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SCS 3: Equationally Compact SENDOs are Retracts of Compact Ones

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EQUATIONALLY COMPACT SENDOs ARE RETRACTS OF COMPACT ONES

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A SENDO is a semilattice $S, \wedge$ together with a semilattice endomorphism $x \mapsto \bar{x} \cdot S \to S$.

BULMAN-FLEMING and I. FLEISCHER have shown: A SENDO is equationally compact *iff* the following properties hold:

(a) Every non-empty subset of S has an infimum.

(b) Every $\uparrow$-directed subset of S has a supremum.

(c) $a \wedge \bigvee d_i = \bigvee (a \wedge d_i)$ for every $a \in S$ and every $\uparrow$-directed subset (d_i) in S.

(d) $\bigwedge \bar{x}_i = \overline{\bigwedge x_i}$ for every family (x_i) in S.

(e) $\bigvee \bar{x}_i = \overline{\bigvee x_i}$ for every $\uparrow$-directed family (x_i) in S.

They ask the question whether every equationally compact SENDO is the retract of a compact one. We now show that this question has a positive answer.

Let S be a SENDO, which is equationally compact.

Let us suppose first, that S has a greatest element 1. Then (b) is a consequence of (a) and S is a complete lattice.

Let $\mathcal{I}(S)$ be the lattice of all ideals of S. For every $I \in \mathcal{I}(S)$ let

$$\bar{I} = \{ x \in S \mid \exists i \in I : \ x \leq \bar{i} \}.$$

Then $\bar{I} \in \mathcal{I}(S)$. We have

(i) $\bigcup \bar{I}_j = \overline{\bigcup I_j}$ for every $\uparrow$-directed family (I_j) in $\mathcal{I}(S)$.

(ii) $\bigcap \bar{I}_j = \overline{\bigcap I_j}$ for every non-empty family (I_j) in $\mathcal{I}(S)$.

The proof of (i) is clear. In (ii) the inclusion $\supseteq$ is also evident. In order to show $\subseteq$ take $x \in \bigcap \bar{I}_j$. Then $x \leq \bar{i}_j$ for some $i_j \in I_j$ (all j). Then $x \leq \bigwedge \bar{i}_j$ (the infimum exist by $\boxed{(a)}$)

$\bigwedge \bar{i}_j = \overline{\bigwedge i_j}$ by $\boxed{(d)}$

As $\bigwedge i_j \in I_j$ for all j, we have $x \in \overline{\bigcap I_j}$, which ends the proof of (ii).

Now $\mathcal{I}(S)$ is an algebraic lattice. Endow $\mathcal{I}(S)$ with the topology that has as subbases the sets of the form $V(K) = \{ I \in \mathcal{I} \mid K \subseteq I \}$ as well as the complements of these sets. This topology is compact and zero-dimensional and the operation $\wedge$ is continuous, i.e. $\mathcal{I}(S)$ is a compact zero-dimensional $\wedge$-semilattice (see HOFMANN-MISLOVE-STRALKA, Springer Lecture Notes). By (i) and (ii), $I \mapsto \bar{I}$ is continuous in $\mathcal{I}(S)$ (see HOFMANN-STRALKA, Dissertationes Math., to appear). Thus $\mathcal{I}(S)$ is a compact zero-dimensional SENDO.

Define $\rho : \mathcal{I}(S) \to S$ by $\rho(I) = \bigvee I$. Then

(iii) $\rho(I_1 \cap I_2) = \rho(I) \wedge \rho(I)$

Indeed $\sup(I_1 \cap I_2) = \sup_{\substack{i_1 \in I_1 \\ i_2 \in I_2}} (i_1 \wedge i_2) = \sup I_1 \wedge \sup I_2$ by $\boxed{(c)}$

...

(iv) $\rho(\bar{I}) = \overline{\rho(I)}$

Indeed, $\bigvee_{i \in I} \bar{i} = \overline{\bigvee_{i \in I} i}$ by $\boxed{(e)}$

Thus ρ is a SENDO-homomorphism from $\mathcal{I}(S)$ onto S. ρ is indeed a retraction to the embedding of I in $\mathcal{I}(S)$ which associates with every $a \in S$ the principal ideal generated by a.

Now consider a SENDO S without 1. Let S^1 be the SENDO obtained by adjoining a 1. If S satisfies (a) through (e), the same is true for S^1. Observe that the greatest element of $\mathcal{I}(S^1)$ is compact, hence isolated. Thus $\mathcal{I}(S) = \mathcal{I}(S^1) \setminus \{S^1\}$ is a compact zero-dimensional SENDO. If I is a proper ideal of S^1, then I is an ideal of S, and, by $\boxed{(b)}$, $\rho(I) = \bigvee I \in S$. Thus ρ maps $\mathcal{I}(S)$ onto S.

This proves the assertion in the title of this note.

REMARK on K.KEIMEL :SENDOs . (khh 2-10-76)

Except for the additional structural element $^-$ (the endomorphism of S), Keimel's argument is

~~2.13~~ ATLAS , 2.13 pp.34,35 (resp. 3.2,p.52 . What is said in 3.1 and 3.2 could have been said for any complete upper continuous semilattice (see ATLAS 2.11).

It should be noted that Keimel's retraction $\rho : J(S) \longrightarrow S$ is in fact a lattice morphism preserving arbitrary sups (ATLAS 2.9, 3.2).

Furthermore, while Keimel notes $\rho(\bar{I}) = \overline{\rho(I)}$, he did not, but perhaps should have, observed that also $\downarrow\bar{x} = \overline{(\downarrow x)}$ for all $x \in S$.

Thus the retraction is indeed a SENDO retraction.

I do not know what the SENDOs are good for. Of course, Keimel's remark is useful in the case $^- = id_S$,relative to which endomorphism every $\underline{S}$ -object is a ~~SENDO~~ SENDO.

Of course, the entire Section 2 of ATLAS in some sense is concerned with SENDOs, namely, with the $\underline{D}$ -objects L ~~which are upper~~ ~~xixxxxx~~ for which K(L) is upper continuous with the compact closure operator as endomorphism. On the other hand, it says nowhere, that the endomorphism $^-$ of a SENDO has to be idempotent.

Final remark. If you ask how far a complete upper continuous S is from a $\underline{CL}$-object, then the answer is this: $S \in \underline{CL}$ iff Keimel's retraction $\rho J(S) \longrightarrow S$ preserves arbitrary infs. (ATLAS 2.14,p.36).