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MIXING COEFFICIENT FOR DISCRETE-TIME STOCHASTIC FLOW

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Abstract. In this paper we give an upper bound for α-mixing coefficient for a one-dimensional discrete-time stochastic flow with respect to the spatial parameter.

1. Introduction

A Harris flow [4] $\{x(u,t),\ t \geq 0\}_{u\in\mathbb{R}}$ describes a system of interacting Brownian particles that starts form each point of the line. The most interested case is a flow with coalescing. In this case particles in a system can meet one another and then stick together. Under certain condition [6] on correlation between particles the image $x(\cdot,t)$ $(t>0)$ of any bounded subset of $\mathbb{R}$ consists of a finite number of points, so called clusters. Properties of distribution of clusters in a Harris flow are of our interest. One of the approach to investigate coalesced flows is to study it with respect to the spatial parameter for fixed time $t>0$. We already proved [2] ergodicity of a Harris flow with respect to the spatial parameter and obtained an upper bound for a mixing coefficient. This gives us opportunity to apply known results like central limit theorem and law of iterated logarithm for number of clusters in the Arratia flow (that is a special case of a Harris flow) [3]. In order to find out limiting distribution of some functional one need to know the speed of decreasing of its variance. But we get into difficulties trying to estimate from below the variance of the functional $\sum_{\theta\in\Theta\cap[0,n]}(x(\theta-,t)-x(\theta,t))^2$, where Θ is the set of jumps of the right continuous mapping $x(\cdot,t)$. To handle with this problem we consider a discrete-time approximation of a flow proposed in [7]. Let $\{\xi_n(u), u\in\mathbb{R}\}_{n\geq 1}$ be a sequence of independent stationary Gaussian processes with $\mathbf{E}\xi(u)=0$ and $\mathbf{E}\xi_k(0)\xi_k(u)=\Gamma(u)$ for every $u\in\mathbb{R}$ and $k\geq 1$. Define the sequence of random processes $\{x_n(u), u\in\mathbb{R}\}_{n\geq 0}$ by the following recurrent equation:

$$\begin{aligned} &x_{k+1}(u) = x_k(u) + \xi_{k+1}(x_k(u)),\\ &x_0(u) = u. \end{aligned} \tag{1.1}$$

The ergodicity of such discrete-time flows was discussed in [1]. In the present paper we obtain an upper bound for the α-mixing coefficient for discrete-time flow

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(1.1). In the proof we essentially use the fact that the increments ξ in the scheme (1.1) are Gaussian processes.

2. Main Result

Let us recall the definition of the $\alpha-$ mixing coefficient for a stationary random process.

Definition 2.1. Let $X(u),\ u \in \mathbb{R}$, be a strictly stationary random process.The *α-mixing coefficient* is defined by

$$\alpha(h) = \sup\left\{ \left|\mathbf{P}\left\{A \cap B\right\} - \mathbf{P}\left\{A\right\}\mathbf{P}\left\{B\right\}\right|, A \in \mathcal{F}^0_{-\infty}, B \in \mathcal{F}^{\infty}_h \right\},$$

where $\mathcal{F}^v_u = \sigma\{X(w),\ u \le w \le v\}$.

We will denote by α_n the mixing coefficient for the process x_n. In the next proposition we get the recurrence inequality for mixing coefficients:

Proposition 2.2. *Consider the sequence of processes* $\{x_n(u),\ u \in \mathbb{R}\}_{n\ge 0}$ *that is built on the sequence of independent centered Gaussian processes* $\{\xi_n(u),\ u \in \mathbb{R}\}_{n\ge 0}$ *by the equation 1.1. Assume that the covariance function* Γ *of processes* ξ_n *is such that* $\operatorname{supp}\Gamma \subset [-M, M]$ *for some* $M > 0$. *Then*

$$\alpha_{n+1}(h) \le \alpha_n(h) + \mathbf{P}\left\{ \inf_{\substack{u<0,\\ v>h}} |x_n(u) - x_n(v)| \le M \right\}.$$

