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Equations of parametric surfaces with base points via syzygies

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Abstract

Let S be a tensor product parametrized surface in \mathbf{P}^3 ; that is, S is given as the image of $\varphi : \mathbf{P}^1 \times \mathbf{P}^1 \rightarrow \mathbf{P}^3$. This paper will show that the use of syzygies in the form of a combination of moving planes and moving quadrics provides a valid method for finding the implicit equation of S when certain base points are present. This work extends the algorithm provided by Cox [Cox, D.A., 2001. Equations of parametric curves and surfaces via syzygies. In: Symbolic Computation: Solving Equations in Algebra, Geometry, and Engineering. Contemporary Mathematics vol. 286, pp. 1–20] for when φ has no base points, and it is analogous to some of the results of Busé et al. [Busé, L., Cox, D., D'Andrea, C., 2003. Implicitization of surfaces in \mathbf{P}^3 in the presence of base points. J. Algebra Appl. 2 (2), 189–214] for the case of a triangular parametrization $\varphi : \mathbf{P}^2 \rightarrow \mathbf{P}^3$ with base points. © 2004 Elsevier Ltd. All rights reserved.

Keywords: Parametrization; Implicit equation; Base points; Local complete intersection; Syzygy; Saturation

1. Introduction

The use of syzygies has been explored in a number of recent works as an alternative to resultants for producing determinantal formulas for the equations of rationally

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parametrized curves and surfaces. Syzygies were first employed in the paper by Sederberg and Chen (1995), where the concept was called the method of moving curves and surfaces. The article by Cox (2003) provides a detailed survey of the current status of the problem of finding the implicit equation of a rational surface $S \subset \mathbf{P}^3$ described implicitly by a map $\varphi : X \rightarrow \mathbf{P}^3$, where X is either \mathbf{P}^2 (referred to as a triangular parametrization in the language of computer aided geometric design (CAGD)) or $\mathbf{P}^1 \times \mathbf{P}^1$ (tensor product parametrization). The reader is referred to this paper and its references for a discussion of the history of the use of syzygies in the implicitization problem, that is, the problem of finding a generator for the ideal $I(S)$ from the knowledge of φ .

1.1. Notation and terminology

In this paper we will only consider the case $X = \mathbf{P}^1 \times \mathbf{P}^1$ of tensor product parametrizations (one of the most popular representations for surfaces in CAGD), so that our parametrization map $\varphi : \mathbf{P}^1 \times \mathbf{P}^1 \rightarrow \mathbf{P}^3$ will have the form

$$\varphi = [a_0(s, u; t, v), a_1(s, u; t, v), a_2(s, u; t, v), a_3(s, u; t, v)], \quad (1)$$

where $a_0, a_1, a_2, a_3 \in R = \mathbb{C}[s, u, t, v]$ are bihomogeneous polynomials of bidegree (m, n) . Moreover, we always assume that $\gcd(a_0, a_1, a_2, a_3) = 1$. Even with the gcd assumption, it can happen that there are points of $\mathbf{P}^1 \times \mathbf{P}^1$ where all of the a_i , $0 \leq i \leq 3$, vanish simultaneously. These are points, referred to as *base points*, where the map φ is not defined. The goal of this paper is to study the implicitization problem when base points are present, but we will start by summarizing how syzygies were employed by Cox et al. (2000) and Cox (2001) to produce a determinantal equation for S in the case of no base points since this summary will be useful in establishing needed notation and terminology. If φ has no base points, that is, $\mathbb{V}(a_0, a_1, a_2, a_3) = \emptyset$ in $\mathbf{P}^1 \times \mathbf{P}^1$, and φ is generically one-to-one onto its image, then the image of φ is a surface $S \subset \mathbf{P}^3$ of degree $2mn$, (Cox, 2001, Theorem 3.1; Cox et al., 2000).

In the polynomial ring $\mathbb{C}[s, u, t, v, x_0, x_1, x_2, x_3] = R[x_0, x_1, x_2, x_3]$, consider the polynomial $\sum_{i=0}^3 A_i x_i$ where $A_i \in R$ ($0 \leq i \leq 3$) are bihomogeneous polynomials, all of the same bidegree. If we fix a point $\mathbf{p} = [s, u; t, v] \in \mathbf{P}^1 \times \mathbf{P}^1$, then $\sum_{i=0}^3 A_i(\mathbf{p})x_i = 0$ is an equation of a plane in \mathbf{P}^3 , provided some $A_i(\mathbf{p}) \neq 0$. When the point \mathbf{p} changes, we will obtain different equations of planes in \mathbf{P}^3 . This suggests the following definition:

Definition 1.1. A moving plane of bidegree (k, l) on \mathbf{P}^3 is a polynomial of the form

$$\sum_{i=0}^3 A_i x_i$$

where, for $0 \leq i \leq 3$, x_i are homogeneous coordinates on \mathbf{P}^3 and $A_i \in R$ are bihomogeneous polynomials of the same bidegree (k, l) . We say the moving plane follows the parametrization φ if

$$\sum_{i=0}^3 A_i(\mathbf{p})a_i(\mathbf{p}) = 0, \quad \text{for all } \mathbf{p} \in \mathbf{P}^1 \times \mathbf{P}^1,$$

which is equivalent to

$$\sum_{i=0}^3 A_i a_i = 0 \in \mathbb{C}[s, u, t, v] \tag{2}$$

where the polynomials a_i ($0 \leq i \leq 3$) are the components of the parametric map φ that defines the surface S .

In the language of commutative algebra, Eq. (2) states that the moving plane $\sum_{i=0}^3 A_i x_i$ follows the parametrization φ if and only if

$$(A_0, A_1, A_2, A_3) \in \text{Syz}(a_0, a_1, a_2, a_3)$$

where $\text{Syz}(a_0, a_1, a_2, a_3)$ denotes the syzygy submodule of R^4 determined by $a_0, a_1, a_2, a_3 \in R$.

Analogously:

Definition 1.2. A moving quadric of bidegree (k, l) is a polynomial

$$\sum_{0 \leq i \leq j \leq 3} A_{ij} x_i x_j \tag{3}$$

which is quadratic in the homogeneous variables x_i ($0 \leq i \leq 3$) and where all of the $A_{ij} \in R$ are bihomogeneous polynomials of the same bidegree (k, l) .

As with moving planes, a moving quadric follows the parametrization φ , if

$$(A_{00}, A_{01}, \dots, A_{33}) \in \text{Syz}(a_0^2, a_0 a_1, \dots, a_3^2),$$

which means that

$$\sum_{0 \leq i \leq j \leq 3} A_{ij}(\mathbf{p}) a_i(\mathbf{p}) a_j(\mathbf{p}) = 0, \quad \text{for all } \mathbf{p} \in \mathbf{P}^1 \times \mathbf{P}^1.$$

We will primarily have occasion to focus on moving planes and moving quadrics of bidegree $(m - 1, n - 1)$ that follow the parametrization φ , which we have assumed has bidegree (m, n) . If $R_{k,l} \subset R$ denotes the bihomogeneous forms of bidegree (k, l) , then the moving planes of bidegree $(m - 1, n - 1)$ that follow φ make up the kernel of the complex linear map

$$MP : R_{m-1,n-1}^4 \xrightarrow{[a_0 \ a_1 \ a_2 \ a_3]} R_{2m-1,2n-1}$$

given by

$$MP(A_0, A_1, A_2, A_3) = \sum_{i=0}^3 A_i a_i.$$

Note that the standard monomial basis of $R_{k,l}$ is

$$\mathcal{B}_{k,l} = \{s^i u^{k-i} t^j v^{l-j} : 0 \leq i \leq k, 0 \leq j \leq l\}$$

so that $\dim_{\mathbb{C}} R_{k,l} = (k + 1)(l + 1)$. With respect to the standard bases $\mathcal{B}_{m-1,n-1}^4$ on $R_{m-1,n-1}^4$ and $\mathcal{B}_{2m-1,2n-1}$ on $R_{2m-1,2n-1}$, the linear map MP is represented by a

$4mn \times 4mn$ matrix that, by abuse of notation, we will also denote by MP . If φ has no base points and is generically one-to-one, then MP is an isomorphism (Cox, 2001, Page 8), so that there are no moving planes of bidegree $(m - 1, n - 1)$. One of our results (Lemma 4.5) is the verification that certain base points of total multiplicity k will have the effect of producing exactly k linearly independent moving planes.

Similarly, the moving quadrics of bidegree $(m - 1, n - 1)$ that follow φ can be identified as the kernel of the map

$$MQ : R_{m-1,n-1}^{10} \xrightarrow{[a_0^2 \ a_0 a_1 \ \dots \ a_3^2]} R_{3m-1,3n-1}$$

given by

$$MQ(A_{00}, A_{01}, \dots, A_{33}) = \sum_{0 \leq i \leq j \leq 3} A_{ij} a_i a_j.$$

As for the case of moving planes, we will identify the map MQ with the $9mn \times 10mn$ matrix which represents MQ in the standard bases. Since

$$\dim R_{m-1,n-1}^{10} - \dim R_{3m-1,3n-1} = 10mn - 9mn = mn,$$

it follows that $\dim \text{Syz}(a_0^2, \dots, a_3^2)_{m-1,n-1} \geq mn$ and

$$\dim \text{Syz}(a_0^2, \dots, a_3^2)_{m-1,n-1} = mn \iff MQ \text{ has maximal rank.}$$

Thus, if MQ has maximal rank, we can choose a basis of exactly mn linearly independent moving quadrics of bidegree $(m - 1, n - 1)$ which follow the parametrization φ . Each of these mn linearly independent moving quadrics Q_k ($1 \leq k \leq mn$) can be written as

$$\begin{aligned} Q_k &= \sum_{0 \leq i \leq j \leq 3} A_{ij} x_i x_j \\ &= \sum_{0 \leq i \leq j \leq 3} \left(\sum_{\alpha=0}^{m-1} \sum_{\beta=0}^{n-1} A_{ij,\alpha\beta} s^\alpha t^\beta \right) x_i x_j \\ &= \sum_{\alpha=0}^{m-1} \sum_{\beta=0}^{n-1} \left(\sum_{0 \leq i \leq j \leq 3} A_{ij,\alpha\beta} x_i x_j \right) s^\alpha t^\beta \\ &= \sum_{\alpha=0}^{m-1} \sum_{\beta=0}^{n-1} Q_{k,\alpha\beta}(x_0, x_1, x_2, x_3) s^\alpha t^\beta \end{aligned} \tag{4}$$

where $Q_{k,\alpha\beta}$ is a quadric in x_i with coefficients in \mathbb{C} . To simplify the notation somewhat, we have identified the bihomogeneous monomial $s^\alpha u^{m-1-\alpha} t^\beta v^{n-1-\beta}$ with its particular dehomogenized form $s^\alpha t^\beta$ obtained by taking $u = v = 1$. We will continue with this convention whenever it simplifies the notation, and is not likely to introduce any confusion. Arrange the $Q_{k,\alpha\beta}$ into a square matrix M of size $mn \times mn$, where the columns of the matrix M are indexed by the monomial basis $\{s^\alpha t^\beta\}_{\alpha=0,\beta=0}^{m-1,n-1}$ of $R_{m-1,n-1}$, and the rows

are indexed by the mn moving quadrics Q_k ($1 \leq k \leq mn$). Since each entry of M is a quadric in x_i , we may write

$$M = [Q_{k,\alpha\beta}],$$

so that the determinant of M , denoted as usual by $|M|$, is a polynomial in the variables x_i of degree $\leq 2mn$. One of the main results of Cox et al. (2000) uses the matrix M to give a determinantal equation for $S = \text{Im}(\varphi)$.

Theorem 1.3. *Suppose that $\varphi : \mathbf{P}^1 \times \mathbf{P}^1 \rightarrow \mathbf{P}^3$ has no base points and is generically one-to-one. If MP has maximal rank, then so does MQ and furthermore, the image of φ is defined by the determinantal equation $|M| = 0$.*

Proof. See Cox (2001, Theorem 3.1). \square

1.2. Goals and organization

The goal of this paper is to prove a result similar to Theorem 1.3 in which base points of φ are allowed so long as each base point is a local complete intersection and the total multiplicity of all base points does not exceed mn . In the case that $\mathbf{P}^1 \times \mathbf{P}^1$ is replaced by \mathbf{P}^2 , a similar extension has already been done by Busé et al. (2003, Theorem 3.6). The strategy is to replace certain of the moving quadrics in the matrix M with linearly independent moving planes that exist because of the presence of the base points. The proof of the result of Busé, Cox, and D’Andrea requires some deep results of commutative algebra, related to the concept of Castelnuovo–Mumford regularity of a module (Bayer and Mumford, 1993; Mumford, 1966), and a result of Chandler concerning the regularity of the powers of an ideal in certain cases (Chandler, 1997). The notion of regularity is a concept that is only defined for graded modules (or sheaves of modules on \mathbf{P}^n), which is the reason that the arguments of Busé et al. work for triangular parametrizations $\varphi : \mathbf{P}^2 \rightarrow \mathbf{P}^3$, but not for tensor product parametrizations $\varphi : \mathbf{P}^1 \times \mathbf{P}^1 \rightarrow \mathbf{P}^3$. To prove an extension of Busé et al. to the case of a tensor product parametrization, it is useful to have an extension of the concept of regularity of a module, which is traditionally a concept for graded modules, to cover the case of bigraded modules. This extension was developed in a recent series of papers by Hoffman and Wang (in press-a, in press-b). We will start by summarizing the results needed from these papers, and prove some additional results needed for the application to our implicitization problem. Most notably, we will provide a version of Chandler’s result (Chandler, 1997, Theorem 4) on the regularity of the powers of a homogeneous ideal I in $S = k[X_0, \dots, X_n]$ when $\dim S/I = 0$. The version of this regularity of powers result that is needed is Theorem 2.10. It is worth noting that Theorem 2.10 is not a straightforward extension of Chandler’s result. Chandler’s result follows immediately from the Bayer–Stillman criterion for regularity (Bayer and Stillman, 1987), but Theorem 2.10 requires a more subtle cohomological argument using the biregularity theory developed in Hoffman and Wang (in press-a), and it applies not to the powers I^k , but rather to the saturation of I^k . This is, however, adequate for the application to the implicitization problem.

