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# SCS 1: More Notes on Spread

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# MORE NOTES ON SPREAD

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Notation and references consistent with "NOTES ON NOTES BY JDL". That work is referenced as H-SP (Hofmann on "Spread"). JDL should perhaps now become JDL I

**LEMMA 1.** Let $X$ be an order generating set in a $\mathcal{CL}$-object $T$. Then

$$\mathrm{IRR}\ T \subset \overline{X}$$

*Proof.* Let $y \in \mathrm{IRR}\ T$, $Y = \overline{X} \cap \uparrow y$. By hypothesis $y = \inf Y$. Since $y$ is prime in the $\mathcal{CL}$-object $\uparrow y$, by "THE LEMMA", $y \in Y \subset \overline{X}$.

THEOREM
**PROP. 2.** Let $X$ be a subset of a $\mathcal{CL}$-object $T$. TAE

(1) $\overline{X}$ is generating

(2) $\overline{X}$ is order-generating

(3) $\mathrm{IRR}\ T \subset \overline{X}$

*Proof.* (1) $\Leftrightarrow$ (2) by Thm. 1.10 [H-SP], and (3) $\Rightarrow$ (2) by Prop 1.4 [H-SP]; (2) $\Rightarrow$ (3) by Lemma 1.

**COROLLARY 3.** $\overline{\mathrm{IRR}\ T}$ is the unique smallest closed (order) generating set if $T \in \mathcal{CL}$

A set $X$ (in a semilattice $S$) is algebraically generating if $S$ is the smallest subsemilattice containing $X$.

Prop.2 and Corollary 3 really wrap up the question of generation and order generation in $\underline{CL}$ -objects. Also the ASSERTION on the next page settles my 1 quiry on p. 5 of (H-Sp). There remains the question: $\mathrm{IRR}\ T = \overline{\mathrm{IRR}}\ T$ for $T \in \underline{Z}$ ? Be sure to understand that 1.12,p.4 of (H-Sp) is superseded by Corr.3 Here. KHH

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**PROP. 4.** Let $T \in \mathcal{CL}$ and let $X = \bigcup_{i=1}^{n} T_i$ where each $T_i$ is a subsemilattice containing 1 of $T$ and $n < \infty$. TAE

(1) $\overline{X}$ is generating
(2) $\overline{X}$ is algebraically generating
(3) $\exists$ a continuous epimorphism $m: \prod_{i=1}^{n} \overline{T_i} \twoheadrightarrow T$ defined from the multiplication mapping.

Proof. (3) ⇒ (2) Immediate since $\overline{X} = \overline{T_1} \cup \ldots \cup \overline{T_n}$

(2) ⇒ (1) Alg. gen. ⇒ Ord. gen. Now apply Thm. 1.10 [H-CP]

(1) ⇒ (3) The image of $m$ is a compact subsemilattice containing $\overline{X}$. Hence the image is $T$.

**ASSERTION.** In [Lawson, Lattices with no interval homomorphisms] Example 1 has IRR $T$ order generating (hence IRR $T$ order generating $\not\Rightarrow T \in \mathcal{CL}$). In Example 3 IRR $T$ is not order generating. In fact it can be argued that this example contains a compact sublattice $L$ for which IRR $L = \{1\}$. Hence Prop. 1.4 [H-SP] does not extend to compact semilattices.

We adopt Def. 2.8 of [H-CP] for the "spread" of a $\mathcal{CL}$-object. If $T \in \mathcal{CL}$, one formulation for spread is that $SP(T)$ is the least cardinal $\alpha$ for which there exists $\alpha$ chains in $T$ whose union is a generating set.

**THEOREM 5** Let $T \in \mathcal{CL}$ and let $n$ be a positive integer. TAE.

(1) $SP(T) \le n$;

(2) $T$ is generated by a union of $n$ chains;

(3) $T$ is order-generated by a union of $n$ closed chains;

(4) $T$ is algebraically generated by a union of $n$ closed chains;

(5) $\overline{IRR\, T} \subset \bigcup_{i=1}^{n} C_i$ where each $C_i$ is a compact chain;

(6) $IRR\, T$ has width $\le n$ (Width $\equiv$ largest cardinality of an anti-chain)

(7) $\exists$ a continuous homomorphism $h: \prod_{i=1}^{n} K_i \twoheadrightarrow T$ where each $K_i$ is a compact chain and $Im(h) = T$.

(8) Embedding dimension $(T, \vee) \le n$ (in category $\mathcal{S}$).

(9) If $\hat{T}$ is the dual of $(T_d, \vee)$ [i.e., the lattice of ideals of $(T, \vee)$], then $\hat{T}$ as a $\mathcal{Z}$-object has $SP(\hat{T}) \le n$.