Proof. For arbitrary sets $A_i,\ B_i \in \mathcal{B}(\mathbb{R}),\ i = 1, \ldots, k,\ j = 1, \ldots, m$ define the functions

$$f(\vec{x}) = \prod_{i=1}^{k} \mathbb{1}_{\{x_i \in A_i\}},\ \vec{x} \in \mathbb{R}^k, \quad g(\vec{y}) = \prod_{j=1}^{m} \mathbb{1}_{\{y_j \in B_j\}},\ \vec{y} \in \mathbb{R}^m.$$

Also, for $\vec{u} \in \mathbb{R}^k$ denote $x_n(\vec{u}) = (x_n(u_1), \ldots, x_n(u_k))$. Since the collection of the events

$$\left\{ \bigcap_{i=1}^{k} \{x_n(u_i) \in A_i\},\ A_i \in \mathcal{B}(\mathbb{R}),\ k \ge 1,\ u_i < 0 \right\}$$

generates the $\sigma-$field $\sigma(x_n(u),\ u < 0)$ and the collection of the events

$$\left\{ \bigcap_{j=1}^{m} \{x_n(u_j) \in B_j\},\ B_j \in \mathcal{B}(\mathbb{R}),\ m \ge 1,\ u_j > h \right\}$$

generates the $\sigma-$field $\sigma(x_n(u),\ u > h)$, then

$$\alpha_{n+1}(h) = \sup_{\substack{u_i<0,\\ v_j>h}} \left(|\mathbf{E} f(x_{n+1}(\vec{u})) g(x_{n+1}(\vec{v})) - \mathbf{E} f(x_{n+1}(\vec{u})) \mathbf{E} g(x_{n+1}(\vec{v}))| \right).$$

In order to estimate $\alpha_{n+1}(h)$ we consider

$$\begin{aligned}
&\mathbf{E}f(x_{n+1}(\vec{u}))g(x_{n+1}(\vec{v})) - \mathbf{E}f(x_{n+1}(\vec{u}))\mathbf{E}g(x_{n+1}(\vec{v}))\\
&= \mathbf{E}\Big(\mathbf{E}\big(f(x_{n+1}(\vec{u}))g(x_{n+1}(\vec{v}))\big|x_n\big)1\!\!1_{\{\forall i,j|x_n(u_i)-x_n(u_j)|>M\}}\\
&\quad + \mathbf{E}\big(f(x_{n+1}(\vec{u}))g(x_{n+1}(\vec{v}))\big|x_n\big)1\!\!1_{\{\exists i,j|x_n(u_i)-x_n(u_j)|\le M\}}\Big)\\
&\quad - \mathbf{E}f(x_{n+1}(\vec{u}))\mathbf{E}g(x_{n+1}(\vec{v})). \qquad (2.1)
\end{aligned}$$

Since $\operatorname{supp}\Gamma \subset [-M, M]$, then for u, v such that $|u-v| > M$, the random values $\xi_n(u)$ and $\xi_n(v)$ are independent, so

$$\begin{aligned}
&\mathbf{E}\big(f(x_{n+1}(\vec{u}))g(x_{n+1}(\vec{v}))|x_n\big)1\!\!1_{\{\forall i,j|x_n(u_i)-x_n(u_j)|>M\}}\\
&= \mathbf{E}\big(f(x_n(\vec{u}) + \xi_{n+1}(x_n(\vec{u})))g(x_n(\vec{v}) + \xi_{n+1}(x_n(\vec{v})))|x_n\big)1\!\!1_{\{\forall i,j|x_n(u_i)-x_n(u_j)|>M\}}\\
&= \mathbf{E}\big(f(x_n(\vec{u}) + \xi_{n+1}(x_n(\vec{u})))|x_n\big)\mathbf{E}\big(g(x_n(\vec{v}) + \xi_{n+1}(x_n(\vec{v})))|x_n\big)\\
&\quad \times 1\!\!1_{\{\forall i,j|x_n(u_i)-x_n(u_j)|>M\}}. \qquad (2.2)
\end{aligned}$$