The paper is organized as follows. Section 2 summarizes some of the results needed from Hoffman and Wang (in press-a, in press-b), and proves the theorem on the

biregularity of the saturation of the powers of an ideal (Theorem 2.10). Section 3 gives some additional results on finite subschemes of $\mathbf{P}^1 \times \mathbf{P}^1$ needed for the application to the implicitization theorem, which is then proved in Section 4 (Theorem 4.15).

2. Bigraded regularity and saturation

We will start by recalling the definition and some of the results concerning bigraded regularity and saturation as developed in Hoffman and Wang (in press-a). Our main goal in this section is a bound on the bigraded regularity of the saturation of a power of an ideal. This result is inspired by results of Chandler (1997).

In this section we will work over the polynomial ring

$$R = K[x_0, \dots, x_m, y_0, \dots, y_n]$$

where K is an infinite field, and $m, n \geq 1$. We will make R into a bigraded K -algebra in the normal manner by assigning the bidegree $(1, 0)$ to the x_i variables and the bidegree $(0, 1)$ to the y_j variables. Moreover, we will partially order \mathbb{Z}^2 by the rule $(k, l) \leq (r, s)$ if $k \leq r$ and $l \leq s$. As usual, we let $R_{k,l}$ denote the K -subspace of R consisting of bihomogeneous polynomials of bidegree (k, l) , and if M is a bigraded R -module, then $M_{k,l}$ is the (k, l) bihomogeneous part of M . Let

$$\mathfrak{m} = \langle x_0, \dots, x_m \rangle \cap \langle y_0, \dots, y_n \rangle = \langle \{x_i y_j : 0 \leq i \leq m, 0 \leq j \leq n\} \rangle \subset R.$$

The ideal \mathfrak{m} is known as the *irrelevant ideal* of R . Moreover we note that the local cohomology modules $H_{\mathfrak{m}}^i(M)$ are naturally bigraded R -modules. For the basic notation and properties of local cohomology, consult Brodmann and Sharp (1998), Hermann et al. (1988, Chapter VII), or the original notes of Grothendieck (1967).

Definition 2.1 (See Hoffman and Wang, in press-a, Definition 3.1). We say that a bigraded R -module M is (p, p') -regular if for all $i \geq 0$,

$$H_{\mathfrak{m}}^i(M)_{k,k'} = 0 \text{ whenever } (k, k') \in \text{Reg}_{i-1}(p, p'),$$

where $\text{Reg}_j(p, p') = \{(x, y) \in \mathbb{Z}^2 : x \geq p - j, y \geq p' - j, x + y \geq p + p' - j - 1\}$.

Remark 2.2. The definition of (p, p') -regular given here coincides with what is called *weakly* (p, p') -regular in Hoffman and Wang (in press-a). In that paper, a concept known as *strongly* (p, p') -regular is also introduced and studied. Since we will not need this stronger version in this paper, we shall simply use the term (p, p') -regular for what would normally be referred to as weakly (p, p') -regular.

The definition given above for (p, p') -regularity is an extension to bigraded modules of the concept of Castelnuovo regularity for graded modules as found, for example, in Ooishi (1982). The concept of regularity was originally defined for coherent sheaves of modules on projective space by Mumford (1966), and this version of regularity is also treated in the bigraded case in Hoffman and Wang (in press-a). The reader is referred to this paper for a precise comparison of the two concepts. However, in case the bigraded R -module M is an ideal $I \subset R$ generated by bihomogeneous polynomials and $\mathcal{I} \subset \mathcal{O}_X$ (where $X = \mathbf{P}^m \times \mathbf{P}^n$)

is the corresponding sheaf of ideals in the structure sheaf \mathcal{O}_X , the equivalence is expressed by the following result. For a brief introduction to the Proj construction in the multigraded situation, see Hyry (1999, Section 1). In analogy with the ordinary graded case, $\mathcal{I}(k, k')$ denotes the twisting of \mathcal{I} in bidegree (k, k') .

Proposition 2.3. *With the above notation, the bihomogeneous ideal $I \subset R$ is (p, p') -regular if and only if the natural map*

$$I_{p,p'} \rightarrow H^0(X, \mathcal{I}(p, p'))$$

is an isomorphism and for all $i \geq 1$,

$$H^i(X, \mathcal{I}(k, k')) = 0 \text{ whenever } (k, k') \in \text{Reg}_i(p, p').$$

Moreover, if I is (p, p') -regular, then the natural map

$$I_{d,d'} \rightarrow H^0(X, \mathcal{I}(d, d'))$$

is an isomorphism for all $(d, d') \geq (p, p')$.

Proof. See Hoffman and Wang (in press-a, Proposition 3.5). \square

Definition 2.4. Let M be a bigraded submodule of a finitely generated free R -module F . The saturation of the module M , denoted by M^{sat} or $\text{sat}(M)$ is the submodule of F defined by

$$M^{\text{sat}} = \{f \in F : \mathfrak{m}^k f \subset M, \text{ for some } k\}.$$

The submodule M is said to be saturated if $M = M^{\text{sat}}$, while M is (p, p') -saturated if

$$M_{k,k'}^{\text{sat}} = M_{k,k'} \text{ for all } (k, k') \geq (p, p').$$

Lemma 2.5. *Let $R = K[x_0, \dots, x_m, y_0, \dots, y_n]$ where $(m, n) \geq (1, 1)$ and let M be a bigraded submodule of a free R -module F of finite rank. Then*

- (1) $H_{\mathfrak{m}}^0(M) = 0$, and
- (2) $H_{\mathfrak{m}}^1(M) \cong M^{\text{sat}}/M$.

Proof. $H_{\mathfrak{m}}^0(M) = \cup_n (0 :_M \mathfrak{m}^n) = 0$ since M is a submodule of a free module F . The long exact cohomology sequence for

$$0 \longrightarrow M \longrightarrow F \longrightarrow F/M \longrightarrow 0$$

has a segment

$$H_{\mathfrak{m}}^0(F) \longrightarrow H_{\mathfrak{m}}^0(F/M) \longrightarrow H_{\mathfrak{m}}^1(M) \longrightarrow H_{\mathfrak{m}}^1(F).$$

Since F is free and $(m, n) \geq (1, 1)$, it follows that $\text{grade}_F(\mathfrak{m}) \geq 2$, so that $H_{\mathfrak{m}}^i(F) = 0$ for $i = 0, 1$ (see Brodmann and Sharp (1998, Theorem 6.2.7, Page 109)). Thus there is an isomorphism

$$H_{\mathfrak{m}}^1(M) \cong H_{\mathfrak{m}}^0(F/M) = M^{\text{sat}}/M. \quad \square$$

Proposition 2.6. *Let M be a bigraded submodule of a free R -module F of finite rank and let \mathcal{M} be the corresponding coherent sheaf of modules on $X = \mathbf{P}^m \times \mathbf{P}^n$. Then*

$$M_{k,l}^{\text{sat}} = H^0(X, \mathcal{M}(k, l)).$$

Proof. For any finitely generated bigraded R -module M there is an exact sequence (see Hyry (1999, Corollary 1.5)):

$$0 \rightarrow H_{\mathbf{m}}^0(M) \rightarrow M \rightarrow \bigoplus_{(a,b) \in \mathbb{Z}^2} H^0(X, \mathcal{M}(a, b)) \rightarrow H_{\mathbf{m}}^1(M) \rightarrow 0.$$

Since M and M^{sat} generate the same sheaf \mathcal{M} on X , we can apply this exact sequence with M replaced by M^{sat} . Lemma 2.5 shows that $H_{\mathbf{m}}^i(M^{\text{sat}}) = 0$ for $i = 0, 1$, and the result follows. \square

Corollary 2.7. *If M is a bigraded submodule of a free R -module F of finite rank, then M is (p, p') -saturated if and only if $H_{\mathbf{m}}^1(M)_{k,k'} = 0$ for all $(k, k') \geq (p, p')$. Moreover, if M is (p, p') -regular, then M is (p, p') -saturated.*

The converse of the last statement is true in the case of dimension 0:

Lemma 2.8. *Let $I \subset R$ be a bihomogeneous ideal with $\dim R/I = 0$, where \dim refers to Krull dimension. Then the following are equivalent:*

- (1) I is (p, p') -saturated.
- (2) I is (p, p') -regular.
- (3) $I_{k,k'} = R_{k,k'}$ for all $(k, k') \geq (p, p')$.

Proof. ((1) \Leftrightarrow (3)) This is clear since I is (x, y) -primary.

((2) \Rightarrow (1)) Corollary 2.7.

((1) \Rightarrow (2)) According to Definition 2.1, we need to show that

$$H_{\mathbf{m}}^i(I)_{k,k'} = 0 \text{ whenever } (k, k') \in \text{Reg}_{i-1}(p, p'). \tag{*}$$

Since $H_{\mathbf{m}}^0(I) = 0$, (*) is certainly true for $i = 0$, and since I is (p, p') -saturated, $H_{\mathbf{m}}^1(I)_{k,k'} = I_{k,k'}^{\text{sat}}/I_{k,k'} = 0$ for all $(k, k') \in (p, p') + \mathbb{Z}_+^2 = \text{Reg}_0(p, p')$, where $\mathbb{Z}_+ = \{x \in \mathbb{Z} : x \geq 0\}$. Thus (*) is satisfied for $i = 1$. Now consider the case $i \geq 2$. The long exact cohomology sequence of the exact sequence

$$0 \longrightarrow I \longrightarrow R \longrightarrow R/I \longrightarrow 0$$

contains the segment

$$H_{\mathbf{m}}^{i-1}(R/I) \longrightarrow H_{\mathbf{m}}^i(I) \longrightarrow H_{\mathbf{m}}^i(R) \longrightarrow H_{\mathbf{m}}^i(R/I).$$

Since $\dim R/I = 0$, it follows that $H_{\mathbf{m}}^i(R/I) = 0$ for $i \geq 1$, so that if $i \geq 2$, we conclude that $H_{\mathbf{m}}^i(I) = H_{\mathbf{m}}^i(R)$. By Hoffman and Wang (in press-a, Proposition 4.3 and Corollary 4.5), R is $(0, 0)$ -regular, and by Hoffman and Wang (in press-a, Theorem 3.4), it follows that R is (p, p') -regular for all $(p, p') \geq (0, 0)$. Therefore, $H_{\mathbf{m}}^i(I)_{k,k'} = H_{\mathbf{m}}^i(R)_{k,k'} = 0$ for all $(k, k') \in \text{Reg}_{i-1}(p, p')$. Thus, (*) is also satisfied for $i \geq 2$, and hence I is (p, p') -regular. \square

With this background out of the way we can proceed with a discussion of the results on regularity of the powers of a bihomogeneous ideal that will be needed for the implicitization problem.

Proposition 2.9. *Let $I \subset R$ generated by bihomogeneous forms of bidegree $\leq (r, r')$, and assume that I is (p, p') -regular. If $\dim R/I = 0$, then I^e is (l, l') -regular for some $(l, l') \leq ((e - 1)r + p, (e - 1)r' + p')$.*

Proof. The proof is by induction on e . The result is true for $e = 1$ by assumption. Since $\dim R/I = \dim R/I^e = 0$, we can proceed by induction, and assume that I^{e-1} is $((e - 2)r + p, (e - 2)r' + p')$ -regular.

According to Lemma 2.8, we need to show that $I_{k,k'}^e = R_{k,k'}$ for any $(k, k') \geq ((e - 1)r + p, (e - 1)r' + p')$. For this, it will suffice to show that $M \in I^e$ for every monomial M of bidegree (k, k') , where $(k, k') \geq ((e - 1)r + p, (e - 1)r' + p')$. Thus let M be an arbitrary monomial of bidegree $(k, k') \geq ((e - 1)r + p, (e - 1)r' + p')$. Write M as a product $M = NN'$ where N and N' are monomials of bidegrees (p, p') and $(k - p, k' - p')$, respectively. Suppose that $I = \langle f_1, \dots, f_s \rangle$, where $\text{bideg}(f_i) = (d_i, d'_i) \leq (r, r')$, for all i . Since I is (p, p') -regular, $N \in I$ by Lemma 2.8. Thus we can write $N = \sum_{i=1}^s N_i f_i$, where $\text{bideg}(N_i) = (n_i, n'_i) = (p - d_i, p' - d'_i) \geq (p - r, p' - r')$, so that

$$\text{bideg}(N_i N') \geq (k - r, k - r') \geq ((e - 2)r + p, (e - 2)r' + p').$$

By the induction hypothesis, $N_i N' \in I^{e-1}$, and hence $M = \sum_{i=1}^s N_i N' f_i \in I^{e-1} I = I^e$. By Lemma 2.8, we conclude that I^e is $((e - 1)r + p, (e - 1)r' + p')$ -regular. \square

Theorem 2.10. *Let $I \subset R$ be a bihomogeneous ideal, and assume that*

- (1) $\mathbb{V}(I) \subset \mathbf{P}^m \times \mathbf{P}^n$ is finite;
- (2) I is (p, p') -regular;
- (3) I is generated by forms of bidegree $\leq (r, r')$.

