf{x}}$ can be rewritten as

$$\begin{aligned}
\widetilde{\Lambda}_{n,\mathbf{x}}(u) &= \frac{v_n^2}{a_{s_n}} \sum_{k=1}^n \mathbb{E}\left[\exp\left(u_n \frac{a_{s_n} \Pi_n}{a_{s_k} \Pi_k} T_{s_k}\right) - 1\right] \\
&\quad - \frac{v_n^2}{2a_{s_n}} \sum_{k=1}^n \frac{1}{c_{k,n}^2} \left(\mathbb{E}\left[\exp\left(u_n \frac{a_{s_n} \Pi_n}{a_{s_k} \Pi_k} T_{s_k}\right) - 1\right]\right)^2 \\
&\quad - u v_n \Pi_n \sum_{k=1}^n \Pi_k^{-1} a_{s_k}^{-1} \mathbb{E}\left[T_{s_k}\right]. \qquad (5.15)
\end{aligned}$$

First case: $v_n \to \infty$. A Taylor's expansion implies the existence of $c'_{k,n}$ between 0 and $u_n \frac{a_{s_n}\Pi_n}{a_{s_k}\Pi_k} T_{s_k}$ such that

$$\begin{aligned}
&\mathbb{E}\left[\exp\left(u_n \frac{a_{s_n} \Pi_n}{a_{s_k} \Pi_k} T_{s_k}\right) - 1\right] \\
&= u_n \frac{a_{s_n} \Pi_n}{a_{s_k} \Pi_k} \mathbb{E}\left[T_{s_k}\right] + \frac{1}{2} \left(u_n \frac{a_{s_n} \Pi_n}{a_{s_k} \Pi_k}\right)^2 \mathbb{E}\left[T_{s_k}^2\right] \\
&\quad + \frac{1}{6} \left(u_n \frac{a_{s_n} \Pi_n}{a_{s_k} \Pi_k}\right)^3 \mathbb{E}\left[T_{s_k}^3 e^{c'_{k,n}}\right].
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
&\widetilde{\Lambda}_{n,\mathbf{x}}(u)\\
&= \frac{1}{2}u^2 a_{s_n}\Pi_n^2\sum_{k=1}^n \Pi_k^{-2}a_{s_k}^{-2}\mathbb{E}\left[T_{s_k}^2\right]+\frac{1}{6}u^2 u_n a_{s_n}^2\Pi_n^3\sum_{k=1}^n \Pi_k^{-3}a_{s_k}^{-3}\mathbb{E}\left[T_{s_k}^3 e^{c'_{k,n}}\right]\\
&\quad -\frac{v_n^2}{2a_{s_n}}\sum_{k=1}^n\frac{1}{c_{k,n}^2}\left(\mathbb{E}\left[\exp\left(u_n\frac{a_{s_n}\Pi_n}{a_{s_k}\Pi_k}T_{s_k}\right)-1\right]\right)^2.
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
&\mathbb{E}\left[T_{s_k}^2\right]\\
&= \mathbb{E}\left[\left(Y_{s_k}(\mathbf{x})-Y_{s_k}(-\mathbf{x})\right)^2\right]\\
&= \mathbb{E}\left[\left(K\left(\frac{\mathbf{x}-X_{s_k}}{h_{s_k}}\right)-K\left(\frac{-\mathbf{x}-X_{s_k}}{h_{s_k}}\right)\right)^2\right]\\
&= \mathbb{E}\left[K^2\left(\frac{\mathbf{x}-X_{s_k}}{h_{s_k}}\right)\right]+\mathbb{E}\left[K^2\left(\frac{-\mathbf{x}-X_{s_k}}{h_{s_k}}\right)\right]\\
&\quad -2\mathbb{E}\left[K\left(\frac{\mathbf{x}-X_{s_k}}{h_{s_k}}\right)\right]\mathbb{E}\left[K\left(\frac{-\mathbf{x}-X_{s_k}}{h_{s_k}}\right)\right]\\
&= h_{s_k}^d\left\{\int_{\mathbb{R}^d}K^2(z)\left[f\left(\mathbf{x}+\mathbf{z}h_{s_k}\right)+f\left(\mathbf{x}-\mathbf{z}h_{s_k}\right)\right]dz\right.\\
&\quad\left. -2\int_{\mathbb{R}^d}K(\mathbf{z})K\left(\mathbf{z}-2\frac{\mathbf{x}}{h_{s_k}}\right)f\left(\mathbf{x}-\mathbf{z}h_{s_k}\right)d\mathbf{z}\right\}.
\end{aligned}$$