Denote by $\tilde{f}(x_n(\vec{u})) = \mathbf{E}\big(f(x_n(\vec{u}) + \xi_{n+1}(x_n(\vec{u})))|x_n\big)$ and let $\tilde{g}(x_n(\vec{v})) = \mathbf{E}\big(g(x_n(\vec{v}) + \xi_{n+1}(x_n(\vec{v})))|x_n\big)$. In this term, gathering together (2.1) and (2.2), we get:

$$\begin{aligned}
&|\mathbf{E}f(x_{n+1}(\vec{u}))g(x_{n+1}(\vec{v})) - \mathbf{E}f(x_{n+1}(\vec{u}))\mathbf{E}g(x_{n+1}(\vec{v}))|\\
&= \Big|\mathbf{E}\Big(\tilde{f}(x_n(\vec{u}))\tilde{g}(x_n(\vec{v}))1\!\!1_{\{\forall i,j|x_n(u_i)-x_n(u_j)|>M\}}\Big) - \mathbf{E}\tilde{f}(x_n(\vec{u}))\mathbf{E}\tilde{g}(x_n(\vec{v}))\\
&\quad + \mathbf{E}\Big(\mathbf{E}f(x_{n+1}(\vec{u}))g(x_{n+1}(\vec{v}))|x_n)1\!\!1_{\{\exists i,j|x_n(u_i)-x_n(u_j)|\le M\}}\Big)\Big|\\
&\le |\mathbf{E}\tilde{f}(x_n(\vec{u}))\tilde{g}(x_n(\vec{v})) - \mathbf{E}\tilde{f}(x_n(\vec{u}))\mathbf{E}\tilde{g}(x_n(\vec{v}))|\\
&\quad + \big|\mathbf{E}\big(\mathbf{E}(f(x_{n+1}(\vec{u}))g(x_{n+1}(\vec{v}))|x_n) - \tilde{f}(x_n(\vec{u}))\tilde{g}(x_n(\vec{v}))\big)1\!\!1_{\{\exists i,j|x_n(u_i)-x_n(u_j)|\le M\}}\big|\\
&\le \alpha_n(h) + \mathbf{P}\left\{\exists i, j\, |x_n(u_i) - x_n(u_j)| \le M\right\}.
\end{aligned}$$

Taking supreme over all $\vec{u} \in \mathbb{R}^k$ and $\vec{v} \in \mathbb{R}^m$, $k \ge 1$, $m \ge 1$, we get the conclusion of the proposition. ☐

Corollary 2.3. *Assume that the covariance function Γ is such that* $\operatorname{supp}\Gamma \subset [-M, M]$ *for some $M > 0$. Then*

$$\alpha_{n+1}(h) \le \alpha_1(h) + \sum_{k=1}^{n-1} \mathbf{P}\left\{\inf_{\substack{u<0,\\ v>h}} |x_k(u) - x_k(v)| \le M\right\}.$$

To obtain an upper bound for the sum in the previous corollary we start with $k = 1$. This term includes an infimum of a Gaussian process and next proposition gives an upper bound for the term.

Proposition 2.4. *Let $\{\xi(u)\}_{u\in\mathbb{R}}$ be a stationary Gaussian process with $\mathbf{E}\xi(u) = 0$ and $\Gamma(u) = \mathbf{E}\xi(u)\xi(0)$. Assume that the covariance function Γ is such that:*

(1) Γ *decreasing on* $[0, \infty)$;
(2) $\Gamma'(u) \ge -1$ *for* $u > 1$;
(3) $1 - \Gamma(u) \sim u^\alpha$ *as* $u \to 0$ *with* $\alpha > 0$;
(4) $\operatorname{supp}\Gamma \subset [-M, M]$ *for some* $M > 0$.

Then for $h > M + d$

$$\mathbf{P}\left\{\inf_{\substack{u<0,\\ v>h}} |\xi(u)-\xi(v)+u-v| \le M\right\} \le C\,\frac{\exp\left\{-\frac{(h-d-M)^2}{4}\right\}}{\sqrt{\pi}(h-d-M)^3},$$

where $d = 4\sqrt{2}\int_0^{\frac{1}{\sqrt{2}}}\sqrt{\log\left(\left(g(\cdot)+\cdot\right)^{-1}\left(\frac{1}{4}u^2\right)\right)^{-2}}du$ *and* $g = 1-\Gamma$.