Then $J = \text{sat}(I^e)$ is $((e - 1)r + p, (e - 1)r' + p')$ -regular.

Proof. We will let $X = \mathbf{P}^m \times \mathbf{P}^n$ and Z will denote the finite subscheme $\mathbb{V}(I)$. The proof is by induction on e .

Suppose $e = 1$. In this case, it is necessary to show $J = \text{sat}(I)$ is (p, p') -regular. By Proposition 2.3, it is sufficient to show that

$$J_{p,p'} \cong H^0(X, \mathcal{J}(p, p')), \tag{5}$$

and for all $i \geq 1$,

$$H^i(X, \mathcal{J}(k, k')) = 0 \text{ whenever } (k, k') \in \text{Reg}_i(p, p'), \tag{6}$$

where \mathcal{J} denotes the sheaf of ideals generated by the ideal J . Eq. (5) follows from Proposition 2.6. Since I and $J = \text{sat}(I)$ determine the same sheaf of ideals \mathcal{J} , Eq. (6) follows from Proposition 2.3 and the assumption that I is (p, p') -regular. Thus the theorem holds for $e = 1$.

Now assume that $e \geq 2$. The sheaf of ideals \mathcal{J} generated by the ideal J is \mathcal{I}^e , where \mathcal{I} is the sheaf of ideals generated by I . According to Proposition 2.6, we conclude that

$H^0(X, \mathcal{I}^e(k, k')) = J_{k, k'}$ for all (k, k') . Thus the first condition in Proposition 2.3 for (l, l') regularity of J , where $(l, l') = ((e - 1)r + p, (e - 1)r' + p')$, is satisfied. To verify the second condition, we must show that for all $i \geq 1$,

$$H^i(X, \mathcal{I}^e(k, k')) = 0 \text{ whenever } (k, k') \in \text{Reg}_i(l, l').$$

Tensor the exact sequence

$$0 \longrightarrow \mathcal{I}^e \longrightarrow \mathcal{I}^{e-1} \longrightarrow \mathcal{I}^{e-1}/\mathcal{I}^e \longrightarrow 0$$

with $\mathcal{O}(k, k')$ and consider the resulting cohomology sequence. Since the support of $\mathcal{I}^{e-1}/\mathcal{I}^e$ is contained in Z , which is 0-dimensional, it follows that $H^i(X, (\mathcal{I}^{e-1}/\mathcal{I}^e)(k, k')) = 0$ for $i \geq 1$. Therefore,

$$H^i(X, \mathcal{I}^e(k, k')) = H^i(X, \mathcal{I}^{e-1}(k, k')) \text{ for all } i \geq 2,$$

and the latter group vanishes by induction for all

$$\begin{aligned} (k, k') \in \text{Reg}_i((e - 2)r + p, (e - 2)r' + p') \\ \supset \text{Reg}_i((e - 1)r + p, (e - 1)r' + p'). \end{aligned}$$

Thus, we have the required vanishing for $i \geq 2$. Now look at the sequence

$$\begin{aligned} H^0(X, \mathcal{I}^{e-1}(k, k')) &\xrightarrow{\eta} H^0(X, (\mathcal{I}^{e-1}/\mathcal{I}^e)(k, k')) \\ &\longrightarrow H^1(X, \mathcal{I}^e(k, k')) \longrightarrow H^1(X, \mathcal{I}^{e-1}(k, k')). \end{aligned}$$

By induction, the last term vanishes for all $(k, k') \in \text{Reg}_1(l, l')$, so that the next-to-last term will vanish there provided we show that η is onto for those same (k, k') .

To see that η is onto, first note that

$$H^0(X, \mathcal{I}^{e-1}(k, k')) = (I^{e-1})_{k, k'}^{\text{sat}}.$$

Next suppose that $Z = \{\mathbf{p}_1, \dots, \mathbf{p}_s\}$, and note that, since the support is finite, we have

$$H^0(X, (\mathcal{I}^{e-1}/\mathcal{I}^e)(k, k')) = \bigoplus_{\mathbf{p} \in Z} (I^{e-1}\mathcal{O}_{X, \mathbf{p}}/I^e\mathcal{O}_{X, \mathbf{p}})(k, k').$$

Thus, the map $\eta : H^0(X, \mathcal{I}^{e-1}(k, k')) \rightarrow H^0(X, (\mathcal{I}^{e-1}/\mathcal{I}^e)(k, k'))$ can be described as

$$\eta : (I^{e-1})_{k, k'}^{\text{sat}} \rightarrow \bigoplus_{\mathbf{p} \in Z} (I^{e-1}\mathcal{O}_{X, \mathbf{p}}/I^e\mathcal{O}_{X, \mathbf{p}})(k, k')$$

where, for each $g \in (I^{e-1})_{k, k'}^{\text{sat}}$,

$$\eta(g) = (g_1, \dots, g_s)$$

with g_i denoting the image of the bihomogeneous form g of bidegree (k, k') in the sheaf $(I^{e-1}\mathcal{O}_{X, \mathbf{p}_i}/I^e\mathcal{O}_{X, \mathbf{p}_i})(k, k')$ supported at \mathbf{p}_i .

Thus, to show that η is surjective for $(k, k') \in \text{Reg}_1(l, l')$ (which is the range of interest to us), it is sufficient to show that for any given element

$$\left(\frac{u_1}{v_1}, \dots, \frac{u_s}{v_s} \right) \in \bigoplus_i (I^{e-1}\mathcal{O}_{X, \mathbf{p}_i}/I^e\mathcal{O}_{X, \mathbf{p}_i})(k, k'),$$

where u_i and v_i are bihomogeneous forms with $\text{bideg } u_i - \text{bideg } v_i = (k, k')$, and $u_i \in I^{e-1}$, we can find a single bihomogeneous $g \in (I^{e-1})_{k,k'}^{\text{sat}}$ such that the image of g in $(I^{e-1}\mathcal{O}_{X,\mathbf{p}_i}/I^e\mathcal{O}_{X,\mathbf{p}_i})(k, k')$ is u_i/v_i for $1 \leq i \leq s$. But $g = u_i/v_i \in (I^{e-1}\mathcal{O}_{X,\mathbf{p}_i}/I^e\mathcal{O}_{X,\mathbf{p}_i})(k, k')$ means that there is a form H_i with $H_i(\mathbf{p}_i) \neq 0$, such that

$$H_i(gv_i - u_i) \in I^e. \tag{7}$$

Hence, to prove that η is surjective, we need to produce g and the forms H_i for $1 \leq i \leq s$ that satisfy Eq. (7).

By assumption 3, I is generated by bihomogeneous elements f_1, \dots, f_r with bidegree $(m_i, m'_i) \leq (r, r')$. We can write

$$u_i = \sum a_{ij} f_j, \quad \text{for some } a_{ij} \in I^{e-2},$$

where a_{ij} is bihomogeneous of bidegree $(k - m_j, k' - m'_j) + \text{bideg } v_i$. Note that $(\alpha, \alpha') = (k - m_j, k' - m'_j) \in \text{Reg}_1((e - 2)r + p, (e - 2)r' + p')$, by our initial choice of (k, k') . Tensor the following exact sequence

$$0 \longrightarrow \mathcal{I}^{e-1} \longrightarrow \mathcal{I}^{e-2} \longrightarrow \mathcal{I}^{e-2}/\mathcal{I}^{e-1} \longrightarrow 0$$

with $\mathcal{O}_X(\alpha, \alpha')$, and consider the resulting cohomology sequence

$$\begin{aligned} H^0(X, \mathcal{I}^{e-2}(\alpha, \alpha')) &\xrightarrow{\psi} H^0(X, (\mathcal{I}^{e-2}/\mathcal{I}^{e-1})(\alpha, \alpha')) \\ &\longrightarrow H^1(X, \mathcal{I}^{e-1}(\alpha, \alpha')) \longrightarrow H^1(X, \mathcal{I}^{e-2}(\alpha, \alpha')). \end{aligned}$$

By our induction hypothesis, the third term vanishes, so that ψ is onto for this (α, α') . This means that for every j , and each

$$\left(\frac{a_{1j}}{v_1}, \dots, \frac{a_{sj}}{v_s} \right) \in \bigoplus_i (I^{e-2}\mathcal{O}_{X,\mathbf{p}_i}/I^{e-1}\mathcal{O}_{X,\mathbf{p}_i})(k - m_j, k' - m'_j)$$

we can find a bihomogeneous $g_j \in (I^{e-2})_{\alpha,\alpha'}^{\text{sat}}$ and forms H_{ij} with $H_{ij}(\mathbf{p}_i) \neq 0$, such that

$$H_{ij}(g_j v_i - a_{ij}) \in I^{e-1} \quad \text{for all } i. \tag{8}$$

We may replace each H_{ij} by $H_i = \prod_j H_{ij}$. Multiply Eq. (8) by f_j and sum the result over j and define $g = \sum g_j f_j \in (I^{e-1})_{k,k'}^{\text{sat}}$. Then we have obtained Eq. (7), as required. \square

3. Finite subschemes of $\mathbf{P}^1 \times \mathbf{P}^1$

This section will be devoted to a presentation of several results which can be proved for bihomogeneous ideals I that define finite subschemes of $\mathbf{P}^1 \times \mathbf{P}^1$, but that do not necessarily have immediate analogues for subschemes of general biprojective spaces. The results proved are analogous to the results proved in Busé et al. (2003) for application to the implicitization problem for maps $\varphi : \mathbf{P}^2 \rightarrow \mathbf{P}^3$. Our results will be similarly applied for the implicitization of maps $\varphi : \mathbf{P}^1 \times \mathbf{P}^1 \rightarrow \mathbf{P}^3$. Since we are restricting ourselves to

this low dimensional case, we will let R be the polynomial ring $\mathbb{C}[s, u, t, v]$ in the variables $s, u, t,$ and $v,$ where, as usual, the bigrading of R is given by setting the bidegree of s and u to be $(1, 0)$ and the bidegree of t and v to be $(0, 1)$. If $I = \langle f_1, \dots, f_r \rangle \subset R$ is an ideal generated by forms all of the same bidegree (m, n) with $m, n \geq 1,$ then there is a rational map $\varphi_I : \mathbf{P}^1 \times \mathbf{P}^1 \rightarrow \mathbf{P}^{r-1}$ defined by

$$\varphi_I = [f_1(s, u; t, v), \dots, f_r(s, u; t, v)].$$

Note that the polynomial ring $\mathbb{C}[s, t, v]$ inherits a bigrading as a subring of $R = \mathbb{C}[s, u, t, v],$ so that a polynomial $f(s, t, v)$ is bihomogeneous with bidegree (m, n) if and only if $f(s, t, v) = \sum_{j=0}^n a_{ij} s^m t^i v^{n-j} = s^m g(t, v),$ where $g(t, v)$ is homogeneous of degree $n.$

Lemma 3.1. *Let $\bar{I} \subset S = \mathbb{C}[s, t, v]$ be an ideal, minimally generated by r bihomogeneous forms of bidegree $(m, n).$ That is, $\bar{I} = s^m J$ where $J \subset \mathbb{C}[t, v]$ is an ideal minimally generated by homogeneous polynomials of degree $n.$ If $\mathbb{V}(J) = \emptyset$ in $\mathbf{P}^1,$ then \bar{I} is (p, p') -regular for all $p \geq m$ and $p' \geq 2n - r + 1.$*

Proof. This follows from Hoffman and Wang (in press-a, Remark 4.12) and Lemma B.1 in Busé et al. (2003). \square

Remark 3.2. Similarly, let $\bar{I} \subset S = \mathbb{C}[s, u, t]$ be an ideal, minimally generated by r bihomogeneous forms of bidegree $(m, n).$ That is $\bar{I} = t^n J$ where J is an ideal minimally generated by homogeneous polynomials in $\mathbb{C}[s, u]$ of degree $m.$ If $\mathbb{V}(J) = \emptyset$ in $\mathbf{P}^1,$ then \bar{I} is (p, p') -regular for all $p \geq 2m - r + 1$ and $p' \geq n.$

Lemma 3.3. *Let $I \subset R = \mathbb{C}[s, u, t, v]$ be minimally generated by $r \geq 4$ bihomogeneous forms of bidegree (m, n) with both $m, n \geq 1.$ Assume that $\dim \text{Im}(\varphi_I) = 2$ and that $\mathbb{V}(I) \subset \mathbf{P}^1 \times \mathbf{P}^1$ is finite. Given $\ell \in R_{1,0},$ let I_ℓ be the image of I in the quotient ring $R/(\ell).$ Then for a generic ℓ, I_ℓ is minimally generated by at least 2 elements.*

Proof. The proof is a straightforward modification of the Bertini theorem argument in Busé et al. (2003, Lemma B.2). See also Wang (2003, Lemma 3.4.3). \square

Remark 3.4. The above result is also true if the given generic element ℓ is chosen from $R_{0,1}.$

The following is the main vanishing theorem needed for our application.

