SCS 5: Notes on Chains in CL-Objects

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NOTES ON CHAINS in CL-OBJECTS (from KHH) 4-19-76

REFERENCE: Letter from KHH to AS of 2-22-76 (not circulated)
Letter from DS to KHH of 3-30-76 (circulated)
Notes on Stralka's congruence extension theorem from KE
Nov-Dec.75 (circulated)

Ever since DIMENSION RAISING one had worked with chains inside
a CL-object to obtain chain quotients. An indication of a
general construction was given in ATLAS which was rightly
criticized by Dana Scott. Shortly before, Al Stralka had asked
me whether in a connected CL-object S one could always find, for
any $x \ll 1$ a morphism $f: S \rightarrow I$ with $f^{-1}(1/2) = \min f^{-1}(1/2)$, I believe I gave an argument under the hypothesis
that $\updownarrow x$ has a countable neighborhood basis, and a counterexample in the absence of this hypothesis.

I would like to record some general remarks.

0. Remark. If $a \ll b$ in a CL-object, then there is an $s$ with
$a \ll s \ll b$ and $a < s$.

1. Definition. A subset $C$ in a CL-object $S$ is a strict chain
if $x, y \in C$ implies that $x \ll y$ or $x = y$ or $y \ll x$.

All singletons are strict chains; $[0, 1]$ is a strict chain.
All chains $\uparrow f$ contained in $K(S)$ are strict chains.

2. Remark. The collection of all strict chains is $\subseteq$-inductive.
Indeed if $\{C_j : j \in J\}$ is a $\subseteq$-totally ordered family of strict
chains and $C = \bigcup_j C_j$, then $C$ is a strict chain, since for
any $x, y \in C$ there is a $j \in J$ with $x, y \in C_j$, whence $x \ll y$ or $x = y$
or $y \ll x$ since $C_j$ is strict.

3. Proposition. Let $C$ be a strict chain in a CL-object $S$. Then
$C$ is contained in at least one maximal strict chain $M$.
Proof. Remark 2 and Zorn's Lemma.[]

A gap in a poset $S$ is an interval $[a, b]$ with $[a, b] = [a, b]$.

4. Proposition. Let $C$ be a maximal strict chain in a CL-object.
Then

(1) $0 \in C$.

(ii) If $[a, b]_C$ is a gap in $C$, then $b \in K(S)$.

(iii) $C$ is complete.
Then there by Remark 0 there is an $s \in S$ with $a \ll s \ll b$, $a < s$. By maximality of $C$ we have $s \ll b$. Thus $b \ll b$, whence $b \in K(S)$, i.e. $t \ll b$ for all $t \in [b]$. Thus if $m$ is a maximal element of $\{a \in \{b \setminus \{h\}\}, then a \ll m$, i.e. $a \in \text{a face of } [m]$. $\uparrow$

(iii) Let $X \subseteq C$ and $x = \sup_C X$.

Case 1. $x \in C$. Good.

Case 2. $x \notin C$. Then for any $c \in C$, $c < x$ there is a $d \in C$ with $c < d \ll x$, since $C$ is strict, we have $c \ll d$, whence $c < d$. By the maximality of $C$ there must then be a $y \in C$ with $x < y$ but $x \nmid y$. If there were a $u \in C$ with $x \leq u < y$, then $u \ll y$ (C is strict) whence $x \ll u$ which is impossible. Thus $y = \min([x] \cap C)$. But then $y = \sup_C X$.

(iv) is trivial.

5. Lemma. For a maximal/chain $C$ in $S$ define $φ : C → S$ by $φ(c) = \sup_S \{x \mid x \ll c\}$. Then $φ$ is an injective homomorphism with $φ(c) \leq c$ for all $c \in C$.