Proof. By subadditivity of the probability, we have:

$$\begin{aligned}&\mathbf{P}\left\{\inf_{\substack{u<0,\\ v>h}} |\xi(u)-\xi(v)+u-v| < M\right\}\\ &\le \sum_{i=0}^{\infty}\sum_{j=[h]}^{\infty}\mathbf{P}\left\{\inf_{u\in[-i-1,-i],\ v\in[j,j+1]} |\xi(u)-\xi(v)+u-v| < M\right\}\end{aligned} \tag{2.3}$$

Let us estimate each term of this expression:

$$\begin{aligned}&\mathbf{P}\left\{\inf_{u\in[-i-1,-i]}\inf_{v\in[j,j+1]} |\xi(u)-\xi(v)+u-v| < M\right\}\\ &\le \mathbf{P}\left\{\exists u\in[-i-1,-i], \exists v\in[j,j+1]:\ \{\xi(u)-\xi(v) > -M+v-u\}\right\}\\ &\le \mathbf{P}\left\{\exists u\in[-i-1,-i], \exists v\in[j,j+1]:\ \{\xi(u)-\xi(v) > j+i-M\}\right\}\\ &= \mathbf{P}\left\{\sup_{\substack{u\in[-i-1,-i],\\ v\in[j,j+1]}} \xi(u)-\xi(v) > j+i-M\right\}.\end{aligned}$$

To estimate last probability we use Corollary 2 from [5], p. 181, which says that if $\{X(t),\ t\in T\}$ is a centered Gaussian function then for $r > 4\sqrt{2}D(T,\frac{\sigma}{2})$

$$\mathbf{P}\left\{\sup_T X(t) > r\right\} \le 1-\Phi\left(\frac{r-4\sqrt{2}D(T,\frac{\sigma}{2})}{\sigma}\right),$$

where $\sigma^2 = \sup_T \operatorname{Var} X(t)$, $D(T,\varepsilon) = \int_0^{\varepsilon}\sqrt{\log N(T,u)}du$, and $N(T,\varepsilon)$ is the least possible number of points in an ε-net for the set (T,ρ_X) with $\rho_X^2(s,t) = \mathbf{E}(X(s)-X(t))^2$. We apply this result for Gaussian random function $X(\vec{t}\,) = \xi(t_1)-\xi(t_2)$ on the set $T_{ij} = [-i-1,-i]\times[j,j+1]$, $\vec{t}\in T$. First of all, let us check that the Dudley integral $D(T,\varepsilon)$ is finite. To this aim consider the metric ρ_X on T: for $\vec{t} = (t_1,t_2)\in T$ and $\vec{s} = (s_1,s_2)\in T$

$$\begin{aligned}\rho_X^2(\vec{t},\vec{s}) &= \mathbf{E}(X(\vec{t}\,)-X(\vec{s}))^2 = \mathbf{E}X^2(\vec{t}\,)+\mathbf{E}X_2(\vec{s})-2\mathbf{E}X(\vec{t}\,)X(\vec{s})\\ &= 2-2\Gamma(t_1-t_2)+2-2\Gamma(s_1-s_2)\\ &\quad -2\big(\Gamma(t_1-s_1)+\Gamma(t_2-s_2)-\Gamma(t_1-s_2)-\Gamma(t_2-s_1)\big).\end{aligned}$$

Denote by $g(x) = 1-\Gamma(x)$. In this terms

$$\rho_X^2(\vec{t},\vec{s}) = 2\big(g(t_1-t_2)+g(s_1-s_2)+g(t_1-s_1)+g(t_2-s_2)-g(t_1-s_2)-g(t_2-s_1)\big).$$

Consider the set $T_{ij} = [-i-1,-i]\times[j,j+1]$ with metric $\rho_0(\vec{t},\vec{s}) = \max\{|s_1-t_1|,|s_2-t_2|\}$. Denote by B_X the ball in the space (T,ρ_X) and by B_0 the ball in the space (T,ρ_0). We say that for $\varepsilon = 2\sqrt{g(\delta)+\delta}$ the next inclusion holds:

$$B_X(\vec{t},\varepsilon) \supset B_0(\vec{t},\delta).$$

Indeed, let $\vec{s} \in B_0(\vec{t},\delta)$, then

$$\begin{aligned}\rho_X^2(\vec{t},\vec{s}) &= 2\big(g(t_1-t_2)+g(s_1-s_2)+g(t_1-s_1)\\ &\quad +g(t_2-s_2)-g(t_1-s_2)-g(t_2-s_1)\big)\\ &\le 2\big(g(t_1-s_1)+g(t_2-s_2)+|g(t_1-t_2)-g(t_1-s_2)|\\ &\quad +|g(s_1-s_2)-g(t_2-s_1)|\big)\\ &\le 2\big(2g(\delta)+g'(\theta_1)|t_2-s_2|+g'(\theta_2)|t_2-s_2|\big),\end{aligned}$$