Theorem 3.5. *Let $I \subset R = \mathbb{C}[s, u, t, v]$ be minimally generated by $r \geq 4$ bihomogeneous forms of bidegree $(m, n).$ Assume that $\dim \text{Im}(\varphi_I) = 2$ and assume that $\mathbb{V}(I) \subset \mathbf{P}^1 \times \mathbf{P}^1$ is finite. If \mathcal{I} is the associated sheaf of ideals on $X = \mathbf{P}^1 \times \mathbf{P}^1,$ then*

- (1) $H^1(X, \mathcal{I}(k, k')) = 0$ for all $(k, k') \geq (2m - 2, 2n - 2),$ and
- (2) $H^2(X, \mathcal{I}(k, k')) = 0$ for all $(k, k') \geq (0, 0).$

Proof. If $Z = \mathbb{V}(I) \subset X = \mathbf{P}^1 \times \mathbf{P}^1,$ there is an exact sequence

$$0 \rightarrow \mathcal{I} \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_Z \rightarrow 0,$$

which, upon taking the tensor product with $\mathcal{O}_X(k, k')$, gives rise to a long exact cohomology sequence that contains the segment

$$\begin{aligned} H^1(X, \mathcal{O}_Z(k, k')) &\rightarrow H^2(X, \mathcal{I}(k, k')) \\ &\rightarrow H^2(X, \mathcal{O}_X(k, k')) \rightarrow H^2(X, \mathcal{O}_Z(k, k')). \end{aligned}$$

Since Z is finite, $H^i(X, \mathcal{O}_Z(k, k')) = 0$ for all $i \geq 1$, so

$$H^2(X, \mathcal{I}(k, k')) = H^2(X, \mathcal{O}_X(k, k')).$$

By the Künneth formula (Sampson and Washnitzer, 1959), for all $(k, k') \geq (0, 0)$,

$$H^2(X, \mathcal{O}_X(k, k')) \cong \bigoplus_{i+j=2} H^i(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(k)) \otimes H^j(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(k')) = 0.$$

The last equality follows from the cohomology of projective space (Hartshorne, 1977, Chapter III, Theorem 5.1). This proves item (2).

To prove the first statement, choose a line $\ell \in R_{1,0}$ such that $\mathbb{V}(\ell) \cap \mathbb{V}(I) = \emptyset$ and $\bar{I} = I_\ell =$ the image of I in $R/\langle \ell \rangle$ is minimally generated by at least two elements. This is possible by Lemma 3.3. Then by Lemma 3.1, we know that \bar{I} is (p, p') -regular for $p \geq m$ and $p' \geq 2n - 1$. If $\bar{\mathcal{I}}$ is the sheaf on $\mathbb{V}(\ell) \cong \mathbf{P}^1$ associated to \bar{I} , then by Proposition 2.3, we have

$$\begin{aligned} \bar{I}_{k,k'} &\cong H^0(\mathbb{V}(\ell), \bar{\mathcal{I}}(k, k')) \text{ for all } (k, k') \geq (m, 2n - 1), \text{ and} \\ H^1(\mathbb{V}(\ell), \bar{\mathcal{I}}(k, k')) &= 0 \text{ for all } (k, k') \geq (m - 1, 2n - 2). \end{aligned} \tag{9}$$

Now, we consider the following exact sheaf sequence:

$$0 \rightarrow \mathcal{O}_{\mathbf{P}^1 \times \mathbf{P}^1}(-1, 0) \rightarrow \mathcal{O}_{\mathbf{P}^1 \times \mathbf{P}^1} \rightarrow \mathcal{O}_{\mathbb{V}(\ell)} \cong \mathcal{O}_{\mathbf{P}^1} \rightarrow 0.$$

Tensoring with $\mathcal{I}(k, k')$ gives the exact sequence:

$$\text{Tor}_1^{\mathcal{O}_{\mathbf{P}^1 \times \mathbf{P}^1}}(\mathcal{I}(k, k'), \mathcal{O}_{\mathbf{P}^1}) \rightarrow \mathcal{I}(k - 1, k') \rightarrow \mathcal{I}(k, k') \rightarrow \mathcal{O}_{\mathbf{P}^1} \otimes_{\mathcal{O}_{\mathbf{P}^1 \times \mathbf{P}^1}} \mathcal{I}(k, k') \rightarrow 0.$$

Note $\mathcal{O}_{\mathbf{P}^1} \otimes_{\mathcal{O}_{\mathbf{P}^1 \times \mathbf{P}^1}} \mathcal{I}(k, k') \cong \bar{\mathcal{I}}(k, k')$. Since $\mathcal{O}_{\mathbf{P}^1}$ is supported on $\mathbb{V}(\ell)$, the *Tor*-sheaf is supported there. Also, for $p \notin \mathbb{V}(I)$, the sheaf $\mathcal{I}(k, k')$ is locally free. Hence the *Tor*-sheaf vanishes at p if $p \notin \mathbb{V}(I)$. Hence the support of the *Tor*-sheaf is contained in $\mathbb{V}(I) \cap \mathbb{V}(\ell)$. By the generic choice of ℓ , $\mathbb{V}(I) \cap \mathbb{V}(\ell) = \emptyset$, so the *Tor*-sheaf vanishes. Thus there is an exact sheaf sequence

$$0 \rightarrow \mathcal{I}(k - 1, k') \rightarrow \mathcal{I}(k, k') \rightarrow \bar{\mathcal{I}}(k, k') \rightarrow 0$$

that gives the following commutative diagram

$$\begin{array}{ccccc} I_{k,k'} & \longrightarrow & \bar{I}_{k,k'} & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow \\ H^0(X, \mathcal{I}(k, k')) & \xrightarrow{\alpha} & H^0(\mathbb{V}(\ell), \bar{\mathcal{I}}(k, k')) & \xrightarrow{\beta} & H^1(X, \mathcal{I}(k - 1, k')) \\ & \longrightarrow & H^1(X, \mathcal{I}(k, k')) & \longrightarrow & H^1(\mathbb{V}(\ell), \bar{\mathcal{I}}(k, k')) \end{array}$$

with exact rows. If $(k, k') \geq (m, 2n - 1)$, Eq. (9) shows that

$$H^1(\mathbb{V}(\ell), \bar{\mathcal{I}}(k, k')) = 0 \quad \text{and} \quad \bar{I}_{k,k'} \cong H^0(\mathbb{V}(\ell), \bar{\mathcal{I}}(k, k')).$$

Therefore, α is onto, and β is zero, which implies that there is an isomorphism

$$H^1(X, \mathcal{I}(k - 1, k')) \cong H^1(X, \mathcal{I}(k, k')), \quad \text{for all } (k, k') \geq (m, 2n - 1).$$

An analogous argument with a generic line $\ell \in R_{0,1}$ produces another isomorphism

$$H^1(X, \mathcal{I}(k, k' - 1)) \cong H^1(X, \mathcal{I}(k, k')), \quad \text{for all } (k, k') \geq (2m - 1, n).$$

Therefore,

$$H^1(X, \mathcal{I}(k - 1, k' - 1)) \cong H^1(X, \mathcal{I}(k, k')), \quad \text{for all } (k, k') \geq (2m - 1, 2n - 1).$$

Since $H^1(X, \mathcal{I}(m, n)) = 0$ if $(m, n) \gg (0, 0)$, we conclude that

$$H^1(X, \mathcal{I}(k, k')) = 0$$

for all $(k, k') \geq (2m - 2, 2n - 2)$. \square

We are now able to prove the following result relating regularity of the ideal I and the degree of the 0-dimensional subscheme $\mathbb{V}(I)$. This is one of the main results needed for the application to the implicitization problem.

Theorem 3.6. *Let $I \subset R$ be minimally generated by $r \geq 4$ bihomogeneous forms of bidegree (m, n) with $m, n \geq 1$. Assume that $Z = \mathbb{V}(I) \subset X = \mathbf{P}^1 \times \mathbf{P}^1$ is finite and $\dim \text{Im}(\varphi_I) = 2$. If $(p, p') \geq (2m - 1, 2n - 1)$, then I is (p, p') -regular if and only if $\dim_{\mathbb{C}}(R/I)_{p,p'} = \deg(\mathbb{V}(I))$, where $\deg(\mathbb{V}(I))$ denotes the degree of the 0-dimensional subscheme $\mathbb{V}(I)$.*

Proof. When $p \geq 2m - 1$ and $p' \geq 2n - 1$, Theorem 3.5 implies

$$H^1(X, \mathcal{I}(p, p')) = 0.$$

Thus, the exact sheaf sequence

$$0 \rightarrow \mathcal{I} \rightarrow \mathcal{O}_{\mathbf{P}^1 \times \mathbf{P}^1} \rightarrow \mathcal{O}_Z \rightarrow 0$$

produces an exact sequence

$$0 \rightarrow H^0(X, \mathcal{I}(p, p')) \rightarrow H^0(X, \mathcal{O}_X(p, p')) \rightarrow H^0(Z, \mathcal{O}_Z(p, p')) \rightarrow 0.$$

This gives the following commutative diagram with exact rows:

$$\begin{array}{ccccccc} 0 & \rightarrow & H^0(X, \mathcal{I}(p, p')) & \rightarrow & H^0(X, \mathcal{O}_X(p, p')) & \rightarrow & H^0(Z, \mathcal{O}_Z(p, p')) \rightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \rightarrow & I_{p,p'} & \rightarrow & R_{p,p'} & \rightarrow & (R/I)_{p,p'} \rightarrow 0. \end{array}$$

We have $R_{p,p'} = H^0(X, \mathcal{O}_X(p, p'))$ and if I is (p, p') -regular, then $I_{p,p'} = H^0(X, \mathcal{I}(p, p'))$. The 5-lemma then shows that $(R/I)_{p,p'} = H^0(Z, \mathcal{O}_Z(p, p'))$, so that

$$\dim_{\mathbb{C}}(R/I)_{p,p'} = \dim_{\mathbb{C}} H^0(Z, \mathcal{O}_Z(p, p')).$$

But

$$\dim_{\mathbb{C}} H^0(Z, \mathcal{O}_Z) = \deg(Z),$$

and since

$$\dim_{\mathbb{C}} H^0(Z, \mathcal{O}_Z) = \dim_{\mathbb{C}} H^0(Z, \mathcal{O}_Z(p, p'))$$

when Z is finite, we conclude that

$$\dim_{\mathbb{C}}(R/I)_{p,p'} = \deg(Z).$$

Conversely, suppose $\dim_{\mathbb{C}}(R/I)_{p,p'} = \deg(Z)$. Since $H^2(X, \mathcal{I}(k, k')) = 0$ for all $k, k' \geq 0$ by [Theorem 3.5](#), it follows from [Proposition 2.3](#) that to show I is (p, p') -regular, we only need to prove that

$$I_{p,p'} \cong H^0(X, \mathcal{I}(p, p')), \quad \text{and} \quad H^1(X, \mathcal{I}(p-1, p'-1)) = 0.$$

If $p \geq 2m - 1$ and $p' \geq 2n - 1$, then $H^1(X, \mathcal{I}(p-1, p'-1)) = 0$ by [Theorem 3.5](#). We know that the natural map $I_{p,p'} \rightarrow H^0(X, \mathcal{I}(p, p'))$ is injective, so it is enough to show that

$$\dim_{\mathbb{C}} I_{p,p'} = \dim_{\mathbb{C}} H^0(X, \mathcal{I}(p, p')).$$

From the exact sequence

$$0 \rightarrow H^0(X, \mathcal{I}(p, p')) \rightarrow R_{p,p'} \rightarrow H^0(Z, \mathcal{O}_Z(p, p')) \rightarrow 0,$$

we conclude that

$$\begin{aligned} \dim_{\mathbb{C}} H^0(X, \mathcal{I}(p, p')) &= \dim_{\mathbb{C}} R_{p,p'} - \dim_{\mathbb{C}} H^0(Z, \mathcal{O}_Z(p, p')) \\ &= \dim_{\mathbb{C}} R_{p,p'} - \deg(Z) = \dim_{\mathbb{C}} R_{p,p'} - \dim_{\mathbb{C}}(R/I)_{p,p'} = \dim_{\mathbb{C}} I_{p,p'}. \end{aligned}$$

Thus $I_{p,p'} \cong H^0(X, \mathcal{I}(p, p'))$ and I is (p, p') -regular. \square

Corollary 3.7. *Under the hypotheses of [Theorem 3.6](#), if I is (p, p') -regular, then $\dim_{\mathbb{C}}(R/I)_{k,k'} = \deg(\mathbb{V}(I))$ for all $(k, k') \geq (p, p')$.*

Proof. If I is (p, p') -regular, then I is (k, k') -regular for all $(k, k') \geq (p, p')$. \square

Example 3.8. If $I = \langle u^2t^2v, u^2t^3 + suv^3, s^2tv^2, s^2v^3 + s^2t^3 \rangle \subset \mathbb{C}[s, u, t, v]$, then $\mathbb{V}(I) = (0, 1; 0, 1) \in \mathbf{P}^1 \times \mathbf{P}^1$. In this case, each generator of I has bidegree $(m, n) = (2, 3)$ and $(2m - 1, 2n - 1) = (3, 5)$. A computation with Singular ([Greuel et al., 2001](#)) shows $\dim_{\mathbb{C}}(R/I)_{3,5} = \deg \mathbb{V}(I) = 2$. Therefore, I is $(3, 5)$ -regular by [Theorem 3.6](#).