Proof. (a) If $a < b$ in $C$ then $a \ll b$ since $C$ is strict. Thus $φ(a) \leq a \ll φ(b)$. If $φ(a) = φ(b)$, then $a = \sup_S \{c \in C : c \ll b\}$ i.e. $[a, b]_C$ is a gap. Thus $b \in K(S)$ by 4 (ii). But then $b \ll b$ whence $b = φ(b)$. This implies $a = b$, a contradiction. Thus $φ$ is injective.

(b) If $a \ll b$, then (i) $a < b$ or (ii) $a = b \in K(C)$. In case (i) we have $a = b \in K(S)$ by 4 (ii), whence $a \ll b$ and $φ(a) = a = b = φ(b)$. Thus $φ(a) \ll φ(b)$. Now suppose $a < b$. If $[a, b]_C$ is a gap, then $b \in K(S)$, whence $φ(a) \leq a \ll φ(b)$. If $[a, b]_C$ is not a gap, then there is a $u \in C$ with $a < u < b$. Then $φ(a) \leq a \ll φ(b) \ll φ(b)$ (as $u \ll b$, since $C$ is strict), whence $φ(a) \ll φ(b)$.

(c) If $x = \sup_C X$, then (i) $x = \max X$ or (ii) $x \notin X$.

In case (i) we have $φ(x) = \max φ(X)$ since $φ$ is monotone. In case (ii) the relation $φ < x$ in $C$ implies the existence of a $u \in X$ with $x < u \ll x$, so $φ < u \ll x$, since $C$ is strict. So $\sup_S φ(X) = \sup_S \{\sup_S \{c \mid c \ll u\} \mid u \in X\} = \sup_S \{c \mid \text{there is a } u \in X \text{ with } c \ll u\} = \sup_S \{c \mid c \ll x\} = φ(x) = φ(\sup_C X)$. $\Box$
According to ATLAS duality, from Lemma 5 the function \( \varphi \) has a morphism \( \varphi^* : S \rightarrow C \) as left adjoint which satisfies the following conditions:

1. \( \varphi(s) \geq c \iff s \geq \varphi(c) \) for all \( s \in S, c \in C \).

2. \( \varphi^*(s) = \sup_c \varphi^{-1}(\downarrow s) = \sup_c (c \in C : \varphi(s) \leq s) \)
   \( = \sup_c (c \in C : d \ll c \Rightarrow d \leq s \text{ for all } d \in C) \)

3. \( c = \varphi^*(\varphi(c)) \leq \varphi(c) \) for all \( c \in C \).

4. \( \varphi(c) = \inf \varphi^{-1}(\uparrow c) = \min \varphi^{-1}(c) \setminus s = \min \{ s \in S : \varphi(s) = c \} \)

Indeed, (1) is the definition of adjointness, (2) is the determination of a left adjoint in terms of its right adjoints and the definition of \( \varphi^* \). (3) follows from the fact that \( \varphi \) is injective \((\text{ATLAS I.12})\) and from \( \varphi(c) \leq c \) for all \( c \).

(4) arises from the determination of a right adjoint in terms of its left adjoint, plus the fact that the left adjoint \( \varphi \) is surjective.

6. Lemma. Under the hypotheses of Lemma 5 the following statements are equivalent for an element \( c \in C \).

(i) \( c = \varphi(c) \).

(ii) \( c = \min \{ s \in S : \varphi(s) = c \} \)

(iii) \( c \) is not isolated from below in \( C \) in the induced topology, i.e. for all \( s \ll c, s \in S \) there is a \( d \ll c \)

\[ d \in \uparrow s \cap C. \]

Proof. (i) \( \iff \) (ii) from (4) above. (i) \( \iff \) (iii) from the definition of \( \varphi \) in Lemma 5.[a]

7. Definition. We (perhaps temporarily) call an element \( s \in S \) accessible, if there is a strict chain \( C \) with \( s = \sup_C C. \]

8. Lemma. If \( s \) has a countable neighborhood basis, then
U_n is open and U_{n+1} ⊆ U_n. Then \[ \inf U_n: n=1,\ldots,n \] is a strict chain C with sup C = s.]

Other points s without a countable basis for their neighborhood may be accessible. E.g., every point of a chain is accessible.