Then, we have

$$\begin{aligned}
\widetilde{\Lambda}_{n,\mathbf{x}}(u) &= f(\mathbf{x})u^2 a_{s_n}\Pi_n^2\sum_{k=1}^n\Pi_k^{-2}a_{s_k}^{-1}\gamma_{s_k}\int_{\mathbb{R}^d}K^2(\mathbf{z})\,d\mathbf{z}\\
&\quad +R_{n,\mathbf{x}}^{(1)}(u)+R_{n,\mathbf{x}}^{(2)}(u),
\end{aligned}\tag{5.16}$$

with

$$\begin{aligned}
\widetilde{R}_{n,\mathbf{x}}^{(1)}(u) &= \frac{1}{2}u^2 a_{s_n}\Pi_n^2\sum_{k=1}^n\Pi_k^{-2}a_{s_k}^{-1}\gamma_{s_k}\\
&\quad \int_{\mathbb{R}^d}K^2(\mathbf{z})\left\{\left[f\left(\mathbf{x}-\mathbf{z}h_{s_k}\right)-f(\mathbf{x})\right]+\left[f\left(\mathbf{x}+\mathbf{z}h_{s_k}\right)-f(\mathbf{x})\right]\right\}d\mathbf{z}\\
\widetilde{R}_{n,\mathbf{x}}^{(2)}(u) &= \frac{1}{2}u^2 a_{s_n}\Pi_n^2\sum_{k=1}^n\Pi_k^{-2}a_{s_k}^{-1}\gamma_{s_k}\\
&\quad \int_{\mathbb{R}^d}K(\mathbf{z})K\left(\mathbf{z}-2\frac{\mathbf{x}}{h_{s_k}}\right)f\left(\mathbf{x}-\mathbf{z}h_{s_k}\right)d\mathbf{z}\\
\widetilde{R}_{n,\mathbf{x}}^{(3)}(u) &= \frac{1}{6}\frac{u^3}{v_n}a_{s_n}^2\Pi_n^3\sum_{k=1}^n\Pi_k^{-3}a_{s_k}^{-3}\mathbb{E}\left[T_{s_k}^3e^{c'_{k,n}}\right]\\
&\quad -\frac{v_n^2}{2a_{s_n}}\sum_{k=1}^n\frac{1}{c_{k,n}^2}\left(\mathbb{E}\left[\exp\left(u_n\frac{a_{s_n}\Pi_n}{a_{s_k}\Pi_k}T_{s_k}\right)-1\right]\right)^2.
\end{aligned}$$

Since $f(\cdot)$ is continuous, we have

$$\lim_{k\to\infty} |f(\mathbf{x} \pm \mathbf{z}h_{s_k}) - f(\mathbf{x})| = 0.$$

Thus a straightforward application of Lebesgue dominated convergence theorem in connection with condition (**M1**) implies that

$$\lim_{k\to\infty} \int_{\mathbb{R}^d} K^2(z)\left\{[f(\mathbf{x} - \mathbf{z}h_{s_k}) - f(\mathbf{x})] + [f(\mathbf{x} + \mathbf{z}h_{s_k}) - f(\mathbf{x})]\right\} d\mathbf{z} = 0.$$

Since $(a_{s_n}) \in \mathcal{GS}(\alpha - ad)$, and

$$\lim_{n\to\infty} (n\gamma_{s_n}) > (\alpha - ad)/2.$$

By an application of Lemma 5.1 we can therefore write

$$a_{s_n}\Pi_n^2 \sum_{k=1}^{n} \Pi_k^{-2} a_{s_k}^{-1} \gamma_{s_k} = \frac{1}{(2 - (\alpha - ad)\xi)} + o(1). \tag{5.17}$$