We will conclude this section with a brief description of a result on syzygies that will be needed in the proof of our implicitization theorem.

Definition 3.9. Let $I = \langle r_1, \dots, r_n \rangle \subseteq R$ be an ideal generated by bihomogeneous elements of R . In analogy with the case of a rational map, we will say that $\mathbb{V}(I)$ is the base point scheme of I .

- (1) The syzygy submodule of I is the submodule of relations among the r_i ($1 \leq i \leq n$) defined by

$$\text{Syz}(r_1, \dots, r_n) = \{(a_1, \dots, a_n) \in R^n : a_1 r_1 + \dots + a_n r_n = 0\}.$$

- (2) A syzygy $(a_1, \dots, a_n) \in \text{Syz}(r_1, \dots, r_n)$ vanishes at the base points of I if, for each i , $a_i \in I^{\text{sat}}$.
- (3) A syzygy $(a_1, \dots, a_n) \in \text{Syz}(r_1, \dots, r_n)$ has bidegree (k, l) provided each a_i has bidegree (k, l) .
- (4) If $\mathbf{e}_i \in R^n$ denotes the standard basis vector with a 1 in the i th position and 0 elsewhere, then a basic Koszul syzygy for the ideal I is one of the form

$$\mathbf{s}_{ij} = r_j \mathbf{e}_i - r_i \mathbf{e}_j, \quad \text{for } i < j.$$

Since $(r_j)r_i + (-r_i)r_j = 0$ for $i \neq j$, it is clear that $\mathbf{s}_{ij} \in \text{Syz}(r_1, \dots, r_n)$. Let $\text{Kos}(r_1, \dots, r_n) \subset \text{Syz}(r_1, \dots, r_n)$ be the submodule generated by the basic Koszul syzygies. We refer to an arbitrary element of $\text{Kos}(r_1, \dots, r_n)$ as a Koszul syzygy.

The following result is the fundamental result relating the Koszul syzygies of the ideal I and the module of syzygies of I which vanish at the base points of I in the special situation that we will need for this paper. Before stating the result, we recall what is meant by local complete intersection.

To say that I is a local complete intersection means that each local ring $\mathcal{I}_{\mathbf{p}}$ of the associated sheaf of ideals \mathcal{I} is a complete intersection ideal. Precisely, if I is an ideal of R generated by bihomogeneous forms, and $Z = \mathbb{V}(I) \subset \mathbf{P}^1 \times \mathbf{P}^1$ is a finite set, then we say that a base point $\mathbf{p} \in Z$ is a local complete intersection (LCI) if the local ring $\mathcal{I}_{\mathbf{p}} \subset \mathcal{O}_{X, \mathbf{p}}$ is a complete intersection ideal, i.e., $\mathcal{I}_{\mathbf{p}}$ is generated by two elements. The ideal I is a local complete intersection provided each base point $\mathbf{p} \in Z$ is a local complete intersection.

Theorem 3.10. *Let a_0, a_1 and $a_2 \in R$ be bihomogeneous polynomials of bidegree (m, n) and suppose that $\mathbb{V}(a_0, a_1, a_2) \subset \mathbf{P}^1 \times \mathbf{P}^1$ is finite, each base point $\mathbf{p} \in \mathbb{V}(a_0, a_1, a_2)$ is a local complete intersection, and that $(A_0, A_1, A_2) \in \text{Syz}(a_0, a_1, a_2)$ is a syzygy of bidegree (k, l) , where $(k - 2m + 1)(l - 2n + 1) \geq 0$. Then (A_0, A_1, A_2) vanishes on the base points of $I = \langle a_0, a_1, a_2 \rangle$, if and only if $(A_0, A_1, A_2) \in \text{Kos}(a_0, a_1, a_2)$, which means that there are h_1, h_2, h_3 of bidegree $(k - m, l - n)$ such that*

$$\begin{aligned} A_0 &= h_1 a_2 + h_2 a_1 \\ A_1 &= -h_2 a_0 + h_3 a_2 \\ A_2 &= -h_1 a_0 - h_3 a_1. \end{aligned}$$

Proof. This result follows from Corollary 3.15 of Hoffman and Wang (in press-b). See Remark 3.16 in that paper. \square

4. Local complete intersection base points of total multiplicity $k \leq mn$

In this section, we will extend the method of moving quadrics to the case where multiple base points are present. Throughout this section, φ will be a map $\varphi : \mathbf{P}^1 \times \mathbf{P}^1 \rightarrow \mathbf{P}^3$ given by $\varphi(s, u; t, v) = [a_0, a_1, a_2, a_3]$ where each $a_i \in R$ is a bihomogeneous polynomial of

bidegree (m, n) , and $I = \langle a_0, a_1, a_2, a_3 \rangle$. For convenience of reference, we will list some conditions on φ related to base points. Some of these conditions will be needed in each of the results of this paper, and they will be referred to by number (B1–B6) as needed.

- B1: The polynomials $a_i(s, u, t, v)$ ($0 \leq i \leq 3$) are bihomogeneous of bidegree (m, n) and are linearly independent over \mathbb{C} .
- B2: The base point scheme $\mathbb{V}(I) \subseteq \mathbf{P}^1 \times \mathbf{P}^1$ consists of a finite number of base points with total multiplicity $k \leq mn$.
- B3: Each base point $\mathbf{p} \in \mathbb{V}(I)$ is a LCI.
- B4: $\dim_{\mathbb{C}}(R/I)_{2m-1, 2n-1} = \deg(\mathbb{V}(I))$.
- B5: The base point scheme $\mathbb{V}(I) = \mathbb{V}(a_0, a_1, a_2)$ and $a_3 \in \text{sat}\langle a_0, a_1, a_2 \rangle$.
- B6: $\dim_{\mathbb{C}} \text{Syz}(a_0, a_1, a_2)_{m-1, n-1} = 0$.

Remark 4.1. Some remarks concerning these conditions:

- (1) The condition B1 simply says that $S = \text{Im}(\varphi)$ is not contained in any plane in \mathbf{P}^3 .
- (2) The finiteness of $\mathbb{V}(I)$ in condition B2 is equivalent to $\text{gcd}(a_0, a_1, a_2, a_3) = 1$, while $k \leq mn$ is equivalent to the degree inequality $\deg S \deg \varphi \geq mn$. The last equivalence is a consequence of the degree formula

$$2mn = \deg \varphi \deg S + \sum_{\mathbf{p} \in \mathbb{V}(I)} e(I, \mathbf{p}),$$

which is similar to Cox (2001, Page 19). For a proof, see Wang (2003, Theorem 4.2.12). In this formula, $e(I, \mathbf{p})$ is the multiplicity of the local ring $\mathcal{I}_{\mathbf{p}}$.

- (3) The above degree formula for the image of the parametrization involves the sum of the multiplicities of the base points. This equals $\deg(\mathbb{V}(I))$ only when $\mathbb{V}(I)$ is a local complete intersection. Hence the need for the condition B3.
- (4) Condition B4 is necessary to be able to apply the regularity condition on I given by Theorem 3.6.
- (5) Conditions B5 and B6 are technical conditions which are needed to be able to apply Theorem 3.10.

Lemma 4.2. *Suppose $a_0, a_1, a_2, a_3 \in \mathbb{C}[s, u, t, v]$ are bihomogeneous of bidegree (m, n) with no common factor, and $\mathbb{V}(a_0, a_1, a_2, a_3)$ is a local complete intersection. If we replace $\{a_i\}_{i=0}^2$ with generic linear combinations of $\{a_i\}_{i=0}^3$, then we have $\mathbb{V}(a_0, a_1, a_2) = \mathbb{V}(a_0, a_1, a_2, a_3)$ as subschemes of $\mathbf{P}^1 \times \mathbf{P}^1$, and $a_3 \in \text{sat}\langle a_0, a_1, a_2 \rangle$.*

Proof. The result is proved in Busé et al. (2003, Theorem A.1, Corollary A.2) for the case of homogeneous polynomials in $k[x, y, z]$, but the argument works verbatim in the case of bihomogeneous polynomials. \square

Remark 4.3. A consequence of Lemma 4.2 is that if the parametrization $\varphi : \mathbf{P}^1 \times \mathbf{P}^1 \rightarrow \mathbf{P}^3$ satisfies conditions B1–B4, then after a generic linear change of coordinates T of \mathbf{P}^3 , the resulting parametrization $T \circ \varphi$ will satisfy B1–B5. That is, if we allow generic linear changes of coordinates of the image space, then B1–B4 still hold and B5 is a consequence of B1–B4.

Recall that MP denotes both the map

$$MP : R_{m-1,n-1}^4 \xrightarrow{[a_0 \ a_1 \ a_2 \ a_3]} R_{2m-1,2n-1}$$

given by

$$(A_0, A_1, A_2, A_3) \mapsto \sum_{i=0}^3 A_i a_i,$$

and the $4mn \times 4mn$ matrix which represents this map in the standard monomial bases on $R_{m-1,n-1}^4$ and $R_{2m-1,2n-1}$. If we replace $\{a_i\}_{i=0}^3$ by $\{a'_i\}_{i=0}^3$ where each a'_i is a generic linear combinations of $\{a_i\}_{i=0}^3$, then the rank of the coefficient matrix MP will not change. Thus, the number of linearly independent moving planes is also not affected by a generic linear change of coordinates in the image space \mathbf{P}^3 .

Let

$$MC : R_{m-1,n-1}^3 \xrightarrow{[a_0 \ a_1 \ a_2]} R_{2m-1,2n-1}$$

be the map given by

$$(A_0, A_1, A_2) \mapsto \sum_{i=0}^2 A_i a_i.$$

MC is represented by a matrix, also denoted MC of size $4mn \times 3mn$, and $\text{Ker}(MC) = \text{Syz}(a_0, a_1, a_2)_{m-1,n-1}$. Thus

$$\dim_{\mathbb{C}} \text{Syz}(a_0, a_1, a_2)_{m-1,n-1} = \dim_{\mathbb{C}} \text{Ker}(MC),$$

and the following fact is clear.

Lemma 4.4. *If $\varphi : \mathbf{P}^1 \times \mathbf{P}^1 \rightarrow \mathbf{P}^3$ then MC has maximal rank ($=3mn$) if and only if φ satisfies condition B6.*

We start our analysis with the following lemma, which indicates that base points of total multiplicity k produce exactly k linearly independent moving planes of bidegree $(m - 1, n - 1)$.

Lemma 4.5. *If $\varphi : \mathbf{P}^1 \times \mathbf{P}^1 \rightarrow \mathbf{P}^3$ satisfies the base point conditions B1–B4, then*

$$\dim_{\mathbb{C}} \text{Syz}(I)_{m-1,n-1} = k.$$

Proof. Consider the following exact sequence:

$$\begin{aligned} 0 \rightarrow \text{Syz}(I)_{m-1,n-1} \rightarrow R_{m-1,n-1}^4 \xrightarrow{[a_0 \ a_1 \ a_2 \ a_3]} R_{2m-1,2n-1} \\ \rightarrow (R/I)_{2m-1,2n-1} \rightarrow 0. \end{aligned}$$

We have

$$\begin{aligned} \dim_{\mathbb{C}} \text{Syz}(I)_{m-1,n-1} &= \dim_{\mathbb{C}}(R/I)_{2m-1,2n-1} - \dim_{\mathbb{C}} R_{2m-1,2n-1} \\ &\quad + 4 \dim_{\mathbb{C}} R_{m-1,n-1} \\ &= \dim_{\mathbb{C}}(R/I)_{2m-1,2n-1}. \end{aligned}$$

Since at each base point, $\mathbb{V}(I)$ is a local complete intersection, we have

$$\sum_{\mathfrak{p} \in \mathbb{V}(I)} e(I, \mathfrak{p}) = \deg(\mathbb{V}(I)) = k.$$

Thus, $\dim_{\mathbb{C}}(R/I)_{2m-1,2n-1} = \deg(\mathbb{V}(I)) = k$, and hence

$$\dim_{\mathbb{C}} \text{Syz}(I)_{m-1,n-1} = k. \quad \square$$

Remark 4.6. Under the hypotheses of Lemma 4.5, the condition

$$\dim_{\mathbb{C}} \text{Syz}(I)_{m-1,n-1} = k$$

means that there are exactly k linearly independent moving planes of bidegree $(m-1, n-1)$ which follow the parametrization φ .

Our next goal is to prove that, under suitable conditions on the base point scheme $\mathbb{V}(I)$,

$$\dim_{\mathbb{C}} \text{Syz}(I^2)_{m-1,n-1} = mn + 3k.$$

We will start by proving the following two lemmas.