since g increasing on $[0,\infty)$ and $|s_1-t_1|<\delta$, $|s_2-t_2|<\delta$ by assumption. In the last expression $\theta_1 \in \big[|t_1-t_2| \wedge |t_1-s_2|; |t_1-t_2| \vee |t_1-s_2|\big]$ and $\theta_2 \in \big[|s_1-s_2| \wedge |t_2-s_1|; |s_1-s_2| \vee |t_2-s_1|\big]$, so $\theta_1 \ge |t_1-t_2| \wedge |t_1-s_2| \ge h-\delta$ and $\theta_2 \ge |s_1-s_2| \wedge |t_2-s_1| \ge h-\delta$, since $t_2 \in [j,j+1]$, $j \ge h$, $t_1 \in [-i-1,-i]$, $i \ge 0$ and $|s_1-t_1|<\delta$, $|s_2-t_2|<\delta$.

By assumption, $g'(x) \le 1$ for $x>1$ so $\rho_X^2(\vec{t},\vec{s}) \le 4(g(\delta)+\delta)$ for $h>1+\delta$ and from this we have

$$B_0(\vec{t},\delta) \subset B_X\left(\vec{t}, 2\sqrt{g(\delta)+\delta}\right).$$

From this conjecture we conclude that

$$N_X(\varepsilon) \le N_0\left((g(\cdot)+\cdot)^{-1}\left(\frac{\varepsilon^2}{4}\right)\right),$$

where $N_X(r)$ and $N_0(r)$ are the least possible number of points in an r-net for the set (T,ρ_X) and (T,ρ_0) respectively and $(g(\cdot)+\cdot)^{-1}$ is the inverse function for the mapping $\delta \mapsto g(\delta)+\delta$. Using obtained inequality forN_X and N_0 we get the upper bound for the Dudley integral for the metric space (T,ρ_X):

$$\begin{aligned}D((T,\rho_X),\varepsilon) &= \int_0^\varepsilon \sqrt{\log N_X(u)}du \le \int_0^\varepsilon \sqrt{\log N_0\left((g(\cdot)+\cdot)^{-1}\left(\frac{u^2}{4}\right)\right)}du\\ &= \int_0^\varepsilon \sqrt{\log\left((g(\cdot)+\cdot)^{-1}\left(\frac{1}{4}u^2\right)\right)^{-2}}du,\end{aligned}$$

since $N_0(\delta) = \left[\frac{1}{\delta^2}\right]$. Using the assumption $g(x) \sim x^\alpha$ as $x \to 0$, we conclude that the right hand side of the last inequality is finite.

For the process X

$$\sigma^2 = \sup_T \operatorname{Var}(X) = \sup_T \mathbf{E}\big(\xi(u)-\xi(v)\big)^2 = \sup_T (2-2\Gamma(u-v)) \le 2.$$

Denote by $d = 4\sqrt{2}\int_0^{\frac{1}{\sqrt{2}}} \sqrt{\log\left((g(\cdot)+\cdot)^{-1}\left(\frac{1}{4}u^2\right)\right)^{-2}}du$ then with Corollary 2 from [5], p. 181 we have: for $j+i-M > d$

$$\mathbf{P}\left\{\sup_{T_{ij}}(\xi(u)-\xi(v)) > j+i-M\right\}$$
$$\leq \frac{1}{\sqrt{\pi}(j+i-M-d)}\exp\left\{-\frac{(j+i-M-d)^2}{4}\right\}.$$

Substitute this upper bound into (2.3) and get:

$$\mathbf{P}\left\{\inf_{\substack{u<0,\\ v>h}} |\xi(u)-\xi(v)+u-v| \leq M\right\}$$
$$\leq \frac{1}{\sqrt{\pi}}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{1}{(h+j+i-M-d)}\exp\left\{-\frac{(h+j+i-M-d)^2}{4}\right\}$$
$$\leq C\frac{1}{\sqrt{\pi}(h-d-M)^3}\exp\left\{-\frac{(h-d-M)^2}{4}\right\}.$$

□

Next theorem is the main result of this paper and it gives an upper bound for the mixing coefficient for discrete-time stochastic flow.