From this we see that

$$\lim_{n\to\infty} \left|\widetilde{R}_{n,\mathbf{x}}^{(1)}(u)\right| = 0.$$

Moreover, it follows from (**M1**) that for any $\varepsilon > 0$, there exists n_0 such that for $n \geq n_0$,

$$\left|K\left(\mathbf{z} - 2\frac{\mathbf{x}}{h_{s_k}}\right)\right| < \varepsilon, \tag{5.18}$$

then,

$$\left|\int_{\mathbb{R}^d} K(z)\, K\left(\mathbf{z} - 2\frac{\mathbf{x}}{h_{s_k}}\right) f(\mathbf{x} - \mathbf{z}h_{s_k})\, d\mathbf{z}\right| < \varepsilon \left\|f\right\|_\infty \int_{\mathbb{R}^d} K(\mathbf{z})\, d\mathbf{z},$$

then, it follows from (5.7), that

$$\lim_{n\to\infty} \left|\widetilde{R}_{n,\mathbf{x}}^{(2)}(u)\right| = 0.$$

Moreover, in view of (5.1), we have

$$|T_{s_k}| \leq 2\left\|K\right\|_\infty,$$

then

$$c'_{k,n} \quad \leq \quad \left|u_n \frac{a_{s_n}\Pi_n}{a_{s_k}\Pi_k} T_{s_k}\right| \leq 2\left|u_n\right| \left\|K\right\|_\infty. \tag{5.19}$$

Since, we have

$$\mathbb{E}\left|T_{s_k}\right|^3 \leq h_{s_k}^d \left\|f\right\|_\infty \int_{\mathbb{R}^d} \left|K^3(\mathbf{z})\right| d\mathbf{z}.$$

It follows from, Lemma 5.1 and (5.19), that, there exists a positive constant c_1 such that, for n large enough,

$$\begin{aligned} &\left|\frac{u^3}{v_n} a_{s_n}^2 \Pi_n^3 \sum_{k=1}^{n} \Pi_k^{-3} a_{s_k}^{-3} \mathbb{E}\left[T_{s_k}^3 e^{c'_{k,n}}\right]\right| \\ &\qquad \leq c_1 e^{2|u_n|\|K\|_\infty} \frac{u^3}{v_n} \left\|f\right\|_\infty \int_{\mathbb{R}^d} \left|K^3(\mathbf{z})\right| d\mathbf{z}, \end{aligned} \tag{5.20}$$

which goes to 0 as $n \to \infty$. An application of Lemma 5.1 ensures that

$$\begin{aligned}
&\left|\frac{v_n^2}{2a_{s_n}}\sum_{k=1}^{n}\frac{1}{c_{k,n}^2}\left(\mathbb{E}\left[\exp\left(u_n\frac{a_{s_n}\Pi_n}{a_{s_k}\Pi_k}T_{s_k}\right)-1\right]\right)^2\right| \\
&\leq \frac{v_n^2}{2a_{s_n}}\sum_{k=1}^{n}\left(\mathbb{E}\left[\exp\left(u_n\frac{a_{s_n}\Pi_n}{a_{s_k}\Pi_k}T_{s_k}\right)-1\right]\right)^2 \\
&= \frac{u^2}{2}\|f\|_\infty^2\, a_{s_n}\Pi_n^2\sum_{k=1}^{n}\Pi_k^{-2}a_{s_k}^{-1}\gamma_{s_k}h_{s_k}^d \\
&\quad +o\left(a_{s_n}\Pi_n^2\sum_{k=1}^{n}\Pi_k^{-2}a_{s_k}^{-1}\gamma_{s_k}h_{s_k}^d\right) \\
&= o\left(1\right). \qquad (5.21)
\end{aligned}$$

The combination of (5.9) and (5.21) ensures that

$$\lim_{n\to\infty}\left|\widetilde{R}_{n,\mathbf{x}}^{(3)}\left(u\right)\right| = 0.$$

Then, it follows from (5.16) and (5.7), that

$$\lim_{n\to\infty}\widetilde{\Lambda}_{n,\mathbf{x}}\left(u\right) = \widetilde{\Lambda}_{\mathbf{x}}^{M}\left(u\right).$$