Lemma 4.7. *If $\varphi : \mathbf{P}^1 \times \mathbf{P}^1 \rightarrow \mathbf{P}^3$ satisfies the conditions B1–B4, and $I = \langle a_0, a_1, a_2, a_3 \rangle$ as usual, then $\text{sat}(I^2)$ is $(3m-1, 3n-1)$ -regular.*

Proof. Consider the following exact sequence:

$$\begin{aligned} 0 \rightarrow \text{Syz}(I^2)_{m-1,n-1} &\rightarrow R_{m-1,n-1}^{10} \xrightarrow{[a_0^2 \cdots a_3^2]} R_{3m-1,3n-1} \\ &\rightarrow (R/I^2)_{3m-1,3n-1} \rightarrow 0. \end{aligned}$$

This implies that

$$\begin{aligned} \dim_{\mathbb{C}} \text{Syz}(I^2)_{m-1,n-1} &= \dim_{\mathbb{C}}(R/I^2)_{3m-1,3n-1} - \dim_{\mathbb{C}} R_{3m-1,3n-1} \\ &\quad + 10 \dim_{\mathbb{C}} R_{m-1,n-1} \\ &= \dim_{\mathbb{C}}(R/I^2)_{3m-1,3n-1} + mn. \end{aligned} \tag{10}$$

Conditions B2, B3, and B4 show that $\dim_{\mathbb{C}}(R/I)_{2m-1,2n-1} = \deg(\mathbb{V}(I)) = k$, and this implies that I is $(2m-1, 2n-1)$ -regular by Theorem 3.6. Since $\mathbb{V}(I)$ is finite, Theorem 2.10 shows that $\text{sat}(I^2)$ is $((2-1)(2m-1) + m, (2-1)(2n-1) + n) = (3m-1, 3n-1)$ -regular, as claimed. \square

For the second lemma, we will need the following result of Herzog (1978, Folgerung 2.2 and 2.4):

Proposition 4.8. *Let $\mathcal{O}_{\mathbf{p}}$ be the local ring of a point $\mathbf{p} \in \mathbf{P}^1 \times \mathbf{P}^1$, and let $\mathcal{I}_{\mathbf{p}} \subseteq \mathcal{O}_{\mathbf{p}}$ be a codimension two ideal. Then*

$$\dim_{\mathbb{C}} \mathcal{I}_{\mathbf{p}}/\mathcal{I}_{\mathbf{p}}^2 \geq 2 \dim_{\mathbb{C}} \mathcal{O}_{\mathbf{p}}/\mathcal{I}_{\mathbf{p}}, \tag{11}$$

and equality holds if and only if $\mathcal{I}_{\mathbf{p}}$ is a complete intersection ideal in $\mathcal{O}_{\mathbf{p}}$.

Lemma 4.9. *If $\varphi : \mathbf{P}^1 \times \mathbf{P}^1 \rightarrow \mathbf{P}^3$ satisfies the conditions B1–B4, then*

$$\dim_{\mathbb{C}} \text{Syz} (I^2)_{m-1, n-1} \geq mn + 3k.$$

Proof. The exact sequence

$$0 \rightarrow (I/I^2)_{r,r'} \rightarrow (R/I^2)_{r,r'} \rightarrow (R/I)_{r,r'} \rightarrow 0$$

shows that $\dim_{\mathbb{C}} (R/I^2)_{r,r'} = \dim_{\mathbb{C}} (R/I)_{r,r'} + \dim_{\mathbb{C}} (I/I^2)_{r,r'}$, for all r, r' . By condition B4 $\dim_{\mathbb{C}} (R/I)_{2m-1, 2n-1} = \deg(\mathbb{V}(I)) = k$, and since $\dim_{\mathbb{C}} (R/I)_{k,k'} \leq \dim_{\mathbb{C}} (R/I)_{l,l'}$ whenever $(k, k') \leq (l, l')$, it follows that $\dim_{\mathbb{C}} (R/I)_{r,r'} = k$ for $r \geq 2m - 1, r' \geq 2n - 1$. Hence,

$$\dim_{\mathbb{C}} (R/I^2)_{r,r'} = k + \dim_{\mathbb{C}} (I/I^2)_{r,r'} \quad \text{for } r \geq 2m - 1, r' \geq 2n - 1.$$

For $r, r' \gg 0$, $\dim_{\mathbb{C}} (I/I^2)_{r,r'} = P_{I/I^2}(r, r')$ where $P_{I/I^2}(r, r')$ is the bigraded Hilbert polynomial of I/I^2 .

If \mathcal{I} is the sheaf of ideals associated to I , then $\mathcal{I}/\mathcal{I}^2$ has zero dimensional support since $\mathbb{V}(I)$ is finite. Therefore, letting $X = \mathbf{P}^1 \times \mathbf{P}^1$,

$$H^0(X, \mathcal{I}/\mathcal{I}^2) = \bigoplus_{\mathbf{p} \in \mathbb{V}(I)} \mathcal{I}_{\mathbf{p}}/\mathcal{I}_{\mathbf{p}}^2$$

and

$$H^0(X, \mathcal{I}/\mathcal{I}^2(r, r')) = \bigoplus_{\mathbf{p} \in \mathbb{V}(I)} (\mathcal{I}_{\mathbf{p}}/\mathcal{I}_{\mathbf{p}}^2) \otimes \mathcal{O}_{\mathbf{p}}(r, r'),$$

for all r, r' and hence

$$\dim_{\mathbb{C}} H^0(X, \mathcal{I}/\mathcal{I}^2) = \dim_{\mathbb{C}} H^0(X, \mathcal{I}/\mathcal{I}^2(r, r')) \quad \text{for all } r, r',$$

while $H^0(X, \mathcal{I}/\mathcal{I}^2(r, r')) = (I/I^2)_{r,r'}$ for all $r, r' \gg 0$ by Hyry (1999, Theorem 1.6). Therefore, for all $r, r' \gg 0$ we have

$$\begin{aligned} P_{I/I^2}(r, r') &= \dim_{\mathbb{C}} (I/I^2)_{r,r'} \\ &= \dim_{\mathbb{C}} H^0(X, \mathcal{I}/\mathcal{I}^2(r, r')) \\ &= \dim_{\mathbb{C}} H^0(X, \mathcal{I}/\mathcal{I}^2) \\ &= \sum_{\mathbf{p} \in \mathbb{V}(I)} \dim_{\mathbb{C}} \mathcal{I}_{\mathbf{p}}/\mathcal{I}_{\mathbf{p}}^2. \end{aligned}$$

Since each base point $\mathbf{p} \in \mathbb{V}(I)$ is a local complete intersection by condition B3, Proposition 4.8 shows that

$$\sum_{\mathbf{p} \in \mathbb{V}(I)} \dim_{\mathbb{C}} \mathcal{I}_{\mathbf{p}}/\mathcal{I}_{\mathbf{p}}^2 = 2 \sum_{\mathbf{p} \in \mathbb{V}(I)} \dim_{\mathbb{C}} \mathcal{O}_{\mathbf{p}}/\mathcal{I}_{\mathbf{p}} = 2 \deg(\mathbb{V}(I)) = 2k, \tag{12}$$

and hence, for $r, r' \gg 0$,

$$\begin{aligned} \dim_{\mathbb{C}}(R/I^2)_{r,r'} &= \dim_{\mathbb{C}}(R/I)_{r,r'} + \dim_{\mathbb{C}}(I/I^2)_{r,r'} \\ &= k + 2 \sum_{p \in \mathbb{V}(I)} \dim_{\mathbb{C}} \mathcal{O}_p/\mathcal{I}_p = 3k. \end{aligned} \tag{13}$$

Since $P_{M^{\text{sat}}}(r, r') = P_M(r, r')$ for any finitely generated bihomogeneous R -module M , it follows that

$$\dim_{\mathbb{C}}(R/I^2)_{r,r'} = \dim_{\mathbb{C}}(R/\text{sat}(I^2))_{r,r'},$$

for $r, r' \gg 0$. This fact, combined with Eq. (13), the fact that $\text{sat}(I^2)$ is $(3m - 1, 3n - 1)$ -regular (Lemma 4.7), and Lemma 3.7, shows that

$$\dim_{\mathbb{C}}(R/\text{sat}(I^2))_{3m-1,3n-1} = 3k.$$

Since $I^2 \subset \text{sat}(I^2)$, we have

$$\dim_{\mathbb{C}}(R/I^2)_{3m-1,3n-1} \geq \dim_{\mathbb{C}}(R/\text{sat}(I^2))_{3m-1,3n-1} = 3k.$$

Therefore, Eq. (10) becomes

$$\dim_{\mathbb{C}} \text{Syz}(I^2)_{m-1,n-1} = mn + \dim_{\mathbb{C}}(R/I^2)_{3m-1,3n-1} \geq mn + 3k. \quad \square$$

Remark 4.10. Under the hypothesis of Lemma 4.9, the condition

$$\dim_{\mathbb{C}} \text{Syz}(I^2)_{m-1,n-1} \geq mn + 3k$$

means that there are at least $mn + 3k$ linearly independent moving quadrics of bidegree $(m - 1, n - 1)$ which follow the parametrization φ .

The construction of the matrix M whose determinant is the implicit equation of $S = \text{Im}(\varphi)$ requires a careful choice of basis of the vector space of moving quadrics, which is facilitated by the following elementary linear algebra lemma. We will first establish the notation.

Let the vector space $V = V_1 \oplus V_2$ be the direct sum of two subspaces V_1 and V_2 , and let $W \subset V$ be a subspace such that $V_1 \cap W = \{0\}$. Then the projection $\pi : V \rightarrow V_2$ along V_1 satisfies $\text{Ker}(\pi) = V_1$, and $\text{Ker}(\pi)|_W = W \cap V_1 = \{0\}$. In particular, $\pi|_W$ is injective, so that $\dim_{\mathbb{C}} W = \dim_{\mathbb{C}} \pi(W) := k$. Let $\mathcal{B} = \{v_1, \dots, v_l\}$ be a given basis of V_2 .

Lemma 4.11. *There is a subset $\mathcal{B}_1 = \{v_{h_1}, \dots, v_{h_k}\} \subset \mathcal{B}$ and a basis $\mathcal{C} = \{w_1, \dots, w_k\}$ of W such that*

$$\pi(w_e) = v_{h_e} + \bar{w}_e, \quad \text{where } \bar{w}_e \in \text{Span}(\mathcal{B} \setminus \mathcal{B}_1).$$

Proof. Let $\{\tilde{w}_1, \dots, \tilde{w}_k\}$ be an arbitrary basis of W . Then

$$\pi(\tilde{w}_i) = \sum_{j=1}^l a_{ij} v_j.$$

Let $A = [a_{ij}]$. Then multiply A on the left by an invertible matrix P so that $PA = Q$, where Q is in reduced row echelon form. Since A is a $k \times l$ matrix which has $\text{Rank } A = k$

(because $\dim_{\mathbb{C}} \pi(W) = k$), there are k columns $h_1 < h_2 < \dots < h_k$ which contain a leading 1 in rows 1 to k , respectively. Let $\mathcal{B}_1 = \{v_{h_1}, \dots, v_{h_k}\}$. Let the basis $\mathcal{C} = \{w_1, \dots, w_k\}$ be defined by

$$\begin{bmatrix} w_1 \\ \vdots \\ w_k \end{bmatrix} = P \begin{bmatrix} \tilde{w}_1 \\ \vdots \\ \tilde{w}_k \end{bmatrix},$$

i.e., $w_e = \sum_{j=1}^k p_{ej} \tilde{w}_j$. Then

$$\begin{aligned} \pi(w_e) &= \sum_{j=1}^k p_{ej} \pi(\tilde{w}_j) \\ &= \sum_{j=1}^k p_{ej} \sum_{r=1}^l a_{jr} v_r \\ &= \text{Row}_e Q \begin{bmatrix} v_1 \\ \vdots \\ v_l \end{bmatrix} \\ &= v_{h_e} + \bar{w}_e \end{aligned}$$

where $\bar{w}_e \in \text{Span}(\mathcal{B} \setminus \mathcal{B}_1)$. \square

If $P = \sum_{i=0}^3 A_i(s, u, t, v)x_i \in R[x_1, x_2, x_3, x_4]$ is any moving plane, and $L(x_0, x_1, x_2, x_3)$ is any homogeneous linear polynomial. Then $P \cdot L$ is a moving quadric. Moreover, if P follows φ , then $P \cdot L$ also follows φ . If \mathcal{P} is a set of moving planes, then $\mathcal{P} \cdot L := \{P \cdot L : P \in \mathcal{P}\}$. Let $\mathcal{P}_{\varphi, m-1, n-1}$ be the set of moving planes of bidegree $m-1, n-1$ which follow φ , i.e., $(A_0, A_1, A_2, A_3)_{m-1, n-1} \in \text{Syz}(a_0, a_1, a_2, a_3)_{m-1, n-1}$.

Lemma 4.12. *Let $\varphi : \mathbf{P}^1 \times \mathbf{P}^1 \rightarrow \mathbf{P}^3$, and assume that φ satisfies condition B6, so that $\text{Syz}(a_0, a_1, a_2)_{m-1, n-1} = \{0\}$. Let $\mathcal{S} = \mathcal{P}_{\varphi, m-1, n-1}$, and let $\dim_{\mathbb{C}} \mathcal{S} = k$. Then $\mathcal{Q} = \sum_{i=0}^3 \mathcal{S}x_i$ is a vector space of moving quadrics which follow φ , with $\dim_{\mathbb{C}} \mathcal{Q} = 4k$.*

Proof. We will apply Lemma 4.11 with the following identifications:

- $V = \sum_{i=0}^3 (R_{m-1, n-1})x_i \cong R_{m-1, n-1}^4$,
- $V_1 = \sum_{i=0}^2 (R_{m-1, n-1})x_i \cong R_{m-1, n-1}^3$,
- $V_2 = (R_{m-1, n-1})x_3 \cong R_{m-1, n-1}$,
- $W = \mathcal{S}$, and
- $\mathcal{S} \cap V_1 = \text{Syz}(a_0, a_1, a_2)_{m-1, n-1} = \{0\}$.