Thus if S = [0, \Omega] with the first uncountable ordinal \Omega, then \Omega does not have a countable neighborhood basis, but is accessible.

Let \( S = 2^X \) (or \( I^X \)) with an uncountable set X. Then 1 is not \( q \)-accessible, for \( s = (u_x)_{x \in X} << 1 \) implies that all but a finite number of the \( u_x \) are 0.

9. THEOREM. Let \( S \in \mathcal{CL} \). Then every strict chain \( C \subseteq S \) (such as \( C_0 = \{1\} \) or \( \{0\} \)) is contained in a maximal strict chain C. Moreover, \( C \subseteq \mathcal{CL} \) and there is a \( \mathcal{CL} \)-morphism \( \varphi: S \rightarrow C \) whose right adjoint is given by \( \varphi(c) \mapsto \sup_S \{d \subseteq C: d << c\} \).

If \( C \) is accessible, then there is a maximal strict chain \( C \subseteq S \) with \( c \in C \) such that \( x = \min \{s \in S: \varphi(s) = c\} \). If \( c \in S \) has a countable neighborhood basis, then \( C \) is accessible.

Proof. By Proposition 3, \( C_0 \) is contained in a maximal strict chain \( C \). By Proposition 4, \( C \subseteq \mathcal{CL} \). By Lemma 5 and the subsequent remarks, \( \varphi:S \rightarrow C \) exists with the specified properties. If \( C \) is accessible, we have a strict chain \( C_0 \) with \( \sup C_0 = \emptyset \).

Then let \( C \) be maximal containing \( C_0 \). Then \( c = \min \{s \in S: \varphi(s) = c\} \) by Lemma 6. By Lemma 8 it suffices that \( C \) has a countable neighborhood basis.[]

10. COROLLARY. Every \( S \in \mathcal{CL} \) is a subdirect product of \( \mathcal{CL} \)-chains (Lawson).

Proof. One has to separate two different points \( s \neq t \) in \( S \) by a chain quotient. It suffices to separate st from s, resp. t. Thus one may assume \( s < t \). Then consider the quotient map \( x \mapsto tx \mod SS : S \rightarrow ts/SS \). We may therefore assume that
If anyone has any comments to the following question, then Stralka (and I) would like to hear it:

Suppose that $S \subseteq CL$ and that $T \subseteq S$ is a closed chain subsemilattice. Is there a canonical way to find in $T$, a subchain which is strict in $T$ and maximal with this property?

- One wishes to throw out systematically intervals $J$ on which $a, b \in J$, $a < b \Rightarrow a \not<< b$. Every such interval is contained in a maximal one. For a maximal such one has $\text{sup } J \in J$. (If not, by maximality of $J$ there is a $1 \in J$ such that $1 \not<< \text{sup } J$. By Remark 0 there is an $s \in S$ with $i \ll s \ll \text{sup } J$. Since $\text{sup } J \notin J$ there is a $j \in J$ with $s \leq j$. Thus $1 \not<< J$, a contradiction to the defining property of $J$.) The maximal many such $J$ cover $T$, $\text{max } J$ may be singleton. However, there is no guarantee that they are disjoint. It is conceivable that in $T$ one might have $a < b < c$ with $a \not<< b$ but $a \ll c$.

Does anyone know an example of this situation? Here is one:

If $S = \mathbb{I}^n$ is a cube then $(x_1, \ldots, x_n) \not<< (y_1, \ldots, y_n)$ iff $x_j < y_j$ for $j = 1, \ldots, n$, and hence one can easily fabricate examples of chains $T$ with this property.

In 2 dimensions. One the other hand, one knows that in this open intervals example one wishes to throw out such $\text{max } J$ of a chain $T$ which are stationary in at least one coordinate. This may not be the end of throwing out things even in this simple situation.

P.S.: Scott independently discovered the error in ATLAS which Keimel pointed out and gave the same example to illustrate it (letter 4-12-76)