Second case: $(v_n) \equiv 1$. We obtain from (5.15) that

$$\begin{aligned}
&\widetilde{\Lambda}_{n,\mathbf{x}}\left(u\right) \\
&= \frac{1}{a_{s_n}}\sum_{k=1}^{n}\mathbb{E}\left[\exp\left(u\frac{a_{s_n}\Pi_n}{a_{s_k}\Pi_k}T_{s_k}\right)-1\right] \\
&\quad -\frac{1}{2a_{s_n}}\sum_{k=1}^{n}\frac{1}{c_{k,n}^2}\left(\mathbb{E}\left[\exp\left(u\frac{a_{s_n}\Pi_n}{a_{s_k}\Pi_k}T_{s_k}\right)-1\right]\right)^2 \\
&\quad -u\Pi_n\sum_{k=1}^{n}\Pi_k^{-1}a_{s_k}^{-1}\mathbb{E}\left[T_{s_k}\right] \\
&= \frac{1}{a_{s_n}}\sum_{k=1}^{n}h_{s_k}^d\int_{\mathbb{R}^d}\left[\exp\left(u\frac{a_{s_n}\Pi_n}{a_{s_k}\Pi_k}\left[K\left(\mathbf{z}\right)-K\left(\mathbf{z}-\frac{2\mathbf{x}}{h_{s_k}}\right)\right]\right)-1\right]f(\mathbf{x})d\mathbf{z} \\
&\quad -u\Pi_n\sum_{k=1}^{n}\Pi_k^{-1}\gamma_{s_k}\int_{\mathbb{R}^d}\left[K\left(\mathbf{z}\right)-K\left(\mathbf{z}-\frac{2\mathbf{x}}{h_{s_k}}\right)\right]f(\mathbf{x})d\mathbf{z} \\
&\quad -R_{n,\mathbf{x}}^{(3)}\left(u\right)+R_{n,\mathbf{x}}^{(4)}\left(u\right) \\
&= f(\mathbf{x})\frac{1}{a_{s_n}}\sum_{k=1}^{n}h_{s_k}^d\left[\int_{\mathbb{R}^d}\left(\exp\left(uV_{n,k}\left[K\left(\mathbf{z}\right)-K\left(\mathbf{z}-\frac{2\mathbf{x}}{h_{s_k}}\right)\right]\right)-1\right)\right. \\
&\quad \left.-uV_{n,k}\left[K\left(\mathbf{z}\right)-K\left(\mathbf{z}-\frac{2\mathbf{x}}{h_{s_k}}\right)\right]\right]d\mathbf{z} \\
&\quad -\widetilde{R}_{n,\mathbf{x}}^{(3)}\left(u\right)+\widetilde{R}_{n,\mathbf{x}}^{(4)}\left(u\right), \qquad (5.22)
\end{aligned}$$

with

$$\begin{aligned}
V_{n,k} &= \frac{a_{s_n}\Pi_n}{a_{s_k}\Pi_k} \\
\widetilde{R}^{(3)}_{n,\mathbf{x}}(u) &= \frac{1}{2a_{s_n}}\sum_{k=1}^{n}\frac{1}{c_{k,n}^2}\left(\mathbb{E}\left[\exp\left(u\frac{a_{s_n}\Pi_n}{a_{s_k}\Pi_k}T_{s_k}\right)-1\right]\right)^2 \\
\widetilde{R}^{(4)}_{n,\mathbf{x}}(u) &= \frac{1}{a_{s_n}}\sum_{k=1}^{n}h_{s_k}^d\int_{\mathbb{R}^d}\left[\exp\left(u\frac{a_{s_n}\Pi_n}{a_{s_k}\Pi_k}\left[K(\mathbf{z})-K\left(\mathbf{z}-\frac{2\mathbf{x}}{h_{s_k}}\right)\right]\right)-1\right] \\
&\quad \times\left[f(\mathbf{x}-\mathbf{z}h_{s_k})-f(\mathbf{x})\right]d\mathbf{z} \\
&\quad -u\Pi_n\sum_{k=1}^{n}\Pi_k^{-1}\gamma_{s_k}\int_{\mathbb{R}^d}\left[K(\mathbf{z})-K\left(\mathbf{z}-\frac{2\mathbf{x}}{h_{s_k}}\right)\right] \\
&\quad \left[f(\mathbf{x}-\mathbf{z}h_{s_k})-f(\mathbf{x})\right]d\mathbf{z}.
\end{aligned}$$