Let $\mathcal{B} = \{s^\alpha t^\beta x_3 : 0 \leq \alpha \leq m-1, 0 \leq \beta \leq n-1\}$. According to Lemma 4.11, there is a set $B = \{(\alpha_1, \beta_1), \dots, (\alpha_k, \beta_k)\}$ and a basis $\mathcal{C} = \{P_1, \dots, P_k\}$ of \mathcal{S} such that $\pi(P_i)$, which is the part of P_i in $(R_{m-1, n-1})x_3$, has the form

$$\pi(P_i) = s^{\alpha_i} t^{\beta_i} x_3 + \sum_{(\alpha, \beta) \notin B} b_{i, \alpha \beta} s^\alpha t^\beta x_3$$

where $i = 1, 2, \dots, k$. We claim that $\{P_i x_j\}_{i=1, j=0}^{i=k, j=3}$ is a linearly independent set. We need to show that if

$$\sum_{i=1}^k \sum_{j=0}^3 c_{ij} P_i x_j = 0 \tag{14}$$

where $c_{ij} \in \mathbb{C}$, then we must have $c_{ij} = 0$ for all i, j . Since

$$\mathcal{B}'' = \{s^\alpha t^\beta x_i x_j : 0 \leq \alpha \leq m - 1, 0 \leq \beta \leq n - 1, 1 \leq i \leq j \leq 3\}$$

is a basis of $\bigoplus_{0 \leq i \leq j \leq 3} (R_{m-1, n-1}) x_i x_j$, and since P_i is the only element of \mathcal{C} that contains the term $s^{\alpha_i} t^{\beta_i} x_3$, it follows that $P_i x_j$ is the only term in (14) that contains the basis element $s^{\alpha_i} t^{\beta_i} x_j x_3$ and hence the coefficient of this term, namely c_{ij} , must be 0. Thus $c_{ij} = 0$ for $i = 1, 2, \dots, k$ and $j = 0, 1, 2$, and 3. Therefore, the moving quadrics coming from the moving planes that follow φ are linearly independent, and hence $\dim_{\mathbb{C}} \mathcal{Q} = 4k$. \square

Theorem 4.13. Let $\varphi : \mathbf{P}^1 \times \mathbf{P}^1 \rightarrow \mathbf{P}^3$ be given by $\varphi(s, u; t, v) = [a_0, a_1, a_2, a_3]$ where each $a_i \in R$ is a bihomogeneous polynomial of bidegree (m, n) , and assume that the base point scheme of φ satisfies conditions B1–B6. Then

$$\dim_{\mathbb{C}} \text{Syz} (I^2)_{m-1, n-1} = mn + 3k.$$

Proof. If $M\mathcal{Q} : R_{m-1, n-1}^{10} \rightarrow R_{3m-1, 3n-1}$ is the map such that

$$M\mathcal{Q}(A_{00}, A_{01}, \dots, A_{33}) = \sum_{0 \leq i \leq j \leq 3} A_{ij} a_i a_j,$$

we have that

$$10mn - \text{Rank} (M\mathcal{Q}) = \dim_{\mathbb{C}} \text{Syz} (I^2)_{m-1, n-1}$$

is the number of linearly independent moving quadrics. If

$$\text{Rank} (M\mathcal{Q}) \geq 9mn - 3k,$$

then

$$\dim_{\mathbb{C}} \text{Syz} (I^2)_{m-1, n-1} = 10mn - \text{Rank} (M\mathcal{Q}) \leq mn + 3k.$$

But Lemma 4.9 shows that $\dim_{\mathbb{C}} \text{Syz} (I^2)_{m-1, n-1} \geq mn + 3k$, and hence $\dim_{\mathbb{C}} \text{Syz} (I^2)_{m-1, n-1} = mn + 3k$ will follow, once we have shown that

$$\text{Rank} (M\mathcal{Q}) \geq 9mn - 3k.$$

We now verify that this inequality is valid. Since φ satisfies condition B6, the proof of Lemma 4.12, shows that there is an indexed set

$$B = \{(\alpha_1, \beta_1), \dots, (\alpha_k, \beta_k)\}$$

and a basis of moving planes $\{P_1, \dots, P_k\}$ such that

$$P_i = s^{\alpha_i} t^{\beta_i} x_3 + \sum_{(\alpha, \beta) \notin B} b_{i, \alpha \beta} s^\alpha t^\beta x_3 + \sum_{j=0}^2 \sum_{(\alpha, \beta)} c_{i, \alpha \beta} s^\alpha t^\beta x_j \tag{15}$$

where $i = 1, 2, \dots, k$. As with MP , the matrix representing MQ with respect to the standard bases is also denoted MQ . Thus the columns of MQ are indexed by

$$\Lambda = \{s^\alpha t^\beta x_i x_j : 0 \leq \alpha \leq m - 1, 0 \leq \beta \leq n - 1, 0 \leq i \leq j \leq 3\}.$$

If

$$\Lambda_P = \{s^{\alpha_i} t^{\beta_i} x_j x_3, s^{\alpha} t^{\beta} x_3^2 : 1 \leq i \leq k, 0 \leq \alpha \leq m - 1, 0 \leq \beta \leq n - 1, 0 \leq j \leq 2\},$$

and $\Lambda' = \Lambda \setminus \Lambda_P$, then $|\Lambda'| = 10mn - (mn + 3k) = 9mn - 3k$. Let MQ' be the matrix obtained from MQ by deleting the columns indexed by Λ_P . Thus the nonzero elements of $\text{Ker}(MQ')$ correspond to nontrivial syzygies:

$$\begin{aligned} &A_{00}a_0^2 + A_{01}a_0a_1 + A_{02}a_0a_2 + A_{03}a_0a_3 + A_{11}a_1^2 \\ &+ A_{12}a_1a_2 + A_{13}a_1a_3 + A_{22}a_2^2 + A_{23}a_2a_3 = 0 \end{aligned} \tag{16}$$

where A_{ij} is bihomogeneous of bidegree $(m - 1, n - 1)$ and there are no terms $s^{\alpha_i} t^{\beta_i}$ in $\{A_{i3}\}_{i=0}^2$. Since every term contains $a_0, a_1,$ or a_2 , we obtain:

$$\begin{aligned} &(A_{00}a_0 + A_{01}a_1 + A_{02}a_2 + A_{03}a_3)a_0 \\ &+ (A_{11}a_1 + A_{12}a_2 + A_{13}a_3)a_1 + (A_{22}a_2 + A_{23}a_3)a_2 = 0, \end{aligned}$$

which means that

$$\begin{aligned} &(B_1, B_2, B_3) \\ &= (A_{00}a_0 + A_{01}a_1 + A_{02}a_2 + A_{03}a_3, A_{11}a_1 + A_{12}a_2 + A_{13}a_3, A_{22}a_2 + A_{23}a_3) \end{aligned}$$

is a syzygy of $\langle a_0, a_1, a_2 \rangle$. Each B_i has bidegree $(2m - 1, 2n - 1)$ and

$$B_i \in \langle a_0, a_1, a_2, a_3 \rangle \subseteq \text{sat}\langle a_1, a_1, a_2 \rangle$$

by condition B5. Therefore, each B_i vanishes on the base point scheme of $\langle a_0, a_1, a_2 \rangle$. By condition B5, $\mathbb{V}(a_0, a_1, a_2) = \mathbb{V}(I)$, and thus, by B2,

$$\mathbb{V}(a_0, a_1, a_2) \subset \mathbf{P}^1 \times \mathbf{P}^1$$

is finite and each base point is a local complete intersection (condition B3). Since each a_i has bidegree (m, n) , while each B_i has bidegree $(2m - 1, 2n - 1)$, we have $(2m - 1 - 2m + 1)(2n - 1 - 2n + 1) = 0$. Therefore, all the hypotheses of Theorem 3.10 are satisfied, and we conclude that all syzygies of bidegree $(2m - 1, 2n - 1)$ of $\langle a_0, a_1, a_2 \rangle$ that vanish on the base point scheme are in fact Koszul syzygies. Since (B_0, B_1, B_2) is a syzygy that vanishes on $\mathbb{V}(a_0, a_1, a_2)$, it follows that there are bihomogeneous polynomials $h_1, h_2,$ and h_3 in R of bidegree $(m - 1, n - 1)$ such that:

$$\begin{aligned} &A_{00}a_0 + A_{01}a_1 + A_{02}a_2 + A_{03}a_3 = h_1a_2 + h_2a_1 \\ &A_{11}a_1 + A_{12}a_2 + A_{13}a_3 = -h_2a_0 + h_3a_2 \\ &A_{22}a_2 + A_{23}a_3 = -h_1a_0 - h_3a_1. \end{aligned}$$

We can rewrite the above equations to get:

$$A_{00}a_0 + (A_{01} - h_2)a_1 + (A_{02} - h_1)a_2 + A_{03}a_3 = 0, \tag{17}$$

$$h_2a_0 + A_{11}a_1 + (A_{12} - h_3)a_2 + A_{13}a_3 = 0, \tag{18}$$

$$h_1a_0 + h_3a_1 + A_{22}a_2 + A_{23}a_3 = 0. \tag{19}$$

We know that A_{ij} is bihomogeneous of bidegree $(m - 1, n - 1)$ and there are no $s^{\alpha_i} t^{\beta_i}$ terms in $\{A_{i3}\}_{i=0}^2$. Thus Eqs. (17)–(19) are nontrivial syzygies of $\langle a_0, a_1, a_2, a_3 \rangle$ which correspond to moving planes P with no $s^{\alpha_i} t^{\beta_i} x_3$ term for $1 \leq i \leq k$. But $\{P_1, \dots, P_k\}$ is a basis of moving planes. Any nonzero moving plane $P = c_1 P_1 + \dots + c_k P_k$ must have some nonzero term $s^{\alpha_i} t^{\beta_i} x_3$, since if $c_i \neq 0$, then $s^{\alpha_i} t^{\beta_i} x_3$ appears.

Hence the nontrivial syzygies from Eqs. (17)–(19) cannot exist. Thus $\text{Ker}(MQ') = \{0\}$, so

$$\text{Rank}(MQ) \geq \text{Rank}(MQ') = 9mn - 3k,$$

as required, and hence we conclude that $\dim_{\mathbb{C}} \text{Syz}(I^2)_{m-1, n-1} = mn + 3k$. \square

Remark 4.14. Under the hypothesis of [Theorem 4.13](#), the condition

$$\text{Syz}(I^2)_{m-1, n-1} = mn + 3k$$

means that there are exactly $mn + 3k$ linearly independent moving quadrics of bidegree $(m - 1, n - 1)$ that follow the parametrization φ . Moreover, the proof shows that there are no nontrivial moving quadrics with nonzero coordinates coming only from the basis elements $\Lambda' = \Lambda \setminus \Lambda_P$ (because $\text{Ker}(MQ') = \{0\}$). Hence any nontrivial moving quadric Q must have at least one nonzero coordinate from a term in the set

$$\Lambda_P = \{s^{\alpha_i} t^{\beta_i} x_j x_3, s^{\alpha} t^{\beta} x_3^2 : 1 \leq i \leq k, 0 \leq \alpha \leq m - 1, 0 \leq \beta \leq n - 1, 0 \leq j \leq 2\}.$$

This observation will be key to the proof of [Theorem 4.15](#).

If $\varphi : \mathbf{P}^1 \times \mathbf{P}^1 \rightarrow \mathbf{P}^3$ satisfies the base point conditions B1–B6, then [Lemma 4.5](#) shows that there are exactly k linearly independent moving planes $\mathcal{MP} = \{P_{\gamma}\}_{\gamma=1}^k$ which follow φ , where $k \leq mn$ is the total multiplicity of all base points of φ , and [Theorem 4.13](#) shows that there are exactly $mn + 3k$ linearly independent moving quadrics $\mathcal{MQ} = \{Q_{\tau} : 1 \leq \tau \leq mn + 3k\}$ that follow φ . Each moving plane can be written as

$$P_{\gamma} = \sum_{i=0}^3 A_i x_i = \sum_{\alpha=0}^{m-1} \sum_{\beta=0}^{n-1} P_{\gamma, \alpha \beta}(x_0, x_1, x_2, x_3) s^{\alpha} t^{\beta},$$

and each moving quadric Q_{τ} can be written as (see Eq. (4))

$$Q_{\tau} = \sum_{0 \leq i \leq j \leq 3} A_{ij} x_i x_j = \sum_{\alpha=0}^{m-1} \sum_{\beta=0}^{n-1} Q_{\tau, \alpha \beta}(x_0, x_1, x_2, x_3) s^{\alpha} t^{\beta},$$

where $P_{\gamma, \alpha \beta}(x_0, x_1, x_2, x_3)$ is a homogeneous linear form and $Q_{\tau, \alpha \beta}$ is a homogeneous quadratic form in x_i with coefficients in \mathbb{C} . Our goal is to choose the sets of moving planes

\mathcal{MP} and moving quadrics \mathcal{MQ} in such a way that all k of the moving planes and $mn - k$ of the moving quadrics can be combined into a single $mn \times mn$ matrix (which will depend on the choice of \mathcal{MP} and \mathcal{MQ})

$$M = \begin{bmatrix} P_{\gamma,\alpha\beta}(x_0, x_1, x_2, x_3) \\ Q_{\tau,\alpha\beta}(x_0, x_1, x_2, x_3) \end{bmatrix} \tag{20}$$

such that the equation of the image surface $S = \text{Im}(\varphi)$ is given by the determinantal equation $|M| = 0$, as long as φ is generically one-to-one. The strategy for constructing M is to start with an arbitrary basis \mathcal{MP} of moving planes (consisting of k moving planes), and then choose a basis of moving quadrics \mathcal{MQ} (consisting of $mn + 3k$ moving quadrics) in such a manner that $4k$ of the moving quadrics are obtained by multiplying the moving planes of \mathcal{MP} by each of the coordinate functions x_i ($0 \leq i \leq 3$). If these $4k$ moving quadrics are deleted from the set \mathcal{MQ} , then the remaining $mn - k$ are used for the matrix M of Eq. (20). The justification for this procedure constitutes the proof of our main result.