Proof (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) follows from definitions of spread and Lemma 2.7 [H-SP].

(2) $\Leftrightarrow$ (4) $\Leftrightarrow$ (7) follows from Prop. 4.

4 $\Leftrightarrow$ (3) $\Leftrightarrow$ (5) follows from Prop. 2.

(6) $\Rightarrow$ (2) by Dilworth's theorem (width $X \le n \Rightarrow X = \bigcup_{i=1}^{n} C_i$) and Prop 1.4 [H-SP].

(5) $\Rightarrow$ (6) Immediate.

~~(1) $\Rightarrow$ (8) By Lemma 2.2 [H-SP] $\exists$ a ... $\prod_{i=1}^{n}$ ...~~

~~each $K_i$ is a compact chain and $h$ is~~

(7) $\Rightarrow$ (8) Take $\hat{h} : T \to \prod_{i=1}^{n} K_i$ where $\hat{h}(t) = \inf h^{-1}(t)$.

Then $\hat{h}$ is one-to-one and preserves sups, hence is an embedding of $(T, \vee)$.

(8) $\Rightarrow$ (9) By HMS duality $\exists\, h' : \widehat{\prod_{i=1}^{n} K_i} \left(= \prod_{i=1}^{n} \hat{K}_i\right)$ $\twoheadrightarrow \hat{T}$. $K_i$ a chain $\Rightarrow \hat{K}_i$ is a compact chain.
Since (7) $\Leftrightarrow$ (1), $Sp(\hat{T}) \le n$.

(9) $\Rightarrow$ (7) Again since (1) $\Leftrightarrow$ (1), $\exists$ compact chains

$C_1, \ldots, C_n$ and a continuous homomorphism $h : \prod_{i=1}^{n} C_i \twoheadrightarrow \hat{T}$

which is onto. By ATLAS $\exists$ $p : \hat{T} \twoheadrightarrow T$, a continuous

homomorphism of $\mathcal{CL}$-objects. The composition $p \circ h$ gives (7).

**COROLLARY 6** Let $T \in \mathcal{CL}$. Then

$$SP(T) = \text{Width}(\text{IRR}\,T) = \text{Embedding dim.}(T, \vee) = SP\,\widehat{(T, \vee)}$$

if any of these quantities is finite.

**COROLLARY 7** Let $T \in \mathcal{CL}$ and suppose $T$ is distributive.
Then $SP(T) = Br(T)$.

Proof. $SP(T) \le n \Rightarrow Br(T) \le n$ by (7) of Thm. 5
since a homomorphism cannot raise breadth. Hence
$Br(T) \le SP(T)$ always.
Conversely in the distributive case if
$Br(T) \le n$, it is not difficult to show (see (8) of JDLI or
Lemma 3.2 of H-SP) that Width (PRIME $T$) $\le n$. Since
PRIME $T$ = IRR $T$, $SP(T) \le n$ by (6) of Thm 5. Hence $SP(T) \le Br(T)$.

~~[illegible]~~ Jim Lea writes that he and K. Baker considered some related notions (mainly for distributive lattices). I summarize their definitions and Baker's results. (I do not have the proofs). Let $(L, \wedge, \vee)$ be a lattice.

(1) order dimension $(L) \equiv$ The least cardinal number of linear order relations on $L$ whose intersection, as a subset of $L \times L$ is the given order relation on $L$ = least cardinal for which $L$ as a poset can be embedded in product of chains of that cardinality

(2) Embedding dimension $(L) \equiv$ The least cardinal of chains in which $(L, \vee)$ can be embedding as a subsemilattice (or a sublattice if $L$ is distributive).

(3) Breadth $(L)$

(4) $Cov(L)$, the covering number of $L \equiv$ Greatest number of elements which cover any single element ($b$ covers $a$ if $[a,b] = \{a, b\}$)

(5) width ( IRR $L$)

(6) width $(\pi(L))$ ~~[illegible]~~ $\equiv$ width of prime ideal space = width of PRIME $(L, \vee)$

~~(Facts for [illegible])~~

(7) subcubic dimension $(L) \equiv$ sup of all $n \ni L$ contains a sublattice isomorphic to $2^n$

(8) orthodimension $(L) \equiv$ sup of all $n \ni \exists$ $n$ distinct elements $x_1, \ldots, x_n$ and an element $c \ni x_i \wedge x_j = c$ for $i \neq j$.

(9) Width (Join irreducibles).

Results:

① For ~~[illegible]~~ distributive lattices (1), (2), (3), (7), (8) are equivalent if all are finite

② All are equivalent for finite distributive lattices and apparently (except for 6) for dist. Z-objects (if all are finite).