Moreover, it follows from (5.10), that

$$\lim_{n\to\infty}\left|\widetilde{R}^{(3)}_{n,\mathbf{x}}(u)\right| = 0.$$

Now, since $|e^t-1|\leq|t|\,e^{|t|}$, one can see that

$$\begin{aligned}
\left|\widetilde{R}^{(4)}_{n,\mathbf{x}}(u)\right| &\leq \frac{1}{a_{s_n}}\sum_{k=1}^{n}h_{s_k}^d \\
&\quad \int_{\mathbb{R}^d}\left|\left[\exp\left(u\frac{a_{s_n}\Pi_n}{a_{s_k}\Pi_k}\left[K(\mathbf{z})-K\left(\mathbf{z}-\frac{2\mathbf{x}}{h_{s_k}}\right)\right]\right)-1\right]\right| \\
&\quad \left|\left[f(\mathbf{x}-\mathbf{z}h_{s_k})-f(\mathbf{x})\right]\right|d\mathbf{z} \\
&\quad +|u|\,\Pi_n\sum_{k=1}^{n}\Pi_k^{-1}\gamma_{s_k}\int_{\mathbb{R}^d}\left|\left[K(\mathbf{z})-K\left(\mathbf{z}-\frac{2\mathbf{x}}{h_{s_k}}\right)\right]\right| \\
&\quad \left|f(\mathbf{x}-\mathbf{z}h_{s_k})-f(\mathbf{x})\right|d\mathbf{z} \\
&\leq |u|\,e^{|u|\|K\|_\infty}\Pi_n\sum_{k=1}^{n}\Pi_k^{-1}\gamma_{s_k}\int_{\mathbb{R}^d}\left|\left[K(\mathbf{z})-K\left(\mathbf{z}-\frac{2\mathbf{x}}{h_{s_k}}\right)\right]\right| \\
&\quad \left|f(\mathbf{x}-\mathbf{z}h_{s_k})-f(\mathbf{x})\right|d\mathbf{z} \\
&\quad +|u|\,\Pi_n\sum_{k=1}^{n}\Pi_k^{-1}\gamma_{s_k}\int_{\mathbb{R}^d}\left|\left[K(\mathbf{z})-K\left(\mathbf{z}-\frac{2\mathbf{x}}{h_{s_k}}\right)\right]\right| \\
&\quad \left|f(\mathbf{x}-\mathbf{z}h_{s_k})-f(\mathbf{x})\right|d\mathbf{z} \\
&\leq |u|\left(e^{|u|\|K\|_\infty}+1\right)\Pi_n\sum_{k=1}^{n}\Pi_k^{-1}\gamma_{s_k}\int_{\mathbb{R}^d}\left|\left[K(\mathbf{z})-K\left(\mathbf{z}-\frac{2\mathbf{x}}{h_{s_k}}\right)\right]\right| \\
&\quad \left|f(\mathbf{x}-\mathbf{z}h_{s_k})-f(\mathbf{x})\right|d\mathbf{z}.
\end{aligned}$$

Since, Lemma 5.1 ensures that, the sequence $\left(\Pi_n\sum_{k=1}^{n}\Pi_k^{-1}\gamma_{s_k}\right)$ is bounded, then, the dominated convergence theorem ensures that

$$\lim_{n\to\infty}\widetilde{R}^{(4)}_{n,\mathbf{x}}(u) = 0.$$

Then, it follows from (5.18) and (5.22), that

$$\lim_{n\to\infty}\widetilde{\Lambda}_{n,\mathbf{x}}(u) = \lim_{n\to\infty} f(\mathbf{x})\frac{\gamma_{s_n}}{h_{s_n}^d}\sum_{k=1}^{n} h_{s_k}^d\int_{\mathbb{R}^d}\left[\left(\exp\left(uV_{n,k}K(\mathbf{z})\right)-1\right)\right. \left.-uV_{n,k}K(\mathbf{z})\right]d\mathbf{z}. \tag{5.23}$$