Theorem 4.15. *Let $\varphi : \mathbf{P}^1 \times \mathbf{P}^1 \rightarrow \mathbf{P}^3$ be a parametrization of a surface $S = \text{Im}(\varphi) \subset \mathbf{P}^3$. If φ is generically one-to-one and satisfies base point conditions B1–B6, then a basis of moving planes \mathcal{MP} and a basis of moving quadrics \mathcal{MQ} can be chosen so that S is defined by the determinantal equation $|M| = 0$, where M is the $mn \times mn$ matrix of Eq. (20).*

Proof. By Lemma 4.5, $\dim_{\mathbb{C}} \text{Syz} (a_0, a_1, a_2, a_3)_{m-1, n-1} = k$, and the proof of Lemma 4.12 shows that there is an indexed set $B = \{(\alpha_1, \beta_1), \dots, (\alpha_k, \beta_k)\}$ and a basis of moving planes $\mathcal{MP} = \{P_1, \dots, P_k\}$ such that

$$P_i = s^{\alpha_i} t^{\beta_i} x_3 + \sum_{(\alpha,\beta) \notin B} b_{i,\alpha\beta} s^{\alpha} t^{\beta} x_3 + \sum_{j=0}^2 \sum_{(\alpha,\beta)} c_{i,\alpha\beta} s^{\alpha} t^{\beta} x_j \tag{21}$$

where $i = 1, 2, \dots, k$. By Theorem 4.13,

$$\dim_{\mathbb{C}} \text{Syz} (I^2)_{m-1, n-1} = mn + 3k.$$

We now describe how to produce a convenient basis of $\text{Syz} (I^2)_{m-1, n-1}$.

Let $V_{A'}, V_{A_P}$ be subspaces of $V = \bigoplus_{0 \leq i \leq j \leq 3} (R_{m-1, n-1}) x_i x_j \cong R_{m-1, n-1}^{10}$ with bases A', A_P respectively, where (as in the proof of Theorem 4.13)

$$\begin{aligned} A &= \{s^{\alpha} t^{\beta} x_i x_j : 0 \leq \alpha \leq m - 1, 0 \leq \beta \leq n - 1, 0 \leq i \leq j \leq 3\}, \\ A_P &= \{s^{\alpha_i} t^{\beta_i} x_j x_3, s^{\alpha} t^{\beta} x_3^2 : 0 \leq i \leq k, 0 \leq \alpha \leq m - 1, \\ &\quad 0 \leq \beta \leq n - 1, 0 \leq j \leq 2\}, \end{aligned}$$

and $A' = A \setminus A_P$. Then $V = V_{A'} \oplus V_{A_P}$ and the proof of Theorem 4.13 (namely, the proof that $\text{Ker}(MQ') = \{0\}$) shows that $\text{Syz} (I^2)_{m-1, n-1} \subset V$ satisfies

$$\text{Syz} (I^2)_{m-1, n-1} \cap V_{A'} = \{0\}.$$

We conclude that if $\pi : V \rightarrow V_{A_P}$ given by $\pi(v_1 + v_2) = v_2$ is the projection onto V_{A_P} along $V_{A'}$, then $\pi|_{\text{Syz} (I^2)_{m-1, n-1}}$ is an isomorphism, since

$$\dim_{\mathbb{C}} \text{Syz} (I^2)_{m-1, n-1} = \dim_{\mathbb{C}} V_{A_P} = mn + 3k.$$

Thus $\mathcal{MQ} = \pi^{-1}(A_P)$ is a basis of moving quadrics that follow φ .

Let $Q_{x_j x_3, i} = \pi^{-1}(s^{\alpha_i} t^{\beta_i} x_j x_3)$, for $1 \leq i \leq k$, and $0 \leq j \leq 3$. Since $x_j P_i \in \text{Syz}(I^2)_{m-1, n-1}$, and $\pi(x_j P_i) = s^{\alpha_i} t^{\beta_i} x_j x_3$ (see Eq. (21)), the fact that $\pi|_{\text{Syz}(I^2)_{m-1, n-1}}$ is an isomorphism shows that $x_j P_i = Q_{x_j x_3, i}$. Thus, we have identified the set of moving quadrics in \mathcal{MQ} which arise from multiplication of the moving planes in \mathcal{MP} by the homogeneous coordinate functions x_j ($0 \leq j \leq 3$). These are excluded when forming the matrix M .

Let $Q_{\gamma \delta} = \pi^{-1}(s^\gamma t^\delta x_3^2)$, where

$$(\gamma, \delta) \in \{(\alpha, \beta) : 0 \leq \alpha \leq m - 1, 0 \leq \beta \leq n - 1\} \setminus \{(\alpha_i, \beta_i) : 1 \leq i \leq k\} := C_P.$$

These $mn - k$ moving quadrics in the basis \mathcal{MQ} do not come from the moving planes of \mathcal{MP} by multiplication by $\{x_i\}_{i=0}^3$. Thus, they can be combined with the k moving planes \mathcal{MP} to produce the matrix M . Hence

$$M = \begin{bmatrix} P_{i, \alpha \beta}(x_0, x_1, x_2, x_3) \\ Q_{\gamma \delta, \alpha \beta}(x_0, x_1, x_2, x_3) \end{bmatrix} \tag{22}$$

where $1 \leq i \leq k$ and $(\gamma, \delta) \in C_P$, and the columns are indexed by the monomial basis $s^\alpha t^\beta$ of $R_{m-1, n-1}$ with $0 \leq \alpha \leq m - 1, 0 \leq \beta \leq n - 1$.

The k moving planes P_i have the form

$$P_i = s^{\alpha_i} t^{\beta_i} x_3 + \sum_{(\alpha, \beta) \notin B} b_{i, \alpha \beta} s^\alpha t^\beta x_3 + \sum_{j=0}^2 \sum_{(\alpha, \beta)} c_{i, \alpha \beta} s^\alpha t^\beta x_j,$$

while the $mn - k$ moving quadrics $Q_{\gamma \delta}$ for $(\gamma, \delta) \in C_P$ have the form

$$Q_{\gamma \delta} = x_3^2 s^\gamma t^\delta + \text{terms not involving } x_3^2.$$

That is, the term $s^{\alpha_i} t^{\beta_i} x_3$ occurs in P_i , but in no other P_j for $j \neq i$, while the term $x_3^2 s^\gamma t^\delta$ occurs in $Q_{\gamma \delta}$, but no other term of the form $x_3^2 s^{\gamma'} t^{\delta'}$ occurs in $Q_{\gamma \delta}$. Thus the matrix M of Eq. (22) will have k linear rows and $mn - k$ quadratic rows in the variables x_0, x_1, x_2 and x_3 . Moreover, we can order the rows and columns in such a way that all of the x_3^2 terms (one for each quadratic row) occur on the last $mn - k$ diagonals, while the first k diagonals have the term x_3 coming from the terms $s^{\alpha_i} t^{\beta_i} x_3$ in P_i ($1 \leq i \leq k$). Thus, after appropriate ordering of the rows and columns, M will have the form

$$M = \begin{bmatrix} x_3 + \cdots & & & & & & \\ & \ddots & & & & & \\ & & x_3 + \cdots & & & & \\ & & & x_3^2 + \cdots & & & \\ & & & & \ddots & & \\ & & & & & & x_3^2 + \cdots \end{bmatrix}.$$

There are k linear rows and $mn - k$ quadratic rows, so the determinant of M contains the term x_3^{2mn-k} , which occurs in the multiplication of the diagonal entries. Since the x_3^2 term appears only in the last $mn - k$ diagonal entries, and in the upper left $k \times k$ block, the term

x_3 appears only on the diagonal, it follows that $2mn - k$ is the highest power of x_3 that can appear in $|M|$, and this power appears with nonzero coefficient. Thus $|M|$ is not identically zero. Since M contains $mn - k$ rows of quadratic terms in x_i and k rows of linear terms in x_i , the total degree of $|M|$ is $2mn - k$. By construction, the rows of M represent moving quadrics and moving planes that follow the surface, and hence, when x_i is replaced by a_i it follows that the columns of M are linearly dependent. Therefore, $|M|$ vanishes for points on the surface. From the degree formula

$$\text{deg}(\varphi) \text{deg}(S) = 2mn - \sum_{\text{base points}} \text{multiplicity of the base point}$$

and the fact that φ is generically one to one, and the total multiplicity of all base points is k , we conclude that $\text{deg}(S) = 2mn - k$. But this is the same degree as $|M|$ so $|M| = 0$ must be the implicit equation of the image of φ . \square

Example 4.16. Consider $\varphi : \mathbf{P}^1 \times \mathbf{P}^1 \rightarrow \mathbf{P}^3$ given by the following parametrization:

$$a_0 = u^2tv + s^2tv, \quad a_1 = u^2t^2 + suv^2, \quad a_2 = s^2v^2 + s^2t^2, \quad a_3 = s^2tv.$$

Here $m = n = 2$, $\mathbb{V}(a_0, a_1, a_2, a_3) = (0 : 1; 0 : 1)$ and the multiplicity of this single base point is one. This base point is a local complete intersection and $\text{deg } \mathbb{V}(I) = 1$. Using Singular (Greuel et al., 2001), one can verify that the base point conditions B1–B6 are satisfied, since $\dim_{\mathbb{C}}(R/I)_{3,3} = 1$, $a_3 \in \text{sat}(a_0, a_1, a_2)$ and $\dim_{\mathbb{C}} \text{Syz}(a_0, a_1, a_2)_{1,1} = 0$. Also, by Singular, we find one moving plane of bidegree $(1, 1)$ which is

$$-x_2 + tx_3 + sx_1 + st(x_3 - x_0)$$

and three linearly independent moving quadrics of bidegree $(1, 1)$ which are complementary to $x_i(-x_2 + tx_3 + sx_1 + st(x_3 - x_0))$ for $i = 0, 1, 2, 3$:

$$\begin{aligned} &x_0x_3 + t(x_0x_2 + x_1x_3 + x_2x_3) + s(-x_0x_3 + x_3^2) \\ &(x_1x_3 - x_2x_3) + t(x_0x_3 + 2x_3^2) + s(-x_0x_2 + x_1x_3 + x_2x_3) \\ &(x_2^2 - x_3^2) + t(-x^2x_3) + s(x_0x_3 - x_1x_2 - x_3^2) + stx_1x_3 \end{aligned}$$

Thus the matrix M is

$$M = \begin{bmatrix} -x_2 & x_3 & x_1 & x_3 - x_0 \\ x_0x_3 & x_0x_2 + x_1x_3 + x_2x_3 & -x_0x_3 + x_3^2 & 0 \\ x_1x_3 - x_2x_3 & x_0x_3 + 2x_3^2 & -x_0x_2 + x_1x_3 + x_2x_3 & 0 \\ x_2^2 - x_3^2 & -x^2x_3 & x_0x_3 - x_1x_2 - x_3^2 & x_1x_3 \end{bmatrix}$$

and

$$\begin{aligned} |M| = &-x_0^3x_2^4 + x_0^2x_1^2x_2^2x_3 + x_0^2x_2^4x_4 + x_0x_1^2x_2^2x_3^2 \\ &+ 2x_0x_1x_2^3x_3^2 + x_0x_2^4x_3^2 - x_0^4x_3^3 - 2x_0^2x_1^2x_3^3 - x_1^4x_3^3 \\ &+ 3x_0^2x_1x_2x_3^3 - 2x_1^3x_2x_3^3 + 3x_0^2x_2^2x_3^3 - 2x_1^2x_2^2x_3^3 - 2x_1x_2^3x_3^3 \\ &- x_2^4x_3^3 + 5x_0^3x_4^3 + x_0x_1^2x_3^4 - 7x_0x_1x_2x_3^4 - 6x_0x_2^2x_3^4 - 9x_0^2x_3^5 \\ &+ x_1^2x_3^5 + 4x_1x_2x_3^5 + 3x_2^2x_3^5 + 7x_0x_3^6 - 2x_3^7. \end{aligned}$$

Thus the theorem gives $|M| = 0$ as the implicit equation of $S = \text{Im}(\varphi)$. Note $|M|$ is a polynomial of degree 7 which is the same as the degree of the parametrized surface.

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