Theorem 2.5. *Assume that the covariance function* Γ *is such that:*

1. Γ *decreasing on* $[0,\infty)$*;*
2. $\Gamma'(u) \geq -1$ *for* $u > 1$*;*
3. $1-\Gamma(u) \sim u^{\alpha}$ *as* $u \to 0$ *with* $\alpha > 0$*;*
4. $\operatorname{supp}\Gamma \subset [-M, M]$ *for some* $M > 0$.

Then for $h > d+M$ *and any* $k > 1$*:*

$$\mathbf{P}\left\{\inf_{\substack{u<0,\\ v>h}} |x_k(u)-x_k(v)| \leq M\right\} \leq kU\left(\frac{h-M}{k}\right)$$

and for $h > n\left(\frac{8}{\alpha}+d\right)+M$

$$\alpha_n(h) \leq \alpha_\xi(h) + \sum_{k=1}^{n-1} kU\left(\frac{h-M}{k}\right),$$

where $U(x) = C\frac{1}{\sqrt{\pi}(x-d)^3}\exp\left\{-\frac{(x-d)^2}{4}\right\}$.

Proof.

$$\mathbf{P}\left\{\inf_{\substack{u<0,\\ v>h}} |x_k(u)-x_k(v)| \le M\right\}$$
$$= \mathbf{E}\Bigg(1\!\!\mathrm{I}_{\left\{\inf_{\substack{u<0,\\ v>h}} |x_{k-1}(u)-x_{k-1}(v)|\le c_{k-1}\right\}} \mathbf{P}\left\{\inf_{\substack{u<0,\\ v>h}} |x_k(u)-x_k(v)| \le M \Big| x_{k-1}\right\}$$
$$+ 1\!\!\mathrm{I}_{\left\{\inf_{\substack{u<0,\\ v>h}} |x_{k-1}(u)-x_{k-1}(v)|> c_{k-1}\right\}} \mathbf{P}\left\{\inf_{\substack{u<0,\\ v>h}} |x_k(u)-x_k(v)| \le M \Big| x_{k-1}\right\}\Bigg). \tag{2.4}$$

Denote by U the upper bound from the proposition 2.4, i.e.,

$$U(x) = C\frac{1}{\sqrt{\pi}(x-d)^3}\exp\left\{-\frac{(x-d)^2}{4}\right\}.$$

In this terms for $h > r + d$

$$\mathbf{P}\left\{\inf_{\substack{u<0,\\ v>h}} |\xi(u)-\xi(v)+u-v| \le r\right\} \le U(h-r).$$

Then for $c_{k-1} > M + d$

$$\mathbf{P}\left\{\inf_{\substack{u<0,\\ v>h}} |\xi_k(x_{k-1}(u))-\xi_k(x_{k-1}(v))+x_{k-1}(u)-x_{k-1}(v)| \le M \Big| x_{k-1}\right\}$$
$$\times 1\!\!\mathrm{I}_{\left\{\inf_{\substack{u<0,\\ v>h}} |x_{k-1}(u)-x_{k-1}(v)|> c_{k-1}\right\}}$$
$$\le U(c_{k-1}-M) 1\!\!\mathrm{I}_{\left\{\inf_{\substack{u<0,\\ v>h}} |x_{k-1}(u)-x_{k-1}(v)|> c_{k-1}\right\}}.$$

Substitute this upper bound into 2.4:

$$\mathbf{P}\left\{\inf_{\substack{u<0,\\ v>h}} |x_k(u)-x_k(v)| \le M\right\} \le \mathbf{P}\left\{\inf_{\substack{u<0,\\ v>h}} |x_{k-1}(u)-x_{k-1}(v)| \le c_{k-1}\right\}$$
$$+ U(c_{k-1}-M)\mathbf{P}\left\{\inf_{\substack{u<0,\\ v>h}} |x_{k-1}(u)-x_{k-1}(v)| > c_{k-1}\right\}$$
$$\le \mathbf{P}\left\{\inf_{\substack{u<0,\\ v>h}} |x_{k-1}(u)-x_{k-1}(v)| \le c_{k-1}\right\} + U(c_{k-1}-M).$$