In the case when $(v_n)\equiv 1$, (**L1**)-(**L4**) hold.
By using the fact that

$$\frac{\Pi_n}{\Pi_k} = \prod_{j=k+1}^{n}\left(1-\gamma_{s_j}\right) = \frac{k}{n},$$

we readily infer that we have

$$V_{n,k} = \frac{a_{s_n}\Pi_n}{a_{s_k}\Pi_k} = \left(\frac{k}{n}\right)^{ad}.$$

Consequently, it follows from (5.23) and routine calculation that

$$\begin{aligned}&\lim_{n\to\infty}\Lambda_{n,\mathbf{x}}(u)\\ &= f(\mathbf{x})\int_{\mathbb{R}^d}\left[\int_0^1 t^{-ad}\left(\exp\left(ut^{ad}K(\mathbf{z})\right)-1\right)-ut^{ad}K(\mathbf{z})\right]dtd\mathbf{z}\\ &= \Lambda_{\mathbf{x}}^{L,1}(u).\end{aligned}$$

In the case when $(v_n)\equiv 1$, (**L1**), (**L2′**)-(**L4**) hold.
We have

$$\frac{\Pi_n}{\Pi_k} = \frac{\gamma_{s_n}}{\gamma_{s_k}}\frac{h_{s_k}^d}{h_{s_n}^d}.$$

Then, we have

$$V_{n,k} = 1.$$

Consequently, it follows from (5.23) that

$$\begin{aligned}\lim_{n\to\infty}\Lambda_{n,\mathbf{x}}(u) &= f(\mathbf{x})\int_{\mathbb{R}^d}\left[\left(\exp\left(uK(\mathbf{z})\right)-1\right)-uK(\mathbf{z})\right]d\mathbf{z}.\\ &= \Lambda_{\mathbf{x}}^{L,2}(u).\end{aligned}$$

Therefore Lemma 5.4 is proved. □

Proof of Proposition 5.6. We will use similar arguments as in the proof of Proposition 5.2. In our framework, when $v_n\equiv 1$ and $\gamma_{s_n}=n^{-1}$, we take

$$\widetilde{Z}_n = \left(f_n(\mathbf{x})-f_n(-\mathbf{x})-\mathbb{E}\left[f_n(\mathbf{x})-f_n(-\mathbf{x})\right]\right),$$

$\nu_n = nh_{s_n}^d$ with $h_{s_n}=cn^{-a}$ where $a\in\,]0,1/d[$ and $\Gamma=\Lambda_{\mathbf{x}}^{L,1}$. In this case, the Legendre transform of $\Gamma=\Lambda_{\mathbf{x}}^{L,1}$ is the rate function

$$I_{a,\mathbf{x}}: t\to f(\mathbf{x})I_a\left(\frac{t}{f(\mathbf{x})}+1\right),$$

that is strictly convex by Proposition 1 in [26]. Furthermore, when $v_n\equiv 1$ and

$$\gamma_{s_n} = h_{s_n}^d\left(\sum_{k=1}^{n}h_{s_k}^d\right)^{-1},$$

we take

$$\widetilde{Z}_n = \left(f_n(\mathbf{x}) - f_n(-\mathbf{x}) - \mathbb{E}\left[f_n(\mathbf{x}) - f_n(-\mathbf{x})\right]\right),$$

$\nu_n = nh_{s_n}^d$ with $h_{s_n} \in \mathcal{GS}(-a)$ where $a \in]0, 1/d[$ and $\Gamma = \Lambda_{\mathbf{x}}^{L,2}$. In this case, the Legendre transform of $\Gamma = \Lambda_{\mathbf{x}}^{L,2}$ is the rate function

$$I_{\mathbf{x}} : t \to f(\mathbf{x}) I\left(\frac{t}{f(\mathbf{x})} + 1\right),$$

that is strictly convex by Proposition 1 in [27]. Otherwise, when, $v_n \to \infty$, we take

$$\widetilde{Z}_n = v_n\left(f_n(\mathbf{x}) - f_n(-\mathbf{x}) - \mathbb{E}\left[f_n(\mathbf{x}) - f_n(-\mathbf{x})\right]\right),$$

$\nu_n = h_{s_n}^d / \left(\gamma_{s_n} v_n^2\right)$ and $\Gamma = \widetilde{\Lambda}_{\mathbf{x}}^M$; Γ^* is then the quadratic rate function $\mathbf{J}_{a,\alpha,\mathbf{x}}$ defined in (3.2) and thus Proposition 5.2 follows. □

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