Continue the inequality with the same arguments, we get: for $c_{j-1} > c_j + d$ and $h > c_1 + d$

$$\begin{aligned}&\mathbf{P}\left\{\inf_{\substack{u<0,\\ v>h}} |x_k(u) - x_k(v)| \le M\right\}\\ &\le \mathbf{P}\left\{\inf_{\substack{u<0,\\ v>h}} |x_{k-2}(u) - x_{k-2}(v)| \le c_{k-2}\right\} + U(c_{k-2} - c_{k-1}) + U(c_{k-1} - M)\\ &\le \mathbf{P}\left\{\inf_{\substack{u<0,\\ v>h}} |x_1(u) - x_1(v)| \le c_1\right\} + U(c_1 - c_2) + \ldots + U(c_{k-2} - c_{k-1})\\ &\quad + U(c_{k-1} - M) \le U(h - c_1) + U(c_1 - c_2) + \ldots + U(c_{k-1} - M).\end{aligned}$$

If we take $c_0 = h$, $c_j = c_0 - j\frac{h-M}{k}$ and assuming $\frac{h-M}{k} > d$ we obtain

$$\mathbf{P}\left\{\inf_{\substack{u<0,\\ v>h}} |x_k(u) - x_k(v)| \le M\right\} \le kU\left(\frac{h-M}{k}\right).$$

Substitute this inequality to the upper bound for $\alpha_n(h)$ from the Corollary 2.3 we get: for $h > nd + M$

$$\alpha_n(h) \le \alpha_1(h) + \sum_{k=1}^{n-1} kU\left(\frac{h-M}{k}\right).$$

□

Corollary 2.6. *Let $h_n \sim n^a$ as $n \to \infty$ with $a > 1$ Then*

$$\alpha_n(h) \le u_n \sim n^{(5-3a)} \exp\left\{-\frac{n^{2(a-1)}}{4}\right\} \quad \textit{as } n \to \infty$$

Proof. The theorem 2.5 gives an upper bound for α_n : for $h > n\left(\frac{8}{\alpha} + d\right) + M$

$$\alpha_n(h) \le \alpha_\xi(h) + \sum_{k=1}^{n-1} kU\left(\frac{h-M}{k}\right),$$

where $U(x) = C\frac{1}{\sqrt{\pi}(x-d)^3}\exp\left\{-\frac{(x-d)^2}{4}\right\}$. Let us estimate $\sum_{k=1}^{n-1} kU\left(\frac{h-M}{k}\right)$

$$\begin{aligned}\sum_{k=1}^{n-1} kU\left(\frac{h-M}{k}\right) &\le \int_1^n U\left(\frac{h-M}{x}\right)dx \\ &= C\int_1^n \frac{1}{\sqrt{\pi}(\frac{h-M}{x}-d)^3}\exp\left\{-\frac{\left(\frac{h-M}{x}-d\right)^2}{4}\right\}dx \\ &= \frac{C}{\sqrt{\pi}}\int_{\frac{h-M}{n}-d}^{h-M-d}\frac{h-M}{y^3(y+d)^2}e^{-\frac{y^2}{4}}dy \\ &\le \frac{h-M}{\left(\frac{h-M}{n}-d\right)^5}e^{-\frac{\left(\frac{h-M}{n}-d\right)^2}{4}}\left(h-M-\frac{h-M}{n}\right).\end{aligned}$$

Under assumption $h \sim n^a$ the last expression is equivalent to

$$n^{(5-3a)}\exp\left\{-\frac{n^{2(a-1)}}{4}\right\}$$

as $n \to \infty$. Corollary is proved. □

Note that in [2] we get the following upper bound for the mixing coefficient for the Harris flow with local characteristic Γ such that $\mathrm{supp}\Gamma \subset [-M, M]$: for $h > M$

$$\alpha(h) \le 2\sqrt{\frac{2}{\pi}}\int_{h-c}^{\infty} e^{-x^2/2}dx \le (h-c)^{-1}\exp\left\{-\frac{(h-c)^2}{2}\right\}.$$

So, we can see that $\alpha(h_n)$ decrease to zero faster than $\alpha_n(h_n)$ with $h_n = n^a$, $a > 1$.

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